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SHORT COMMUNICATIONS

ON THE SOLUTION OF PLANE FLOW OF GRANULAR MEDIA FOR JUMP NON-HOMOGENEITY

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SUMMARY

This paper deals with the plane plastic equilibrium of a cohesive frictional material in the case of a jump non-homogeneity. Three types of interface separating rigid-plastic bodies with different material parameters are distinguished: (i) a perfect adhesion contact, (ii) a thin layer of a material different from the adjacent bodies, (iii) a combined frictional-adhesive contact. A sliding along the interface may occur if certain conditions are satisfied. For the first two types of joint, the direction and propagation of the velocity jump vector along the interface follow from the classical theory of plane plastic flow. In the third case, the potential sliding rule relating the velocity jump vector to the limit condition of the joint, or to the potential different from the limit condition, may be adopted. The direction of the velocity jump vector results from such a rule.

The problem of stability of a slope, being continuously non-homogeneous in one region, and homogeneous in another, is shown in an example. The two regions of the slope are separated by the interface, a thin layer of a different material, (type ii). The associated flow law for both regions of the slope and for the material of the interface is assumed.

INTRODUCTION

Non-homogeneity of continuous media is usually defined as a dependence of the mechanical properties on location. Two types of non-homogeneity of a body can be distinguished:[1] non-homogeneity occurring as varying values of material parameters, and non-homogeneity corresponding to the varying constitutive equations (structural non-homogeneity). The present paper deals with some problems of the plastic equilibrium of cohesive frictional media with material parameter jump non-homogeneity. However, for ease of comprehension, a brief presentation of the basic equations for continuously non-homogeneous material follows in this section.

Let us assume that the yield function of a material is specified by the Mohr–Coulomb condition (tension is taken positive):

$$f(\sigma_x, \sigma_y, \tau_{xy}, x, y) = [(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]^{1/2} + (\sigma_x + \sigma_y)\sin\phi - 2c\cos\phi = 0 \qquad (1)$$

where the angle of internal friction ϕ and cohesion c are the functions of the co-ordinates x and y. In plane strain condition† the local equilibrium equations and the condition (1) lead

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† The velocity vectors are parallel to the plane xOy in a rectangular Cartesian system, and the co-ordinate y coincides with the direction of gravity.

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to a set of two quasi-linear, hyperbolic type equations that can be solved using the standard method of characteristics.[2] The following solution results:

$$\frac{dy}{dx} = \tan\left\{\psi \mp \left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right\}; \quad s_1, s_2 \tag{2}$$

$$\begin{aligned} dp + 2p\tan\phi\, d\psi = {} & \left((Ac)\cot\phi - \frac{c}{\sin^2\phi}A\phi + p\phi_{,y} + \gamma\tan\phi\right)dx \\ & + \left(-(Fc)\cot\phi + \frac{c}{\sin^2\phi}F\phi - p\phi_{,x} + \gamma\right)dy;\ s_1 \\ dp - 2p\tan\phi\, d\psi = {} & \left((Bc)\cot\phi - \frac{c}{\sin^2\phi}B\phi + p\phi_{,y} - \gamma\tan\phi\right)dx \\ & + \left((Dc)\cot\phi - \frac{c}{\sin^2\phi}D\phi + p\phi_{,x} + \gamma\right)dy;\ s_2 \end{aligned} \tag{3}$$

where p is defined as:

$$p = -(\sigma_x + \sigma_y)/2 + c\cot\phi \tag{4}$$

and ψ is the angle between major (algebraically) principal stress direction and the x-axis. A, B, D and F denote the following operators:

$$A = (\ )_{,x} + \tan\phi(\ )_{,y} \qquad D = \tan\phi(\ )_{,x} + (\ )_{,y}$$

$$B = (\ )_{,x} - \tan\phi(\ )_{,y} \qquad F = \tan\phi(\ )_{,x} - (\ )_{,y}$$

and

$$(\ )_{,x} \equiv \frac{\partial}{\partial x}(\ ) \qquad (\ )_{,y} \equiv \frac{\partial}{\partial y}(\ ).$$

In a different form, this solution was given by Sobotka[3] and Salencon.[4]

Let the principal axes of the stress and strain-rate tensors coincide, and the strain-rate components be related to the stress components by the flow law associated with the yield condition (1). Such assumptions lead to a set of linear hyperbolic equations for the velocity components. The following solution results:[4]

$$\frac{dy}{dx} = \tan\left\{\psi \mp \left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right\}; \quad \alpha, \beta \tag{5}$$

$$\begin{aligned} dV_\alpha - (V_\alpha\tan\phi + V_\beta\sec\phi)\, d\left(\psi - \frac{\phi}{2}\right) = 0; \quad \alpha \\ dV_\beta + (V_\beta\tan\phi + V_\alpha\sec\phi)\, d\left(\psi + \frac{\phi}{2}\right) = 0; \quad \beta \end{aligned} \tag{6}$$

where V_α and V_β are the projections of the velocity vector on the α and β lines.

