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Limit Torque for a Frictional Joint*

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ABSTRACT

A friction limit moment (torque) is derived for a one-bolt, two-member structural joint. Once the structural load reaches the limit moment, the frictional resistance in the joint is overcome and relative rotation of the structural members can occur. A framework based on the kinematical approach of limit analysis is proposed to calculate the limit torque. While only the simple case of calculations is presented in this paper, more complex joints can be analyzed using the approach proposed. The friction limit loads derived using this approach can be represented as a surface in a generalized stress space, analogously to the yield condition of plasticity. Upon the structural load reaching the threshold (limit) friction load, the process of relative movement of structural elements (rotation and translation) becomes possible until the joint locks up due to structural constraints. Results have application in analysis of structures with gaps in their joints, particularly struc-

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tures with deliberately introduced gaps. The response of such structures to loads can differ significantly from those where gaps are not present, and some benefits can be gained, e.g., an increase in the range of the elastic response. Application also can be envisioned in systems that are required to dissipate energy.

I. INTRODUCTION

The problem considered here has practical implications for mechanical systems where frictional joints are used to arrange the members in particular positions. Such systems are used, e.g., in robotics, the simplest example of which is a multi-arm extension lamp. Friction effects also can be used in structural joints to dissipate energy when the structure is subjected to dynamic loads. The main motivation here is to develop an engineering technique that makes it possible to include frictional effects in structural systems with "slackened" connections. In recent years quasi-static load processes of frictionless slackened structures have been considered by Gawecki [1, 2]. This paper relates to structural systems with gaps, and aims toward the description of the irreversible behavior of such systems.

The relative motion in joints with gaps has a complex kinematics which, in most cases, can be decomposed into two fundamental components, rigid translation and rotation. Whereas rigid translation is relatively simple to analyze when Amontons-Coulomb friction is involved, the rotation mode appears to be complicated. The limit torque cannot be determined without consideration of the distribution of the interfacial pressure or without utilizing the kinematical theorem of limit analysis as applied to dry friction.

This paper considers a joint with a freedom of rotation inhibited by frictional interaction. An ideal model of a joint is considered, where the structural connection contains two circular collar plates linked by a single bolt, as shown in Fig. 1.

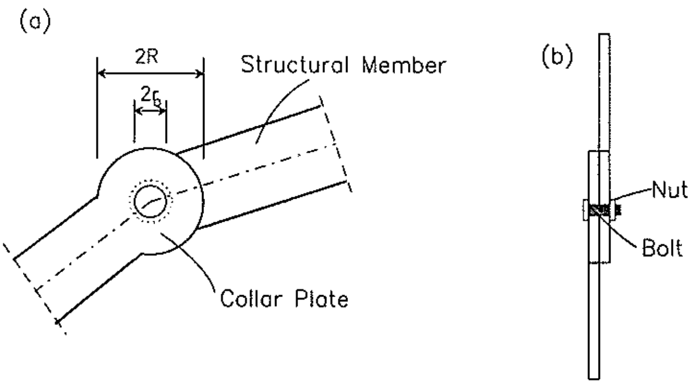

Fig. 1 A simple joint with freedom of rotation inhibited by friction.

Rotation is allowed about the bolt with the frictional resistance induced on the interface between the plates. Fundamental relations are reviewed in the next section, followed by an analysis of the collapse of the perfect joint model. Application of the results to more complex multi-bolt connections is then indicated, after which the paper is summarized with some final remarks.

II. DESCRIPTION OF DRY FRICTION IN TERMS OF PLASTICITY THEORY

The simple dry friction (Amontons-Coulomb) condition is considered between metal plates of a joint with the limit shear force P_t, proportional to the normal load P_n (compression is defined positive) as

$$f(P_n, P_t) = |P_t| - \mu P_n = 0 \tag{1}$$

where μ is the friction coefficient, as shown in Fig. 2. The rate of relative displacement $[\dot{u}]_i$ is assumed to be governed by the normality rule

$$[\dot{u}]_i = \dot{\lambda} \frac{\partial f(P_i)}{\partial P_i} \tag{2}$$

where P_i is the force vector with components P_n and P_t, and $\dot{\lambda}$ is a nonnegative multiplier. Such a rule, in conjunction with frictional problems, was suggested earlier by Mróz and Drescher [3], Collins [4], and Michalowski and Mróz [5].

