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AN APPROXIMATE SOLUTION TO A PROBLEM OF PSEUDO-STEADY FLOW OF STRAIN-HARDENING MATERIAL

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Abstract—An approximate technique is presented for estimating an external force in steady or pseudo-steady flow of strain-hardening material, for which the deformation can be simplified to a rigid-body motion mechanism. The rigid blocks are separated from each other by layers of the material undergoing intensive shear. The strain-hardening effect is incorporated into the analysis by assuming the flow shear stress as an exponential function of equivalent strain. This technique, similar in its nature to the upper bound approach of limit analysis, cannot be rigorously regarded as such since the distribution of the shear stress and the boundaries in pseudo-steady state problems cannot be specified *a priori*. It is shown that the rate of work-dissipation within a shear layer depends on the total increase of the tangential velocity and is independent of its thickness, as in the case of perfectly plastic solids. The results of computations for a pseudo-steady pyramidal indentation of a strain-hardening solid are presented and compared to the experimentally measured forces reported by other authors. Computations for a pseudo-steady flattening of a tetrahedral asperity are also briefly reported.

NOTATION

δ	thickness of a shear layer
x, y	local co-ordinate system, x being tangent to the shear layer
$\dot{D}$	work-rate per unit area of a shear layer
k	shear yield stress
k_0, b, a	parameters of the stress–strain relation
$\bar{\varepsilon}$	equivalent strain
$[V], [V]^*$	tangential velocity increase across a shear layer and its total value
V_y	material velocity component normal to the shear layer
θ	included semi-angle of a pyramid
F	indentation force
F^0	indentation force for a frictionless pyramid
h	depth of indentation

1. INTRODUCTION

The well-established slip-line field theory for plane strain (and axisymmetrical) deformations of rigid–perfectly-plastic solids cannot be easily generalized for strain-hardening materials. Consequently, the solutions available for rigid–perfectly-plastic solids cannot be utilized when distinct strain-hardening properties of the material are involved. In the present paper an approximate technique is proposed for evaluating an external force in steady or pseudo-steady plastic flow processes, with account taken for isotropic strain-hardening.

An attempt to solve the problem of flow of an isotropic strain-hardening material was made by Halling and Mitchell [1], who adapted the known perfectly-plastic solution in order to obtain a first-order estimation of the yield stress distribution. As noted by the authors, this approach ignores the velocity discontinuities, as they produce a jump in the yield stress, and, consequently, it leads to an unacceptable situation where the derivative of the mean stress across the characteristic (the discontinuity) needs to be infinite in order to satisfy the appropriate equilibrium condition. A semi-experimental approach to solving the problem of flow of an isotropically hardening material was shown by Farmer and Oxley [2]. In order to find the stresses throughout the deformation region in the plane extrusion problem they integrated the equilibrium equations along the experimentally determined characteristics. Distribution of the yield stress was obtained from the stress–strain curve using the experimentally determined strain field. It was demonstrated in [2] that the velocity field for the hardening material is 'smooth'. In place of discontinuities predicted in a perfectly-plastic solution, finite width bands with high velocity gradients were observed. Indeed, it was shown by Collins [3] that in steady flow of a hardening material the velocity and strain-rate must be continuous across a slip-line, and in an unsteady process discontinuities can appear only instantaneously and must diffuse out into shear bands of a finite width. The process of 'opening up' of the velocity and strain-rate discontinuities has not yet been described theoretically, and thus the only available analytical solutions for strain-hardening materials are those in which discontinuities do not arise (cf. [4]).

The technique presented in this paper is based on the balance of work used internally and done by an external force in the process for which the mechanism of deformation is assumed. This approach cannot be rigorously regarded as an upper bound estimation as the distribution of the yield stress and the variation of the boundary are not specified *a priori*, but follow from the assumed mechanism of deformation. The criterion of minimum applied effort is used to find the geometry of the mechanism which yields the most probable force of indentation (when the upper bound theorem is not applicable this criterion may not always be true). The proposed method differs from other approximate approaches based on assumed mechanisms of deformation [5] in that the deformation of the strain-hardening material is concentrated in narrow zones which may be regarded as shear bands (shear layers). The method can thus be applied to deformation processes with complicated geometries and three-dimensional problems for which the rigid-block mechanisms of deformation are relatively easy to construct. Such mechanisms contain sharp velocity discontinuities (interpreted here as shear layers), hence they cannot be considered as kinematically admissible for strain-hardening materials. However, the problem of kinematical admissibility seems to be insignificant since the other requirements of the upper bound theorem are not satisfied, and therefore, the method can be regarded only as an approximate one.

The rate of work-dissipation per unit area of the shear layer within a strain-hardening material is derived in the next section. In Section 3 two examples are presented in which the proposed technique is used. A mechanism of pyramidal indentation (first example) is described and the computational outcome is compared to the experimental results reported by other authors. Computational results for the process of flattening a pyramid are also shown.