Let us introduce a non-associated flow law and a potential being the function of the same form as the yield condition (1), but with ϕ replaced by an angle ϕ^* $(0 \le \phi^* < \phi)$, which describes the dilatation of the material. This implies that the equations of the velocity characteristics and the relations along them have the same form as in the case of associated flow law, but with ϕ replaced by ϕ^*. If specified value of ϕ^* is equal to zero, no volume changes in deformation process occur.

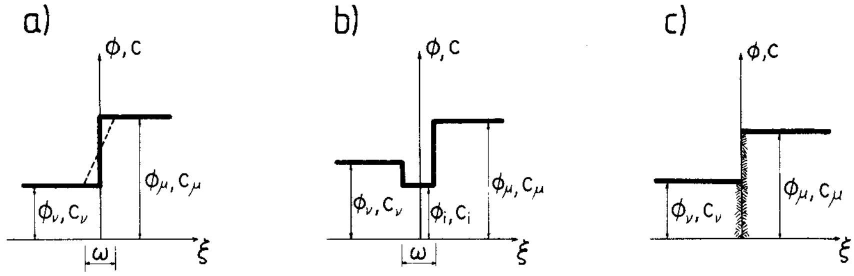

Figure 1. (a) perfect adhesion contact, (b) interface as a thin layer of the third material, (c) frictional-adhesive contact

JUMP NON-HOMOGENEITY

In this section three types of jump non-homogeneity will be distinguished. The first one is conceived as a limit case of a thin layer of the material with monotonic non-homogeneity (Figure 1(a)). This corresponds to the perfect adhesion contact[1] between two rigid-plastic bodies. The failure condition of such interface may be written in the form:

$$|\tau| = \inf(-\sigma \tan \phi_\nu + c_\nu, -\sigma \tan \phi_\mu + c_\mu) \tag{7}$$

where σ and τ are the normal and tangential components of the stress vector $\mathbf{T}$ acting on the interface. The slip along the interface may occur if the contact line is collinear to the velocity characteristic of one of the joined regions. Direction and propagation of the velocity jump vector is governed by the equations of the classical theory of plane plastic flow.[5]

The second type of non-homogeneity, being visualized as a violent decrease and increase of the material parameters ϕ and c, is shown in Figure 1(b). This occurs when two bodies are joined by a layer of thickness ω of a third material. If sliding within such an interface is expected the material of the joint has to be plastified and the velocity characteristics of one family have to be parallel to the interface (Figure 2(b)). Hence, the inclination ε (Figure 2(b)) of the stress vector acting on the line tangential to the interface can be found at each point of the joint. The angle ε is a statical boundary condition to be satisfied in the solution of statical problems.

If slip within the interface occurs, it has the nature of the velocity discontinuity along the velocity characteristic. Hence, the inclination and propagation of the velocity jump vector result from the classical theory of plane plastic flow.[5]

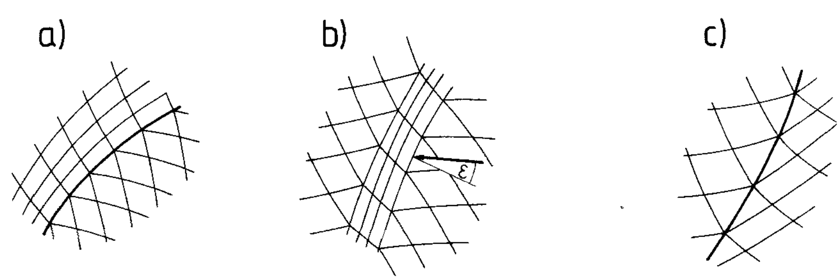

Figure 2. Velocity characteristics in the case of sliding along the interface, (a) perfect adhesion contact, (b) interface as a thin layer of the third material, (c) frictional-adhesive contact

The material of the joint may not, however, be plastified if it is stronger than that of one or both joined regions. Such a case corresponds to the perfect adhesion contact, i.e. sliding within one of the joined regions can occur.