It can be argued that the kinematics of mechanical systems with frictional interfaces derived from the normality (or associative) rule of Eq. 2 is not realistic, since it predicts a velocity component normal to the friction-sliding interface (see Fig. 2). The assumption of normality is discussed here in some detail.

The controversy of using the normality rule arises in the mechanics of frictional materials (such as granular media), where dilatancy occurs during deformation,

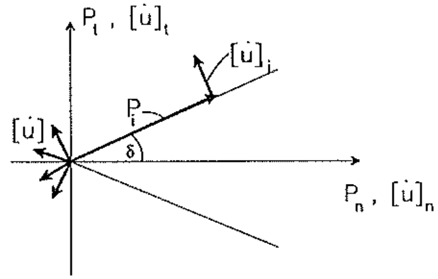

Fig. 2 Dry friction condition.

but at a lesser rate than that suggested by the normality rule. The non-associativity of deformation of granular materials has an effect on solutions to boundary-value problems. It can be clearly shown that the traction vector on a rupture layer within a frictional (granular) material conforms to the friction condition (Mohr-Coulomb criterion) only when deformation is governed by the normality rule. In keeping with the non-associative flow law, the traction on a rupture surface (velocity discontinuity) does not satisfy the friction condition.

An analogy can be suggested between a rupture surface within a granular material and a solid surface with dry friction. It may be noted, however, that such an analogy holds only when the normality rule is enforced for both. In the case of the non-associative rule, the traction on a rupture surface within a frictional material is no longer inclined at the angle of internal friction to the normal of that surface, whereas traction on a solid frictional surface is independent of the sliding rule.

The consequence of the last statement is that the solution to a problem involving solid surface friction must be independent of the particular sliding law, whereas the flow rule does affect the solution for a material with internal friction. Indeed, by the principle of virtual work, the limit force in a boundary-value problem involving dry friction must be independent of the specific motion rule. It is convenient, however, to use the associative rule since, for the dry friction condition (see Fig. 2), normality leads to a zero rate of energy dissipation. Otherwise, the distribution of the contact stress must be known in order to calculate the energy dissipation rate.

In conclusion, even though the associative flow rule (sliding rule) may not be physically reasonable, its application is fully justifiable in solving mechanics problems with solid surface friction.

The upper bound theorem of limit analysis will be used in the next section to calculate estimates of the limit generalized forces (moment and normal force) applied to a frictional joint with a limited freedom of rotation. This theorem states that *in any kinematically admissible mechanism the rate of energy dissipation is not less than the rate of work of the externally applied load* (see, e.g., Ref. 6). Denoting the kinematically admissible strain rate by $\dot{\epsilon}^k_{ij}$, this theorem takes the form

$$\int_V \sigma_{ij}\dot{\epsilon}^k_{ij}dV \geq \int_{S_u} T_i\dot{u}_i dS_u \qquad (3)$$

where σ_{ij} is the stress tensor associated with the admissible kinematics, V is the volume of deforming material, S_u is the boundary where velocity $\dot{u}_i$ is given, and the surface traction T_i is unknown. Note that in structural systems with friction, the term on the left side of the inequality given by Eq. 3 must include the rate of dissipation on frictional interfaces.

III. FRICTION TORQUE FOR AN IDEAL JOINT

The collapse criterion of a frictional joint with a single bolt is derived in this section. The joint is considered to collapse when the structural elements start to rotate with respect to one another. Three different procedures to calculate the limit torque, based on the kinematical approach of limit analysis, are employed. These procedures differ in the assumed kinematics associated with the frictional rotation of the joint. In these three cases, the collar plates are considered as elastic, perfectly plastic, and rigid, respectively. In each case, the upper bound theorem of limit analysis is used to find the limit torque. In addition, an approximate solution based on an assumption of uniform distribution of stress between the plates is shown.

An ideal two-member connection is considered, as shown in Fig. 1. Resistance to rotation comes from friction on the interface between the two collar plates. A "sandwich" connection with multiple plates will not be considered, but the analysis can easily be extended to such a case. The major idealization of the joint lies in neglecting the stiffness of the bolt nut and in neglecting the torsion of the bolt (or frictional resistance between the nut and the collar) during the incipient rotation.

