Coefficient of Earth Pressure at Rest

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Abstract: The widely used Jaky coefficient of earth pressure at rest, K_0, is revisited. It is demonstrated that this coefficient was derived from an analysis of the stress state in a sand prism that yields an unrealistic stress field. It is also surprising that the at rest stress state is represented as a function of the limit state parameter (internal friction angle). Consequently, one arrives at the conclusion that reasonable predictions made by classical K_0 are somewhat coincidental. Jaky's solution to K_0 is discussed in view of more recent research on the stress fields in prismatic mounds of sand.

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Introduction

Solving geotechnical problems often requires that the initial stress state in the soil be known. The coefficient of earth pressure at rest is frequently used to determine this stress state if geologic information is available about both the load history and the soil type. The coefficient of earth pressure at rest proposed by Jaky (1944) is accepted as the horizontal-to-vertical stress ratio in loose deposits and normally consolidated clays. In its abbreviated, and widely accepted form, this coefficient is written as

$$K_0 = 1 - \sin \phi \tag{1}$$

where ϕ stands for the effective internal friction angle of the soil (for brevity $\phi = \phi'$). For clays, angle ϕ represents the angle obtained from a series of tests on specimens (for instance, triaxial compression tests), each normally consolidated to a different stress. The stress ratio in Eq. (1) represents, of course, an admissible stress state, and it falls between the minimum and maximum coefficients that follow directly from the Mohr–Coulomb yield condition. The extreme values of the horizontal-to-vertical stress ratio are referred to as *active* and *passive* earth pressure coefficients, and they represent the limit (or yielding) states in the soil. Therefore, they must be functions of the strength of the soil, represented in the Mohr–Coulomb yield condition by the effective internal friction angle ϕ. The stress that represents an at rest state has not reached yielding, and it is intriguing that such a state would be fairly well represented by a function of ϕ.

It is a common misconception that the coefficient in Eq. (1) is an empirical result. To the contrary, it was derived (in a more elaborate form) by Jaky (1944) from an analysis of the stress field in a wedge prism of a loose granular material. In his 1944 paper Jaky made a bold statement that "... the experimental evaluation of K_0 is not necessary. The factor K_0 is simply and unambiguously related to the angle of internal friction of granular materials." While the coefficient derived is indeed a fair depiction of the stress ratio in the "natural state," one cannot dismiss the impression that coincidence played a role in rendering this coefficient so close to the true state at rest.

The author has revisited the problem of stress distribution in a wedge-shaped prism of sand because of the recent interest among both physicists and engineers in the stress distribution under sand heaps. Stress distribution in sand mounds became a fashionable research area in the late 1990s, with the focus on a counterintuitive observation that the stress at the base can exhibit a local minimum (or a "dip") at the center of a conical or a wedge-shaped sand prism. The stress depression (local minimum) is a result of arching, but predictions of the degree to which the soil will arch are not easily made. The significance of this phenomenon was probably overstated in the *Science* article of Watson (1996): "... sand pile pressure dip is to granular mechanics what Fermat's Last Theorem was to number theory." Nevertheless, it is interesting to place Jaky's (1944) work in the context of the other research on stress states in sand piles. While the appearance of the stress dip may be a curiosity problem to engineers, arching associated with it is a phenomenon of interest and importance in geotechnical engineering.

The term "sand pile" is used in this note to describe a mound, a heap, or a prism of sand, and not a sand column, as it is often used in foundation engineering. Particular attention will be paid to long sand mounds, such as that in Fig. 1(a), that render the stress state to be a function of two space coordinates (plane strain).

The early experiments on the distribution of the stress underneath piles of sand were described by Hummel and Finnan (1920) who found that, if a sand is deposited from a point source (a funnel), then a distribution of the base stress beneath a conical pile of sand has a depression at the center. Similarly, a stress dip occurs under a wedge prism deposited from a line source (a hopper). This was confirmed by other published results, most recently by Vanel et al. (1999), who indicated that the deposition process has a significant effect on the stress distribution in sand mounds. More experimental results are listed in Michalowski and Park (2005). Efforts toward theoretical description of the stress state in sand piles can be found in Wittmer et al. (1997), Savage (1998), Didwania et al. (2000), and Michalowski and Park (2004).