2. THE RATE OF WORK-DISSIPATION WITHIN A SHEAR LAYER

Consider a steady narrow zone of finite thickness δ (Fig. 1) within which a strain-hardening material undergoes rapid shearing (shear band or shear layer). The material enters the zone with speed $V^{(1)}$ and leaves it with speed $V^{(2)} = V^{(1)} + [V]^*$, $[V]^*$ being the total (tangential) velocity increase. The rate of work-dissipation per unit area of the shear layer in plane xOz (z being perpendicular to the plane of Fig. 1) can be expressed as

$$\dot{D} = \int_0^{[V]^*} k(\bar{\varepsilon})\,\mathrm{d}[V], \tag{1}$$

where the yielding shear stress k is considered to be a function of the equivalent strain $\bar{\varepsilon}$ defined as

$$\bar{\varepsilon} = \sqrt{\tfrac{2}{3}\varepsilon_{ij}\varepsilon_{ij}}\,. \tag{2}$$

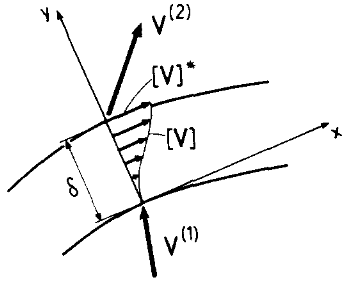

FIG. 1. Steady shear layer of thickness δ.

The function $k(\bar{\varepsilon})$ is here assumed in the form

$$k = k_0 + b(1 - e^{-a\bar{\varepsilon}}), \tag{3}$$

where k_0 is the initial yielding shear stress, b is the increase of the yield stress for a fully-hardened material and a is the third parameter of the stress–strain relation.

Assume first that the increase of tangential velocity $[V]$ has a linear distribution across the shear layer. The equivalent strain across the steady layer is then

$$\bar{\varepsilon} = \frac{2}{\sqrt{3}} \varepsilon_{xy} = \frac{1}{\sqrt{3}} \frac{[V]^*}{V_y} \frac{y}{\delta}, \tag{4}$$

where V_y is the material velocity component normal to the shear layer. The equivalent strain for a non-steady layer (e.g. in a pseudo-steady mechanism of deformation) can also be expressed by (4), however, in such case V_y is the material velocity measured with respect to the moving shear layer (δ = const.). For example, for a layer moving into material at rest V_y is the normal component of the velocity of the layer propagation due to expansion of the mechanism.

The rate of work-dissipation per unit area of the shear layer can be derived using equations (1), (3) and (4) as

$$D = [V]^* \frac{1}{\delta} \int_0^\delta k(\bar{\varepsilon})\,dy = (k_0 + b)[V]^* - \frac{\sqrt{3}b}{a} V_y \left(1 - e^{-\frac{a}{\sqrt{3}}\frac{[V]^*}{V_y}}\right). \tag{5}$$

For rigid-block mechanisms the width δ of the shear layer becomes infinitesimally small; this, however, has no effect on equation (5). A result identical to (5) can be obtained assuming a cosine distribution of $[V]$ across the layer

$$V_x = V_x^{(1)} + [V]^* \frac{1 - \cos \pi \frac{y}{\delta}}{2}, \quad V_y = V_y^{(1)} = \text{const.}, \tag{6}$$

in which strain-rate discontinuities do not arise (for finite widths of the layer).

It is clearly seen from (5) that the rate of work-dissipation is independent of the thickness δ of the shear band. It also follows from (5) that in the limit case, when $V_y \to 0$ (the stationary boundary between material at rest and undergoing deformation), the work-rate is equal to that for a fully hardened material, $\dot{D} = (k_0 + b)[V]^*$, and $\dot{D} = k_0[V]^*$ when $V_y \to \infty$ (independent of the distribution of the tangential velocity across the shear band). These conclusions could, of course, be deduced intuitively.

3. A MECHANISM OF PYRAMIDAL INDENTATION AND COMPUTATIONAL RESULTS

The mechanism considered is shown in Fig. 2(a). It consists of four identical flow areas [only one being shown in Fig. 2(a)], each adhering to one side of a square-base pyramid. The process of indentation is assumed to be pseudo-steady, thus any stage of deformation can be selected for analysis of the kinematic field. The flow area in Fig. 2(a) consists of two rigid-motion regions: ACDF (and ACDF′) with velocity V_1 and ABCF (and ABCF′) with velocity V_2. A hodograph with the geometrical contours of the mechanism in cross-section ABCDO is shown in Fig. 2(b). The two rigid-motion regions are separated from each other by the surfaces ACF and ACF′, and from the material at rest by the surfaces CDF (CDF′) and BCF (BCF′). All these surfaces are interpreted here as shear layers.