First, the connection plates are considered as thin elastic plates, and the contact pressure between them is calculated according to the theory of elasticity [7]. The circular plates are compressed together with a bolt by force N, as shown in Fig. 3(a). The collapse of the joint is associated with the relative rotation of the collar plates. Equation 3 is used to calculate the collapse torque. However, in order to calculate the energy dissipation rate on the left side of the inequality of Eq. 3, the distribution of the interface pressure must be known. This distribution is calculated, first considering the joint as two thin circular elastic plates of radius R with an opening of radius r_0, subjected to a deformation mode consistent with the normality rule in Eq. 2. The latter requires the sliding velocity vector on the interface to be inclined at the angle of friction to that interface at each point. The deformed platens must then assume conical shapes such that the ratio of separation

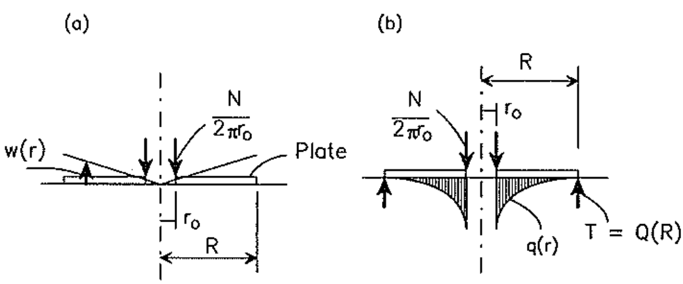

Fig. 3 A thin collar plate: (a) Conical deformation; (b) Distribution of contact pressure.

distance $w(r)$ to the relative tangential sliding is equal to the coefficient of friction μ (see Fig. 3(a)).

The equation for the elastic deflection of an axisymmetric thin plate is

$$\frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r^2}\frac{d^2w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr} = \frac{q}{D} \tag{4}$$

where w is deflection of the plate, q is contact pressure, and D is plate stiffness ($D = Eh^3/12(1 - \nu^2)$, E and ν are Young's modulus and Poisson's ratio, respectively, and h is plate thickness). A small deflection of the plate compatible with the normality rule of relative frictional sliding is described as

$$w(r) = \mu\omega r \tag{5}$$

where ω is a small angle of rotation. The plate is supported at the internal circumference r_0 by total reaction N; hence the shear force at r_0 must be equal to $Q = N/2\pi r_0$. The shear force can be expressed as

$$Q(r) = -D\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right) \tag{6}$$

Utilizing Eqs. 4–6 and noting that $Q = N/2\pi r_0$ at r_0, relations for the distribution of the contact pressure $q(r)$ and the shear force $Q(r)$ are obtained as

$$q(r) = \frac{Nr_0}{2\pi r^3}, \qquad Q(r) = \frac{Nr_0}{2\pi r^2} \tag{7}$$

The distribution of the contact pressure is shown in Fig. 3(b). Note that the shear force Q at the outer edge $r = R$ is not equal to zero, and needs to be counterbalanced with a contact force distributed along the outer perimeter, $T = Q(R)$. This force needs to be accounted for later, in addition to contact pressure $q(r)$, when calculating energy dissipation during frictional sliding. Moreover, nonzero bending moments appear at $r = r_0$ and $r = R$. This is a consequence of the assumed conical deformation mode of the plates used here.

Having derived the equation that describes the distribution of normal contact pressure $q(r)$ and $Q(R)$, Eq. 3 is used to find the upper bound to collapse torsional moment M_f about the connection bolt. Calculations can be performed using two different modes of deformation, one consistent with the normality rule (also used to derive the contact stress distribution), and the other an instantaneous mode, where the relative rotation of the two parts of the joint occurs without any separation component (non-associative motion rule). Both lead to identical solutions for the limit fictional moment M_f. Using the associative rule, the energy dissipation rate during relative rotation is equal to zero, but the effect of contact stresses and force

T at $r = R$ on the separation velocity must be accounted for. In the second case, no separation velocity is allowed. Thus, the contact stresses and T do no work, but the energy dissipation rate is now nonzero. Consequently, both mechanisms lead to the same limit moment M_f. The latter is used here.