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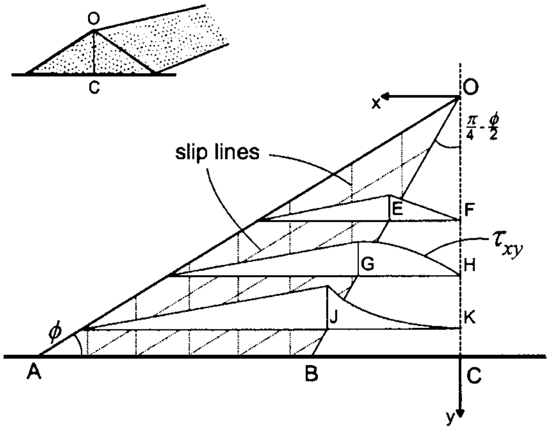

Fig. 1. Wedge-shaped sand prism: (a) schematic of a long mound (plane strain analysis); and (b) cross section of a symmetric half

Solutions with fully plastic stress states were considered earlier by Booker (1969).

It appears that some of the more recent considerations of the stress state in piles of sand resemble the theoretical effort of Jaky (1944). As will be shown in this note, the stress field in the sand prism that led Jaky to derivation of his coefficient is not supported by experiments, yet the coefficient itself was proved to be a good representation of the natural state in loose deposits and normally consolidated clays.

The theoretical background for K_0 is shown in the next section, followed by a discussion in the context of other results. Modification of K_0 to account for overconsolidated soils is only briefly mentioned. The note ends with brief concluding remarks.

Coefficient K_0

The derivation of coefficient K_0 is presented in this section in the context of other possible solutions. The translation of the original paper of Jaky (1944) can be found in Hayat (1992).

Jaky (1944) considered a sand prism of loose granular soil, inclined at ϕ to the horizontal, and asserted that the stress on vertical plane OC, Fig. 1(b), is the pressure at rest. This in itself is a far reaching assumption. In the introductory portion of the paper Jaky indicates that the coefficient of pressure at rest is associated with one-dimensional strain state (supporting structure "does not shift sideways, tilt or tip over ..."), whereas sand heaps certainly are not a result of a one-dimensional strain (or deposition) process.

The stress state in region ABO was assumed to be at its limit, with uniform directions of the principal stresses, and the major principal stress being parallel to OB. This stress state is identical to that considered by Rankine (1857) for an infinite slope in the limit state (see Michalowski and Park 2004), and it is admissible as long as the base AC of the prism is sufficiently rough. The stress components in region ABO then become

$$\sigma_x = \gamma(y\cos\phi - x\sin\phi)\cos\phi$$

$$\sigma_y = \gamma(y - x\tan\phi)(1+\sin^2\phi) \tag{2}$$

$$\tau_{xy} = \gamma(y\cos\phi - x\sin\phi)\sin\phi$$

where γ=unit weight of the sand. The same stress state cannot be extended into triangle BCO, as it would violate equilibrium at the symmetry plane. Three components of stress need to be determined in region BCO, but only two differential equations of equilibrium are available. Hence, an additional piece of information is needed to solve for the stresses in BCO, and Jaky (1944) chose to predetermine the shear stress distribution τ_{xy}. The shear stress needs to match that in Eq. (2) along OB, and it needs to drop down to zero at the symmetry plane OC. An obvious first attempt would be an assumption of a linear distribution, as indicated along line EF, Fig. 1(b); this yields the following stress field in BCO

$$\sigma_x = \gamma y(1-\sin\phi)$$

$$\sigma_y = \gamma y(1-\sin\phi) + 2\gamma x\sin\phi\tan\phi \tag{3}$$

$$\tau_{xy} = \gamma x\sin\phi$$

It is not clear whether Jaky did or did not try this distribution, but if he tried, he found that the ratio of horizontal-to-vertical stress along OC ($x=0$) is equal exactly to 1 (hydrostatic stress). It comes as a surprise that the principal stress directions in BCO are constant, and that they are the same as those in region ABO

$$\psi_{BCO} = \psi_{ABO} = \frac{\pi}{4} + \frac{\phi}{2} \tag{4}$$

where ψ=the angle of inclination of the major principal stress to axis x[$\tan 2\psi = 2\tau_{xy}/(\sigma_x - \sigma_y)$]. Consequently, the stress state must be hydrostatic on OC for equilibrium to hold, and the symmetry plane is not a unique principal direction. Clearly, as $\sigma_x/\sigma_y=1$ on OC, this is not a case that leads to acceptable K_0.