The mechanism considered is similar to that suggested elsewhere for tetrahedral indentation of rock-like materials [6]. It differs from the mechanism considered earlier by Haddow and Johnson [7] in that the region of continual deformation is replaced by a shear layer. The idea of a pseudo-steady mechanism of indentation originates from the early paper by Hill *et al.* [8], in which a rigorous solution to plane-strain wedge indentation into a rigid–perfectly-plastic material was presented.

The unknown angle η in Fig. 2(b) was determined from the condition of self-similarity (geometrical similarity at any stage) of the process. The respective formulae for dilational

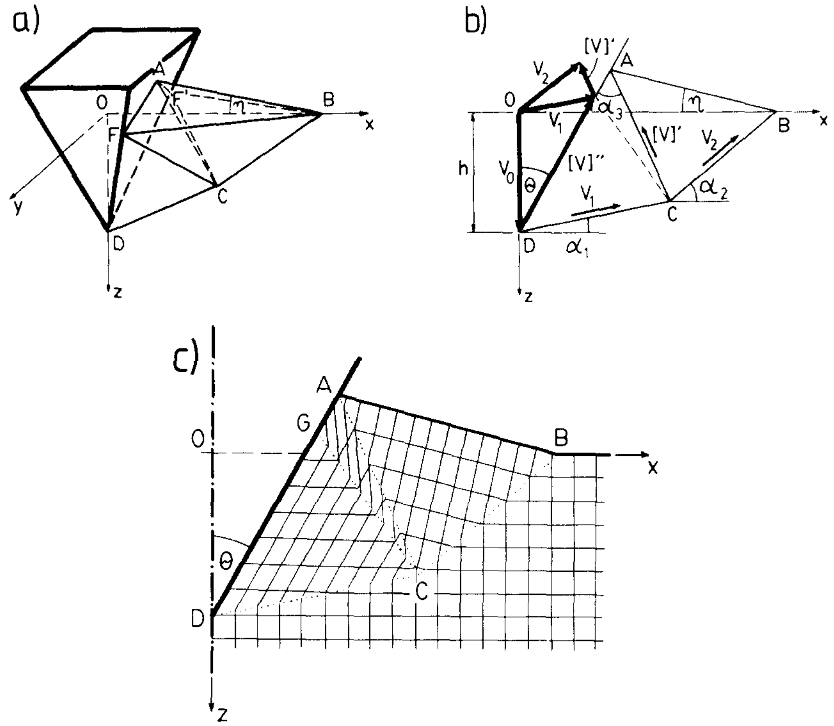

FIG. 2. Mechanism of pyramidal indentation; (a) deformed area adhering to one flank of the pyramid, (b) hodograph, (c) deformation of the originally square material grid in the plane xOz.

material were presented in Appendix I of Ref. [6]. The formulae thereof can be used here after substituting zero for the dilation angle ϕ^*. The approximate value of indentation force F^0 for a frictionless pyramid was computed from the balance of work dissipated within the shear layers and work done by the external (indentation) force. The rate of internal work was computed using equation (5). The criterion of minimum applied effort was used in order to determine the angles α_1, α_2 and α_3 [Fig. 2(b)]. It has to be noted that for the shear layers ACF and ACF′ [Fig. 2(a)] equation (5) must be modified to take into account the strain of the material after it has crossed the layers BCF and BCF′. Figure 2(c) shows the deformation of the originally square material grid in the plane xOz. It is clearly seen from Fig. 2(c) that the material in the triangle region ACG was subjected to deformation twice: once across the shear layer BC and then across AC. The indentation force F for a pyramid with frictional flanks can be determined by adding the rate of work-dissipation due to friction on the flanks of the pyramid, to the rate of internal work (this explicit technique can be applied here due to the symmetry of the assumed mechanism and uniform distribution of the sliding vector $[V]''$ on the pyramid–material interface). The following expression results

$$F = F^0 \bigg/ \left(1 - \frac{[V]''}{V_0} \frac{\mu}{\mu \cos\theta + \sin\theta}\right), \tag{7}$$

where μ is the coefficient of friction and θ is the included semi-angle of the pyramid [for V_0 and $[V]''$ see Fig. 2(b)].