The left side of the inequality given by Eq. 3 represents the rate of energy dissipation, and, for the frictional mechanism considered, can be written as

$$\dot{D} = \mu\dot{\omega} \int_S q(r) r dS + \mu\dot{\omega} 2\pi R^2 Q(R) \tag{8}$$

where S is the area of the contact surface and $\dot{\omega}$ is the rate of relative rotation of the two structural members (see Fig. 1) with respect to one another. The first term in Eq. 8 represents the dissipation rate over the contact surface, and the second term is due to the force distributed along the outer circumference of the interface of the connection plates. Realistically, if frictional sliding occurs only between the connection plates, a bolt holding the joint together is being twisted. The dissipation due to the torsion of the bolt is, however, neglected here.

The term on the right side of Eq. 3 is equal to the product of the unkown collapse torque M_f and the rate of rotation $\dot{\omega}$, plus the normal force (compression force) times the separation rate of the two parts of the joint. Since in the instantaneous mechanism considered the separation rate is zero, the compression force (the axial force in the bolt) does no work during rotation, and the rate of work of external forces $\dot{W}$ is

$$\dot{W} = M_f \dot{\omega} \tag{9}$$

The upper bound of torque M_f can be found by requiring that $\dot{D} = \dot{W}$ (see Eq. 3), and it can be represented in a dimensionless form as

$$\frac{M_f}{NR} = \mu \frac{r_0}{R} \left(1 + \ln \frac{R}{r_0}\right) \tag{10}$$

Frictional sliding without a separation component does not conform to the normality rule. However, as mentioned earlier, such a mechanism leads to a solution to the limit load identical with that based on the associative rule. This comes from the fact that the inequality of Eq. 3 has a structure identical to the principle of virtual work, which leads to a static equilibrium solution independent of the specific flow rule. The only disputable point is the use of the stress distribution obtained from the elasticity solution for a thin plate. At the very least, however, Eq. 10 is a reasonable approximate solution.

The second mechanism of failure is depicted in Fig. 4(a). Collapse of the joint occurs here due to plastic failure of the two circular plates. The two plates of the

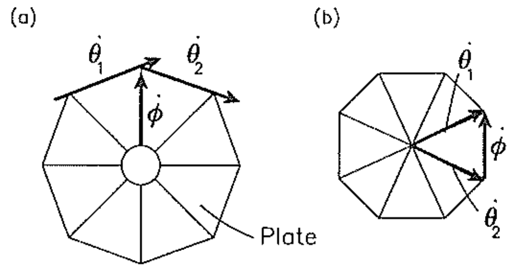

Fig. 4 (a) Yield-line pattern for a collar plate; (b) Velocity hodograph.

connection model adhere to one another, and a kinematically admissible incipient mechanism with relative rotation and axisymmetrical separation is considered where the vector of relative motion at each point of the interface is inclined at an angle of friction $\delta(\tan\delta = \mu)$ to that interface. The kinematics here is identical to that used earlier to calculate the elastic distribution of the contact pressure, but this time the plate is considered to have reached the plastic state (yielding). Such relative motion conforms to the normality rule in Eq. 2. The consequence of such kinematics is that the energy dissipation rate due to frictional sliding becomes zero, since the interfacial stress vector and the relative velocity vector are mutually perpendicular at each point of the interface.

A finite displacement with a separation component between the two parts of the connection subjected to rotation is not realistic, but an incipient mechanism of that kind is kinematically admissible, and can be used in conjunction with the upper bound theorem of limit analysis (for other applications of a similar concept see, e.g., Ref. 4).

In the collapse mechanism considered now, the energy is dissipated within the deforming plates, and is calculated using the yield line theory. Due to the symmetry of the connection model, dissipation is calculated for one plate subjected to a conical deformation (identical to the one shown in Fig. 3(a)). To calculate the energy dissipation rate, the plate is divided into n sectors, as shown in Fig. 4(a), separated by the yield lines. The hodograph in Fig. 4(b) represents the rates of rotation of the plate segments and the rates of relative rotations along the yield lines. In the particular case shown, $n = 8$. Assuming the rate of rotation of a single sector about the outer edge to be $\dot{\theta}$, and using the geometrical relation in Fig. 4(b), the rate of relative rotation along the yield lines $\dot{\phi}$ can be calculated as

$$\dot{\phi} = 2\dot{\theta}\sin\frac{\pi}{n} \tag{11}$$

and the rate of energy dissipation along all the yield lines becomes

$$\dot{D} = 2n(R - r_0)M_0\dot{\theta}\sin\frac{\pi}{n} \tag{12}$$

where M_0 is the plastic moment of the plate (per unit length). The deformation of the plate tends to the continual deformation mode when $n \to \infty$. In limit $n \to \infty$, the dissipation rate in Eq. 12 becomes