While the above-presented solution did not yield a reasonable coefficient K_0, it was considered by Wittmer et al. (1997) as a possible explanation for the occurrence of a stress dip under prismatic sand mounds, and they termed it a *fixed principal axis* (FPA) solution. The FPA solution was derived by Wittmer et al. (1997) by assuming directions of principal stresses rather than distribution of τ_{xy} in region OBC. The distribution of the stress components at the base for a wedge-shaped sand prism with $\phi=30°$ is presented in Fig. 2(a). This distribution exhibits a clear stress depression under the center of the pile (the length scale is normalized so that the base length =1, and the stress norm is γH, H=sand wedge height).

Another intuitive but reasonable assumption for distribution of τ_{xy} is a square-root function, as shown along GH in Fig. 1(b); it has the analytical form

$$\tau_{xy} = \tau_{xy}^{OB}\frac{\sqrt{x}}{\sqrt{x_1}} \tag{5}$$

where τ_{xy}^{OB}=shear stress along line OB [Fig. 1(b)] that is described by the third equation of Eq. (2) for $x=x_1$, and x_1 is a horizontal distance from the symmetry axis OC to line OB

$$x_1 = y\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \tag{6}$$

By integration of the partial differential equations of equilibrium, with τ_{xy} described in Eq. (5), one arrives at the solution to the stress state in region BCO, given in the Appendix [Eq. (11)]. The distribution of the base stress under the prism of sand is illustrated in Fig. 2(b). Although stress τ_{xy} assumed in Eq. (5) seems reasonable, the resulting distribution of σ_y is not, as it has a singularity of order $-1/\sqrt{x}$ when $x\to 0$. Consequently, the stress state in the neighborhood of the symmetry axis becomes inadmissible.

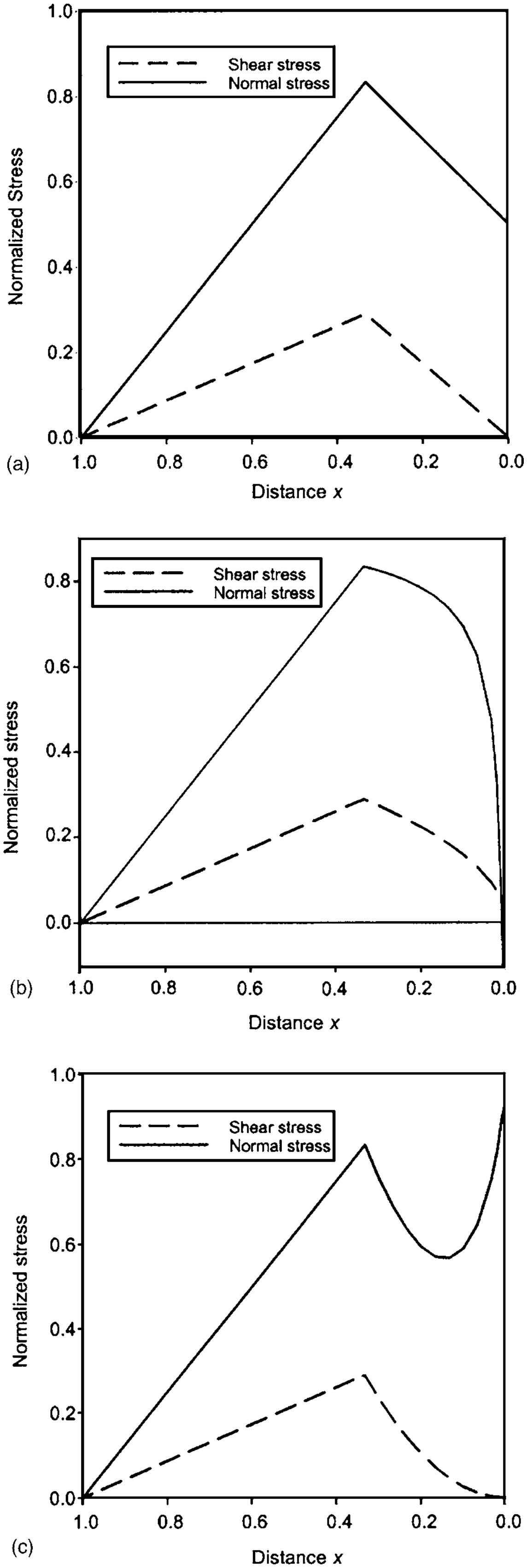

Fig. 2. Distribution of the contact stress at the base of a sand prism for different assumptions of τ_{xy} in region *OBC*: (a) linear distribution; (b) square-root distribution; and (c) Jaky's parabolic distribution