The third case of the jump non-homogeneity is shown in Figure 1(c). This case corresponds to a frictional contact. The limit condition for relative sliding along such a contact can be specified by the function:

$$F(\mathbf{T}) = |\tau| + \sigma \tan \rho - k = 0 \tag{8}$$

where $\tan \rho$ is the coefficient of friction, k is the maximum shear resistance of the contact surface due to adhesion, and τ and σ are the components of the stress vector $\mathbf{T}$ acting on the contact line at the point considered. Hence, in case of sliding along such a contact line, the solution of the boundary value problems must satisfy the statical conditions resulting from (8). To describe the kinematics along the contact line (Figure 2(c)), the associated or non-associated sliding rule, relating the velocity jump vector $[\mathbf{V}]$ along the contact line with the limit condition $F(\mathbf{T})$, or sliding potential $G(\mathbf{T}) \neq F(\mathbf{T})$, may be adopted:[6]

$$[\mathbf{V}] = \dot{\lambda} \frac{\partial F(\mathbf{T})}{\partial \mathbf{T}} \qquad \dot{\lambda} > 0 \tag{9}$$

$$[\mathbf{V}] = \dot{\lambda} \frac{\partial G(\mathbf{T})}{\partial \mathbf{T}} \qquad \dot{\lambda} > 0 \tag{10}$$

Substituting ρ in the limit condition (8) by the angle ρ^*, which governs the separating velocity jump component normal to the contact surface, the useful form of the sliding potential $G(\mathbf{T})$ is obtained. In the case where $\rho^* = 0$ tangential sliding occurs.

It is clear that from (9) or (10) only the direction of the velocity jump vector along the contact line is determined. Propagation of the velocity jump is not a function of the material properties in this case, and follows from the solution of the boundary value problem.

Jump non-homogeneity of a rigid-plastic material generates discontinuity of the stress tensor along the interface. If the interface is assumed to be a thin layer of a third material, the stress state within the interface follows from the yield condition of this material. It can be assumed that the stress state does not vary across the interface since its thickness is relatively small. Hence, the jump of the stress tensor can be evaluated as in the case of an ideal contact joint ($\omega = 0$). It is obvious that only the discontinuity of the stress component σ_η (Figure 3), parallel

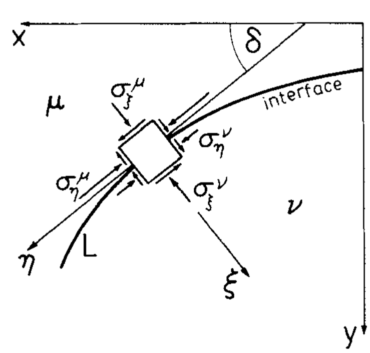

Figure 3. Discontinuity of the stress component σ_η

to the discontinuity line L may appear† (η, ξ—local co-ordinate system, $\eta \| L$ at the point considered). Thus, along the line L, which separates the regions ν and μ we can write:

$$\sigma_\eta^\nu \neq \sigma_\eta^\mu \qquad \sigma_\xi^\nu = \sigma_\xi^\mu \qquad \tau_{\eta\xi}^\nu = \tau_{\eta\xi}^\mu$$

If the limit state occurs on both sides of the jump non-homogeneity, the following equations for the values p_ν (see (4)) and ψ_ν can be derived (p_μ and ψ_μ being known):

$$p_\nu = \frac{1}{\cos^2 \phi_\nu}\{K \pm (K^2 \sin^2 \phi_\nu - p_\mu^2 \sin^2 \phi_\mu \cos^2 \phi_\nu \sin^2 2(\psi_\mu - \delta))^{1/2}\} \tag{11}$$

$$\psi_\nu = \delta - \tfrac{1}{2} \arcsin \left\{\frac{p_\mu \sin \phi_\mu}{p_\nu \sin \phi_\nu} \sin 2(\psi_\mu - \delta)\right\} - (\tfrac{1}{2} + n)\pi \tag{12a}$$

$$\psi_\nu = \delta + \tfrac{1}{2} \arcsin \left\{\frac{p_\mu \sin \phi_\mu}{p_\nu \sin \phi_\nu} \sin 2\,(\psi_\mu - \delta)\right\} + n\pi \tag{12b}$$

where

$$K = p_\mu\{1 + \sin \phi_\mu \cos 2(\psi_\mu - \delta)\} + c_\nu \cot \phi_\nu - c_\mu \cot \phi_\mu$$

and δ is the angle between the tangent to the contact line and the x-axis (Figure 3). The plus sign in (11) and the formula (12a) correspond to the stress discontinuity which vanishes for $\phi_\nu = \phi_\mu$ and $c_\nu = c_\mu$, wheras the minus sign and (12b) correspond to the case where discontinuity appears even for a homogeneous body (if L is not a stress characteristic). The limit equilibrium state on both sides of the jump non-homogeneity may occur when the expression inside the square root in (11) is positive. Otherwise, the real solution cannot be obtained, i.e. the limit state in the μ-region cannot appear.