The diagrams in Fig. 3 show the dimensionless force of indentation vs the included semi-angle of the pyramid for different materials. Parameters k_0, b and a used in computations for steel, copper and aluminium were determined by making use of results of torsion tests reported by Dugdale [9]. Experimental results of pyramidal indentation tests given by Dugdale are also shown in Fig. 3(a), (b) and (c). Respective parameters for lead were computed from the results of the compression test reported by Haddow and Johnson [10], assuming that the lead obeys the Huber–von-Mises' yield criterion. In Fig. 3(d) experimental forces of indentation from the paper by Haddow and Johnson are shown for both strain-hardening lead and for fully hardened lead (black dots). The theoretical solution for fully-hardened lead (broken line) was obtained here as a limiting case, for perfectly-plastic material. Results of computations presented in Fig. 3 are in good agreement with experimental results, particularly for the included semi-angle θ in the range 30–50°. It must be noted that the mechanism proposed by Haddow and Johnson [7] for pyramidal indentation into perfectly-plastic material gives results that are in lesser discrepancy with experiments for angles $\theta < 30°$ and $\theta > 50°$ than the mechanism proposed in this paper and used for fully hardened material. Better agreement could be obtained here through refining the mechanism of deformation, e.g. by replacing the shear layer ACF (ACF′) in Fig. 2 with a fan of layers. The scope of the computed example was, however, only to illustrate the method.

The proposed technique was also applied to a pseudo-steady problem of compressing a square-base pyramid (or tetrahedral asperity) by a rigid, flat plate. Figure 4(a) shows schematically the assumed mechanism of deformation. The dimensionless force vs the included semi-angle of the pyramid, for a hypothetical strain-hardening material, is shown in Fig. 4(b) along with a diagram for perfectly-plastic (fully-hardened) material.

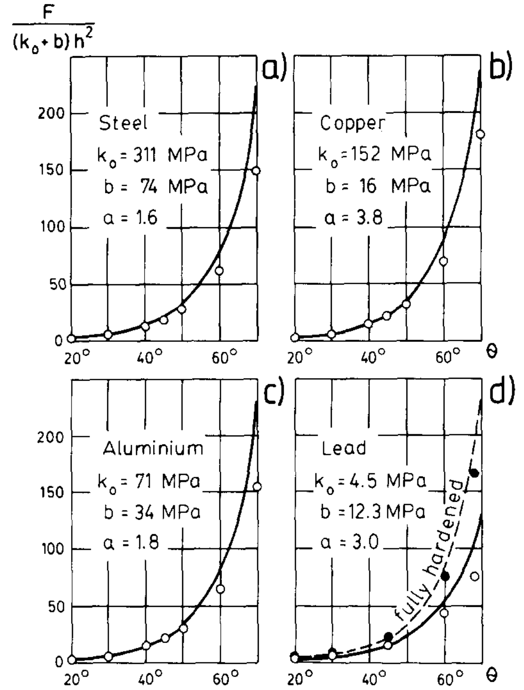

FIG. 3. Dimensionless force of indentation vs included semi-angle of the pyramid; (a) steel, (b) copper, (c) aluminium, (d) lead. Computations performed for smooth pyramids and material parameters for steel, copper and aluminium derived from data reported by Dugdale [9] and for lead by Haddow and Johnson [10]. Experimental forces of indentation therefrom, with corrections for friction having been applied there.

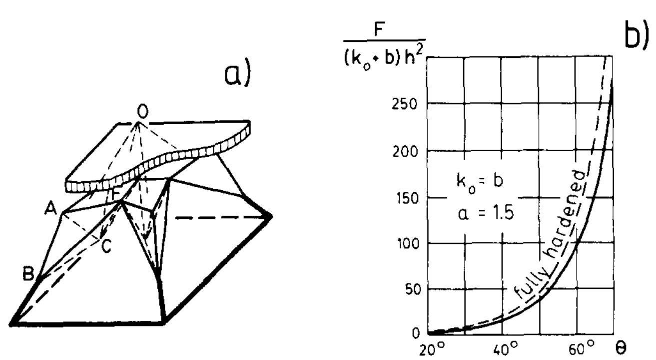

FIG. 4. (a) Scheme of the kinematically admissible deformation mechanism of a pyramid under compression by a flat, rigid plate, (b) dimensionless compression force (smooth plate) vs included semi-angle of the pyramid for strain-hardening material and for perfectly-plastic (fully-hardened) material.

4. CONCLUSIONS

A method was presented for estimating an external force in steady or pseudo-steady deformation processes of strain-hardening materials. The approach suggested may not contribute to better understanding of the flow of strain-hardening materials and is proposed here on account of its simplicity. It may, however, prove a useful technique for solving engineering problems, especially three-dimensional processes where a rigorous solution for perfectly-plastic material cannot be found and thus the perturbation technique for evaluating the strain-hardening effect cannot be applied. It seems that in these cases the technique proposed here has an advantage over the often-used method originated in the early paper by Hill and Tupper [11] and based on using some mean value of yield stress in the flow region. For processes, where the deformation mechanism cannot be given *a priori*, this method would require modification of the mean yield stress in the course of optimizing the geometry of the mechanism (for obtaining the minimum active force).

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