Archives of Mechanics • Archiwum Mechaniki Stosowanej • 30, 3, pp. 259–276, Warszawa 1978

Associated and non-associated sliding rules in contact friction problems

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AN ANALOGY between dry contact friction and perfect plasticity implies that sliding rules of rigid or elastic bodies along a contact surface can be derived by using the velocity rules associated with the limit friction condition $f = 0$, or the other function $g = 0$, by the gradiental rule. The case of orthotropic friction is discussed in detail and it is shown that the model of wedge asperities makes it possible to derive the non-associated sliding rules. The concavity of the limit surface results from the non-associated character of a sliding rule. Further, it is shown that the motion of granular materials can be described by using incremental relations similar to plasticity relations for hardening and softening materials. Here, however, contact hardening or softening occurs. Constitutive relations for contact sliding with account for elastic and plastic deformations are proposed in the last section.

W pracy wykorzystano analogię zachodzącą pomiędzy prawami płynięcia teorii plastyczności a prawami ruchu sztywnych lub sprężystych bloków z kontaktami ciernymi. W przypadku anizotropii tarcia kontaktowego, rozszerzenie koncepcji stowarzyszonego prawa płynięcia pozwala na proste wyprowadzenie ogólnych praw ruchu. Rozpatrując jednak model powierzchni z jednokierunkowymi nierównościami, otrzymuje się niestowarzyszone prawa poślizgu kontaktowego. Niestowarzyszone prawo płynięcia prowadzi w efekcie do wklęsłości powierzchni granicznej w przestrzeni sił zewnętrznych. Na przykładzie zbioru kul i walców pokazano możliwość przyrostowego opisu ruchu ośrodków ziarnistych z uwzględnieniem osłabienia kontaktowego. Równania przyrostowe mają strukturę analogiczną do równań teorii plastyczności materiałów ze wzmocnieniem i osłabieniem. Równania konstytutywne dla poślizgów kontaktowych z uwzględnieniem sprężystych i plastycznych deformacji podane są w ostatnim rozdziale pracy.

В работе использована аналогия имеющая место между законами течения теории пластичности и законами движения жестких или упругих блоков с фрикционными контактами. В случае анизотропии контактного трения расширение концепции ассоциированного закона течения позволяет просто вывести законы движения. Рассматривая однако модель поверхности с однонаправленными неровностями, получаются неассоциированные законы контактного скольжения. Неассоциированный закон течения приводит в эффекте к вогнутости граничной поверхности в пространстве внешних сил. На примере множества шаров и цилиндров показана возможность описания в приростах движения зернистых сред с учетом контактного ослабления. Уравнения в приростах имеют структуру аналогичную уравнениям теории пластичности материалов с упрочнением и ослаблением. Определяющие уравнения для контактных скольжений, с учетом упругих и пластических деформаций, приведены в последней главе работы.

1. Perfect plasticity and contact sliding rules

THE MOTION of a rigid block on a rough surface is usually considered as a prototype of a perfectly plastic material model [1]. In fact, the block motion can be assumed to visualize progressive plastic flow after reaching the yield surface. Thus the velocity components V_x, V_y and the horizontal force components T_x, T_y may be regarded as the analogues of generalized stress and strains $\mathbf{Q}$ and $\mathbf{q}$ for a perfectly plastic material. The associated or non-associated flow rules

$$\dot{\mathbf{q}} = \dot{\lambda}\frac{\partial F(\mathbf{Q})}{\partial \mathbf{Q}} \quad (\dot{\lambda} > 0), \tag{1.1}$$

$$\dot{\mathbf{q}} = \dot{\lambda}\frac{\partial G(\mathbf{Q})}{\partial \mathbf{Q}} \quad (\dot{\lambda} > 0) \tag{1.2}$$

relate the strain rates with the yield condition $F(\mathbf{Q}) = 0$ or the plastic potential $G(\mathbf{Q}) = 0$ which, in general, may be different from the yield condition.

For the associated flow rule, Eq. (1.1), the inverse relations can be derived by using the dissipation function $D(\dot{\mathbf{q}}) = \mathbf{Q}\cdot\dot{\mathbf{q}}$, which is a homogeneous function of strain rates of order one. We have [2]

$$\mathbf{Q} = \frac{\partial D(\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}, \tag{1.3}$$

whereas for the non-associated flow rule, Eq. (1.2), such inverse relation do not occur in general.