THE BOUNDARY VALUE PROBLEM

In Figure 4 the field of the stress characteristics for the problem of stability of non-homogeneous steep slope, with vertical limit load along the top surface, is shown. It was assumed that the angle of internal friction and cohesion are linear functions of depth only:

$$\phi = \phi_0 + ay \qquad c = c_0 + by$$

The following data were selected: $\phi_0 = 18°$, $c_0 = 50{\cdot}0\,\mathrm{kN/m^2}$, $a = -0{\cdot}02\,\mathrm{rad/m}$, $b = 0{\cdot}05\,\mathrm{kN/m^3}$, $\gamma = 20{\cdot}0\,\mathrm{kN/m^3}$. In order to increase the limit load along the boundary OD (see Figure 4) the area AOEF (Figure 5) was filled with a 'supporting' material. This material is homogeneous, with the angle of internal friction $\phi = 20°$, cohesion $c = 10{\cdot}0\,\mathrm{kN/m^2}$ and specific weight $\gamma = 18{\cdot}0\,\mathrm{kN/m^3}$. It is assumed here that a thin layer of the material located along the interface OA (Figure 5), that was previously subjected to weathering, has no cohesion and its angle of internal friction is 15° (contact layer). It is also assumed that in the limit equilibrium state sliding within the contact layer will occur. Thus, one family of the velocity characteristics in this layer has to be parallel to the interface (see Figure 2(b)). The associated flow law for the material of the interface is assumed.

The stress characteristics for the problem of a limit load for the combined slope (Figure 5) are shown in Figure 6(a). The jump of the angle ψ at the singular point E was found from the compatibility condition of the principal directions at point A.

† The normal stress component, perpendicular to the plane of Figure 3 is not taken into consideration, since in plane flow the mean principal stress does not occur in the yield condition.

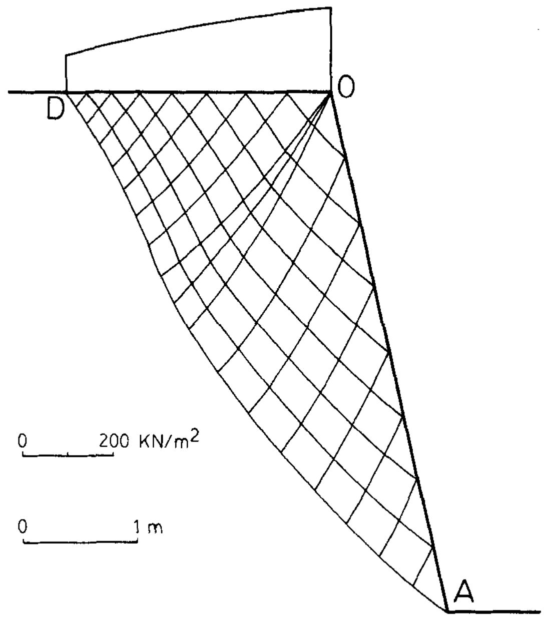

Figure 4. The stress characteristics field of the slope with linear variation of the material parameters with depth

It should be noted that the resulting limit load along the boundary OE (Figure 6(a)) is not vertical, and for selected data varies from 0° at point E to 2° at point O.

The jumps in the values p and ψ along the interface OA were computed from the equations (11) and (12a). As previously, the limit load along OD is assumed to be vertical. A significant increase of the limit load along OD occurs over that of the original slope (see broken line).

It was assumed that slip along OA occurs, so that the inclination of the stress vector acting on the interface could be found. Consequently, it is assumed that the failure of the slope is limited to the OEFA region only. A hodograph of the velocity field, in the case of associated flow law, and constant vertical velocity component of the boundary OE, $V_0 = \text{const.}$, is shown in Figure 6(b). The region OEA (Figure 6(a)) moves as a rigid body with the velocity denoted by O*OEA in the hodograph. The lines AE and AGF are the velocity discontinuity lines. The dotted line in Figure 6(b) represents the velocities along the boundary EF. The rate of work dissipation is non-negative in the whole flow region.