The third distribution of τ_{xy} considered here is a parabolic distribution depicted by the curve along *JK* in Fig. 1(b). This distribution seems to be less intuitive, but this is the one that Jaky (1944) considered

$$\tau_{xy} = \tau_{xy}^{OB} \frac{x^2}{x_1^2} \tag{7}$$

Integration of the equilibrium equations leads now to the stress state in region *OBC* [see Eq. (12) in the Appendix] that yields the base stress illustrated in Fig. 2(c). While this stress state is statically admissible (it satisfies differential equations of equilibrium, it does not violate the Mohr–Coulomb yield condition, and it is consistent with the stress boundary conditions), it is not a stress field that is likely to occur in a sand prism. The distribution of the normal stress in Fig. 2(c) is rather peculiar and it is not confirmed by any known experimental data. Nevertheless, this is the distribution that led to the so well-accepted coefficient of earth pressure at rest.

Newer theoretical data (e.g., Savage 1998; Didwania et al. 2000; Michalowski and Park 2004) indicate that the stress ratio at the symmetry plane of a wedge sand prism may vary in a large range from active to passive coefficient of earth pressure. The known experimental data indicate that the stress distribution can reach the maximum at the symmetry, or, it may exhibit a local minimum (a stress dip) at the center point (see, for instance, Vanel et al. 1999). However, none of the known experimental measurements resembles that in Fig. 2(c).

Jaky (1944) conjectured that the coefficient of pressure at rest is equal to the ratio of the horizontal to the vertical stress on the symmetry plane *OC* ($x=0$) of the sand prism. The vertical stress σ_y in Eq. (12) becomes equal to γy when $x \to 0$, and

$$K_0 = \frac{\sigma_x}{\sigma_y} = (1 - \sin\phi)\frac{1 + \frac{2}{3}\sin\phi}{1 + \sin\phi} \tag{8}$$

In a later paper, Jaky (1948) dropped the fraction term from Eq. (8) without explanation, and the generally accepted form is that in Eq. (1).

An early set of results from tests with both virgin loading and unloading of clay soils in one-dimensional strain state was presented by Brooker and Ireland (1965), who confirmed the usefulness of the formula in Eq. (1), although they found that a slightly modified formula, $K_0 = 0.95 - \sin\phi$, matched the experimental results for clay soils a little better. A large set of experimental data was assembled by Mayne and Kulhawy (1982), who concluded that Eq. (1) is a good representation of the stress coefficient at rest for normally consolidated clays, and it is "moderately valid" for granular soils. Similar views are held by others (e.g., Mesri and Hayat 1993). It is rather surprising that the at rest stress state is well represented as a function of the limit state parameter (internal friction angle).

Remarks

This note's focus is on Jaky's coefficient of earth pressure, and the solutions to the stress state in wedge-shaped sand prisms presented here are those relevant to the original derivation of K_0. A multitude of admissible solutions to the stress field in sand prisms can be found elsewhere (Wittmer et al. 1997; Savage 1998; Michalowski and Park 2004), including those that do not enforce stress symmetry in geometrically symmetric prisms (Didwania et al. 2000). Stress field solutions for conical heaps were not considered, though they can be found in the literature (e.g., Wittmer et al. 1997). These solutions are numerical in nature, and do not yield a convenient form for stresses at the symmetry axis. Radial stress fields form a special class of solutions where the stress magnitude is proportional to the distance from the apex of the sand heap, whether prismatic or conical. For such stress fields the ratio of horizontal-to-vertical stress along the symmetry axis is constant; however, this is not true for all solutions. The multitude of admissible solutions includes horizontal-to-vertical stress ratios spanning from *active* to *passive* earth pressure coefficients. Jaky (1944) selected a very particular stress distribution [Fig. 2(c)] that proved to be a good representation of the at rest pressure, even though no rational criterion for this choice was given.