For the case of Coulomb friction of a rigid block, the limiting friction condition on the isotropic surface has the form

$$f(T_x, T_y, Q) = (T_x^2+T_y^2)^{\frac{1}{2}} - \mu N = 0 \tag{1.4}$$

and the velocities V_x, V_y, V_z can be expressed as follows:

$$V_x = \dot{\lambda}\frac{\partial f}{\partial T_x} = \dot{\lambda}\frac{T_x}{(T_x^2+T_y^2)^{\frac{1}{2}}}, \quad V_y = \dot{\lambda}\frac{\partial f}{\partial T_y} = \dot{\lambda}\frac{T_y}{(T_x^2+T_y^2)^{\frac{1}{2}}}, \quad V_z = 0, \tag{1.5}$$

where

$$\dot{\lambda} = (V_x^2+V_y^2)^{\frac{1}{2}}.$$

Thus for the specified normal force N, the velocities V_x, V_y are derived by the potential flow rules (1.5). But if we regard T_x, T_y, N and V_x, V_y, V_z as conjugate forces and velocities, the potentiality will not occur since the associated velocity rules would require that

$$V_x = \dot{\lambda}\frac{T_x}{(T_x^2+T_y^2)^{\frac{1}{2}}}, \quad V_y = \dot{\lambda}\frac{T_y}{(T_x^2+T_y^2)^{\frac{1}{2}}}, \quad V_z = -\dot{\lambda}\mu, \tag{1.6}$$

that is, the block would have the normal separating velocity to the contact surface.

Hence the limit plasticity theorems cannot be extended to the case of dry friction as this has been demonstrated by DRUCKER [3]. In order to make the kinematic theorem of limit analysis applicable, we may introduce artificially the dilating layer at the contact between the material and the rigid wall; this is equivalent to satisfying the relations (1.6). Such extension of the limit plasticity theorems was discussed independently by MRÓZ and DRESCHER [4] and COLLINS [5].

There exists a class of contact problems where the normal force N is determined from the equilibrium conditions and only tangential sliding of surface should be specified in terms of friction forces at the contact surface. In such problems the normal force can be regarded as a known one and the limit friction condition (1.4) together with the velocity rules (1.5) provide the required contact conditions. Since the normality occurs for T_x, T_y and V_x, V_y, regarding N as fixed, we may easily invert the rules (1.5) by introducing the dissipation function

$$D(V_x, V_y, N) = T_x V_x + T_y V_y = \mu N (V_x^2 + V_y^2)^{\frac{1}{2}}. \tag{1.7}$$

Then the inverse relations are

$$T_x = \frac{\partial D}{\partial V_x} = \mu N \frac{V_x}{(V_x^2+V_y^2)^{\frac{1}{2}}}, \quad T_y = \frac{\partial D}{\partial V_y} = \mu N \frac{V_y}{(V_x^2+V_y^2)^{\frac{1}{2}}}. \tag{1.8}$$

Regarding V_x, V_y as small displacements, the function D defined by Eq. (1.7) can be treated as elastic energy due to contact action; the variational principles of elasticity can

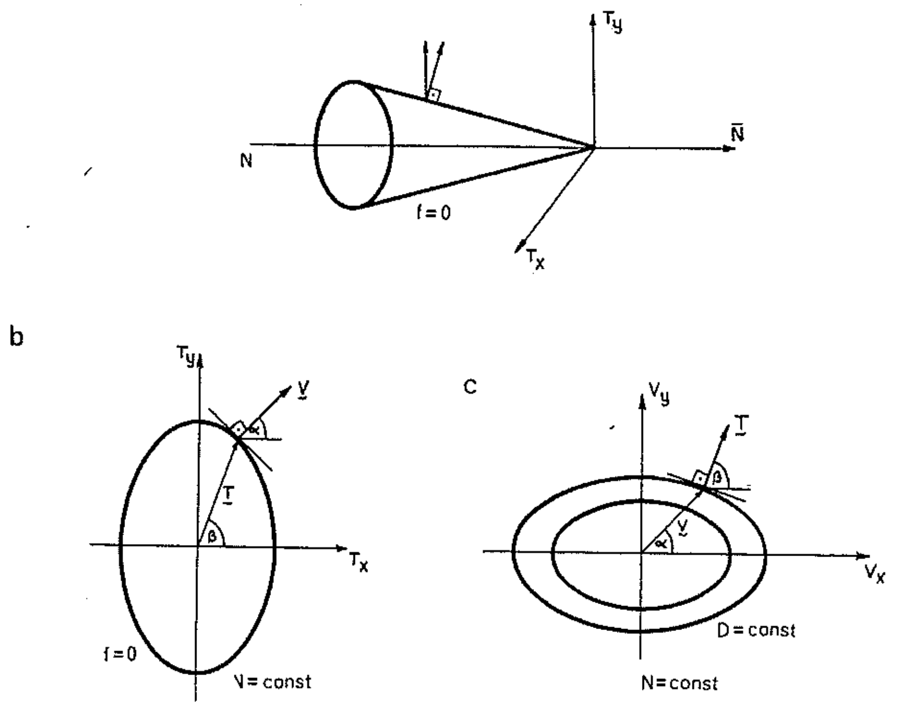

FIG. 1. a) Coulomb limit condition in the force space T_x, T_y, N; b) orthotropic friction condition in the plane N = const; c) dissipation function in the plane V_x, V_y.