$$\dot{D} = 2\pi\dot{\theta}(R - r_0)M_0 \tag{13}$$

where $\dot{\theta} = \mu\dot{\omega}$ for the mechanism considered, and $\dot{\omega}$ is the rate of rotation of the two structural members (Fig. 1) with respect to one another. The rate of work of external forces is

$$\dot{W} = M_f\dot{\omega} - N\mu\dot{\omega}r_0 \tag{14}$$

where N is the joint compression force. The minus sign in Eq. 14 results from the fact that force N acts opposite to the separation velocity $\mu\dot{\omega}r_0$ (compressive force is positive, consistent with Eq. 1). For as long as the same relative motion of the two plates is considered (here, expressed by the normality rule in Eq. 2), both $\dot{D}$ and $\dot{W}$ are expressed by Eq. 13 and Eq. 14, no matter whether the deformation mechanism includes yielding of both plates or yielding is localized to one plate. By equating the dissipation rate in Eq. 13 to the rate of external work in Eq. 14, the following expression is obtained for the limit torque in the joint:

$$\frac{M_f}{NR} = \mu\left(2\pi\frac{M_0}{N}\left(1 - \frac{r_0}{R}\right) + \frac{r_0}{R}\right) \tag{15}$$

The joint is held together by the bolt with a compression nut of radius slightly larger than r_0. Thus, additional energy dissipation may occur due to plastic rotation of the plate along the perimeter of the compression nut. This dissipation rate is equal to

$$\dot{D}_2 = 2\pi r_0\dot{\theta}M_0 = 2\pi r_0\mu\dot{\omega}M_0 \tag{16}$$

If this dissipation is included in the entire energy balance, the limit torque becomes

$$\frac{M_f}{NR} = \mu\left(2\pi\frac{M_0}{N} + \frac{r_0}{R}\right) \tag{17}$$

The moment in Eq. 17 will, of course, always be larger than that in Eq. 15.

The third mechanism of the collapse can be conceived when the plastic stiffness of the connection plates increases significantly, and the two plates separate as rigid bodies. The entire force N must now be distributed along the outer perimeter R, since the normality rule indicates that the contact stresses for $r_0 \le r < R$ are zero (singular point O on the Amontons-Coulomb friction condition; see Fig. 2). Consequently, the limit torque becomes

$$\frac{M_f}{NR} = \mu \tag{18}$$

For reasonable r_0/R, Eq. 18 significantly overestimates the limit torque with respect to previous solutions.

The last solution to the limit torque is based on the assumption that the stress between the plates of the joint is distributed uniformly (this is a standard assumption made in calculations of the threshold moment in nonlubricated thrust bearings). Consequently, for circular connection plates, the moment can be integrated to yield

$$\frac{M_f}{RN} = \frac{2}{3}\mu\left[1 + \frac{\left(\frac{r_0}{R}\right)^2}{1 + \frac{r_0}{R}}\right] \tag{19}$$

This is an approximate solution, which cannot be interpreted as an upper bound to the limit torque.

IV. DISCUSSION OF RESULTS FOR THE IDEAL TWO-MEMBER CONNECTION

The results presented here are theoretical. No experimental tests are available to validate these solutions.

The functions in Eqs. 10, 15, and 19 are shown in Fig. 5 in the range $0 \leq r_0/R \leq 1$. Physical constraints require that r_0/R be larger than zero and smaller than one. Based on the interface pressure distribution given by the theory of thin plates, frictional resistance to rotation of an ideal joint (with thin collar plates; Fig. 1(b)) is very low when the radius of the opening in the joint r_0 is small compared to the diameter of the collar plates R. It increases rapidly, however, with an increase in r_0/R. This dependence is nonlinear, and the friction torque expressed as M_f/NR

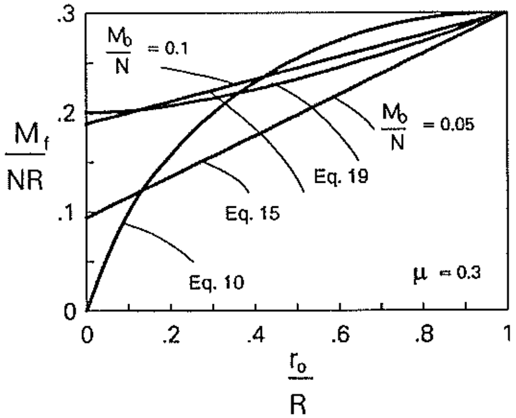

Fig. 5 Upper bounds to the limit torque.