While the coefficient K_0 proposed by Jaky was the result of a purely theoretical exercise, its later modifications to account for overconsolidation fall into the category of empirical corrections. It is remarkable that the earth pressure coefficient at rest for overconsolidated soils (K_0^{OC}) is typically represented as a function of overconsolidation ratio (OCR), but independent of the magnitude of the maximum consolidation stress. Such a suggestion was presented by Schmidt (1966), and it has been widely used in a slightly modified form

$$K_0^{OC} = K_0 \mathrm{OCR}^{\sin\phi} \tag{9}$$

The unique dependence of K_0^{OC} on OCR was criticized by Jefferies et al. (1987), who argued that the measured geostatic stress for Beaufort Sea clays is not a single-value function of OCR. While this criticism prompted a vivid discussion (Mayne and Kulhawy 1988; Mesri and Feng 1988), one might speculate that stress and deformation history may affect K_0^{OC}. Elastic properties are influenced by geologic-time processes, and, because the response to unloading ("rebound") is affected by the elastic properties, K_0^{OC} may reflect some dependence on the history, not just OCR. Geologic features, such as compaction bands, may also affect K_0^{OC}, but this technical note will not venture into these arguments.

Conclusions

The surprising form of the classical coefficient at rest K_0 stems from its dependence on the soil strength parameter ϕ, whereas the stress state at rest is below the soil yielding level. Examination of the original derivation of this coefficient (Jaky 1944) led to two conclusions. First, K_0 was derived from an analysis of the stress distribution in a wedge-shaped sand prism—a problem that is not related to the stress path typical of a one-dimensional strain process associated with the K_0 state. Results of the investigation of admissible stress states in sand prisms (Savage 1998; Vanel et al. 1999; Michalowski and Park 2004) indicate that the horizontal-to-vertical stress ratio at the symmetry axis of a sand prism can span the entire range from the active to the passive state, and it depends on the history of deposition of the granular material as well as the deflection of the base.

The second conclusion relates to a rather peculiar distribution of the base stress that stems from Jaky's solution. Since the problem formulated by Jaky (1944) was indeterminate in the core of the sand heap, the shape of the shear stress distribution was assumed in that part of the prism. Of possible stress distributions, the one taken by Jaky yields a rather unrealistic distribution of the normal stress at the base. Interest in this problem was revived in the last ten years in the context of stress "depression" under the center of sand piles, but none of the experimental test results available today confirmed the peculiar distribution following from Jaky's solution. In view of these comments, it is surprising that the theoretical formula $K_0 = 1 - \sin\phi$ is a good representation of the true stress ratio in soils at rest.

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Appendix

Substituting τ_{xy} from Eq. (5) into equations of equilibrium ($\tau_{xy} = \tau_{yx}$)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \gamma \tag{10}$$

one obtains two differential equations with unknown functions σ_x and σ_y, and, after integration, the set of equations describing the stress state in region *OBC* becomes

$$\sigma_x = \gamma y(1 - \sin\phi)\frac{1 + \frac{4}{3}\sin\phi}{1 + \sin\phi} - \frac{1}{3}\gamma x\left[\frac{x}{y}\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\right]^{1/2}\sin\phi$$

$$\sigma_y = \gamma y\left\{1 - \frac{1}{3}\sin\phi\left[\frac{y}{x}\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\right]^{1/2}\right\} + \gamma x\left[\frac{1}{3}\sin\phi\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) - (1 - \sin\phi)\tan\phi\right]$$

$$\tau_{xy} = \gamma\sin\phi\left[xy\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\right]^{1/2} \tag{11}$$

Integration of Eq. (10), with τ_{xy} as assumed by Jaky (1944) and described in Eq. (7), leads to the following set of equations describing the stress state in region *OBC*

$$\sigma_x = \gamma y(1 - \sin\phi)\frac{1 + \frac{2}{3}\sin\phi}{1 + \sin\phi} + \gamma\frac{x^3}{3y^2}\sin\phi\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$$

$$\sigma_y = \gamma y - 2\gamma x\sin\phi\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\ln\frac{y}{x\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)} - \gamma x\sin\phi\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$$

$$\tau_{xy} = \gamma\frac{x^2}{y}\sin\phi\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \tag{12}$$

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