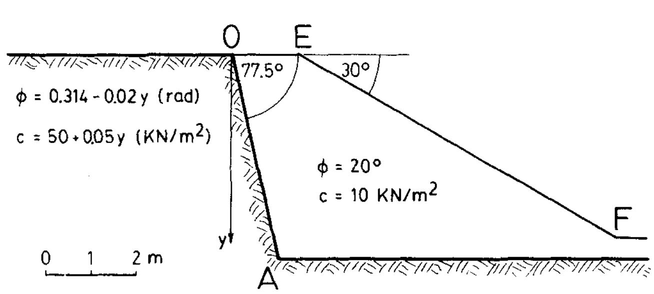

Figure 5. Combined slope

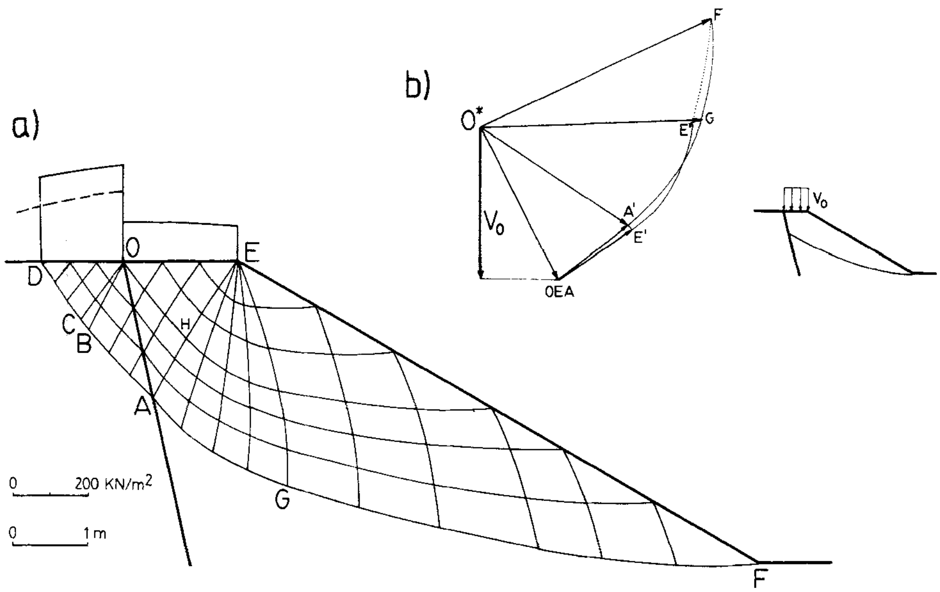

Figure 6 (a) the stress characteristics field for the combined slope, (b) a velocity hodograph in case of associated flow law

In the above example the admissible velocity field was obtained for specified constant velocity along the boundary OE. Such a solution is not possible in general, for arbitrarily chosen velocity distributions along OE, or curvlinear interfaces OA. The direction and propagation of the velocity jump vector along the interface (of the second type, Figure 1(b)) follow from the classical theory of plane plastic flow, hence, the velocities along OH are uniquely defined from the Cauchy problem in the region OAH. Thus, in general, prescribing an arbitrary velocity distribution along OE leads to an over-constrained boundary value problem in the region OHE. No admissible boundary value problem in OHE can be formulated when this occurs. However, a compatible solution can be obtained if only one of the velocity components, or the velocity direction, is specified along OE. This restriction does not arise when the interface OA is assumed to be a jump non-homogeneity of the third type (Figure 1(c)). In this case only the directions of the velocity jump vectors along OA are defined (see equations (9) and (10)). Hence, the Cauchy problem in triangle OEH and then the mixed problem in HOA can be formulated.

CONCLUSIONS

Three types of the contact joints in the case of the jump non-homogeneity may be distinguished: (i) perfect adhesion contact, (ii) a thin contact layer with material parameters different from both joined regions, (iii) combined frictional-adhesive contact. In the first case the slip along the interface may take place when the contact line is collinear to the velocity characteristic of one of the joined regions. In the second case, the slip within the interface may appear if the thin layer of the joint is plastified and the velocity characteristics of one family are parallel to the interface. If the third kind of contact joint takes place, the slip is possible if the inclination of the stress vector, acting on the interface, is equal to the one following from the limit condition for relative sliding. For zero adhesion forces, this condition becomes Coulomb dry friction condition.

In the first two cases, the inclination and propagation of the velocity jump vector along the interface follow from the classical theory of plane plastic flow. In the third case, the associated or non-associated sliding rule, relating the velocity jump vector to the limit condition, or sliding potential different from the limit condition, may be adopted. The direction of the velocity jump vector results from such rules.

Jump non-homogeneity of a rigid plastic material produces discontinuity of the stress tensor along the interface. Two different kinds of discontinuity may appear: discontinuity vanishing when non-homogeneity disappears, and discontinuity accompanied by an additional rotation of the principal directions (this kind of discontinuity may also occur in homogeneous media).

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