(N is joint compression force; R is outer radius of the collar plates) reaches the magnitude of the friction coefficient μ when $r_0/R = 1$.

If the plastic moment of the collar plates M_0 is low, the frictional torque calculated from the mechanism that includes plastic yielding of the collar plates (Eq. 15) will be lower than that based on the stress distribution from the theory of thin plates. For example, when $\mu = 0.3$ and the plastic moment $M_0/N = 0.05$, the mechanism based on the plastic yielding of the connection plates gives a lower estimate of the frictional torque when $r_0/R > 0.14$. Since the calculations are based on the upper-bound approach of limit analysis, the lower moment is a better estimate of the true friction torque. The torque can be increased, of course, by increasing the plastic moment of the plates. For example, when $M_0/N = 0.1$, the plastic yielding of plates would not be expected to occur until $r_0/R > 0.42$. For most realistic connections, the plastic moment M_0 is quite large and, consequently, the solution in Eq. 15 gives a high limit torque value.

If the connection plates are thick, the theory of thin plates is no longer applicable and, according to Eq. 18, M_f/NR will reach the magnitude of μ. The approximate solution in Eq. 19 yields a limit frictional moment larger than that found using the flexible plate approach of Eq. 10 when r_0/R is less than 0.36, and it falls very close to the plastic plate solution when the plastic moment M_0/N of the plate is 0.1. Notice that when the plastic moment of the connection plates M_0/N increases beyond $1/2\pi$, the estimate of frictional torque from Eq. 15 will be larger than μ. The usefulness of these solutions can be questioned, since the limit torque is a function of the normal force in the bolt, and this force is rarely seen in most structures. Such joints, however, may be designed with a known bolt force purposely, e.g., with the intention to control energy dissipation. Application of such structures can be envisioned in earthquake-prone regions and space stations (with no gravity field), for example.

While the joints in traditional metal structures are likely to be relatively rigid, consideration of loads necessary to overcome frictional resistance may prove useful for structures where gaps in joints are designed deliberately. Structures with such clearances in the joints are discussed in the next section.

V. APPLICATION TO SLACKENED CONNECTIONS WITH MULTIPLE BOLTS

Structural systems with gaps at connections exhibit a peculiar response to applied loads, particularly when irreversible deformation occurs. Gaps present at structural joints can originate from long-term service under variable loads or from manufacturing and construction inaccuracies, or they can be designed deliberately, e.g., in order to increase the range of elastic response of a structure [2].

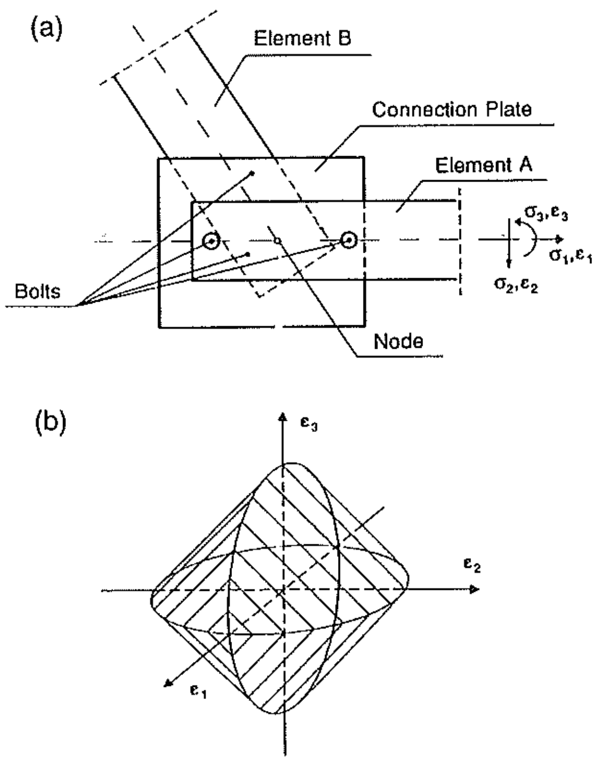

Fig. 6 (a) Multi-bolt joint with a separate connection plate; (b) Clearance strain surface.

A model of a realistic joint with gaps is shown in Fig. 6(a). Assume, for simplicity, that the elements of the structure are rigid, and that the only source of deformation is the gaps at the joints. Two structural members are jointed together with four bolts, all attached to a connection plate. The change of configuration of a structural member, e.g., member A, with respect to the connection plate is described here by generalized strains ϵ_i, $i = 1, 2, 3$. Because of the gaps between the bolts and the openings in the structural members, relative movement of the members and the connection plate is possible to a certain extent. The relative displacements (translation and rotation) are considered here as generalized clearance strains (ϵ_i). The maximum extent of these displacements is described by the clearance surface in generalized strain space. Such a surface for the element A of the joint in Fig. 6(a) is presented in Fig. 6(b) (see also Ref. 2). Note that only for rigid elements are the generalized strains of the members equal to the clearance strains. For elastoplastic elements, the generalized strains ϵ_i have elastic, plastic, and clearance components.

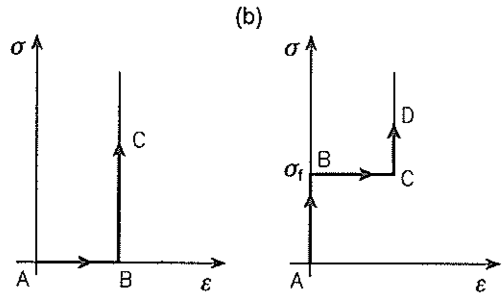

Fig. 7 (a) Stress-strain behavior of a "slackened" joint in the absence of friction; (b) Response of a joint with frictional resistance (rigid elements).

In the absence of friction, the stress-strain behavior of a slackened joint has the shape shown schematically in Fig. 7(a). The symbol σ here represents a generalized stress, e.g., an axial force or a bending moment in the element at the connection, and ϵ is a generalized strain. Segment AB corresponds to the clearance mechanism (closing of gaps), and segment BC to loading of the nondeformable (locked) connection.

When the frictional interface between a structural element and the connection plate is considered, the generalized stress-strain behavior is modified where the deformation starts at some limit stress σ_f related to friction in the joint (Fig. 7(b)). Segment BC now corresponds to the frictional slip mode. The fundamental problem in frictional multiple-bolt connections is evaluation of the generalized stress σ_f. This stress function is analogous to the yield condition of plasticity and can be represented as a surface in the generalized stress space. Such a limit function was presented in the previous section for a simple one-bolt connection with freedom of frictional rotation only. Development of such surfaces for a general case is complicated, and it will be attempted in a subsequent paper. The intent here is to present a concept that would be useful in analysis of such joints.

VI. FINAL REMARKS

The kinematical approach of limit analysis is a useful tool in deriving limit conditions of joints with frictional constraints. Irreversible generalized strains in such joints (relative translation and rotation of members) become possible after the limit condition is reached. The limit condition describes the combinations of generalized forces necessary to initiate the process of closing joint gaps until the joint locks up. This paper is a step toward describing the irreversible behavior of structural systems with gaps at the joints. Application of the limit analysis approach to calculation of the threshold moment associated with frictional rotation for a simple joint is presented. While derivation of the limit surfaces that describe the threshold load for closing gaps in more realistic multiple-bolt joints seems to be a complex task, the framework outlined in this paper is well-suited for the task.

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REFERENCES

1. A. Gawecki, Elastic-plastic beams and frames with unilateral boundary conditions, *J. Struct. Mech.* **14**: 53–76 (1986).
2. A. Gawecki, Elasto-plasticity of slackened systems, *Arch. Mech.* **44**: 363–390 (1992).
3. Z. Mróz and A. Drescher, Limit plasticity approach to some cases of flow of bulk solids, *J. Eng. Ind. Trans ASME* **51**: 357–364 (1969).
4. I. F. Collins, The upper-bound theorem for rigid/plastic solids generalized to include Coulomb friction, *J. Mech. Phys. Solids* **17**: 323–338 (1969).
5. R. L. Michalowski and Z. Mróz, Associated and non-associated sliding rules in contact friction problems, *Arch. Mech.* **30**: 259–276 (1978).
6. D. C. Drucker, H. J. Greenberg, and W. Prager, Extended limit design theorems for continuous media, *Q. Appl. Math.* **9**: 381–389 (1952).
7. S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York, 1959.

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