be extended by adding the term (1.7) to the elastic energy of the body. Such variational principles have been discussed by DUVAUT and LIONS [11], and reviewed by KALKER [12]. In [10] KALKER formulated the maximum dissipation principle for contact sliding which is equivalent to the normality rule (1.5).

The case of anisotropic contact friction was discussed by HUBER [6] and MOSZYŃSKI [7]. In [6] the analogy between the transformation of the plane stress tensor and force acting on the block was applied, whereas in [7] a heuristic assumption was made claiming that the potential velocity rule (1.5) applies also when surface asperities have preferred orientations.

Consider for instance the case of orthotropy and assume that the principal axes of orthotropy coincide with the reference axes x, y. Let the limiting friction condition have the form

$$f(T_x, T_y, N) = \frac{T_x^2}{\mu_x^2}+\frac{T_y^2}{\mu_y^2}-N^2 = 0 \tag{1.9}$$

and the velocity components be

$$V_x = \dot{\lambda}\frac{\partial f}{\partial T_x} = \dot{\lambda}\frac{2T_x}{\mu_x^2}, \quad V_y = \dot{\lambda}\frac{\partial f}{\partial T_y} = \dot{\lambda}\frac{2T_y}{\mu_y^2}, \quad V_z = 0, \tag{1.10}$$

where

$$\dot{\lambda} = \frac{1}{2}N^{-1}[(\mu_x V_x)^2+(\mu_y V_y)^2]^{\frac{1}{2}}. \tag{1.11}$$

The dissipation function is expressed as follows:

$$D = T_x V_x+T_y V_y = 2\dot{\lambda}N^2 = N[(\mu_x V_x)^2+(\mu_y V_y)^2]^{\frac{1}{2}}. \tag{1.12}$$

The inverse relations are generated by the dissipation function, that is,

$$T_x = \frac{\partial D}{\partial V_x} = N\frac{\mu_x^2 V_x}{[(\mu_x V_x)^2+(\mu_y V_y)^2]^{\frac{1}{2}}}, \quad T_y = \frac{\partial D}{\partial V_y} = N\frac{\mu_y^2 V_y}{[(\mu_x V_x)^2+(\mu_y V_y)^2]^{\frac{1}{2}}}. \tag{1.13}$$

Denoting the inclination angle of the force **T** to the x-axis by β and that of the velocity vector **V** by α, Fig. 1, it follows, from Eq. (1.10) or Eq. (1.13), that

$$\frac{\operatorname{tg}\alpha}{\operatorname{tg}\beta} = \frac{\mu_x^2}{\mu_y^2}. \tag{1.14}$$

Let us denote by T_α the value of force **T** associated with the velocity vector **V** inclined at the angle α to the x-axis. In view of Eq. (1.13) we have

$$T_\alpha = \mu_\alpha N, \quad \mu_\alpha = \left(\frac{\mu_x^4\cos^2\alpha+\mu_y^4\sin^2\alpha}{\mu_x^2\cos^2\alpha+\mu_y^2\sin^2\alpha}\right)^{\frac{1}{2}}, \tag{1.15}$$

where μ_α is a directional coefficient of friction. The value of μ_α defined by Eq. (1.15) is different from that derived in [6], namely,

$$\mu_\alpha = \mu_1\cos^2\alpha+\mu_2\sin^2\alpha \tag{1.16}$$

or the value

$$\mu_\alpha = (\mu_x^2\cos^2\alpha+\mu_y^2\sin^2\alpha)^{\frac{1}{2}} \tag{1.17}$$

following from the assumption that the limit condition (1.9) can be reduced to Eq. (1.4) by using the transformed variables $T_x' = T_x/\mu_x$, $T_y' = T_y/\mu_y$ and applying the relation (1.8), that is,

$$T_x' = \frac{T_x}{\mu_x} = N\frac{V_x}{(V_x^2+V_y^2)^{\frac{1}{2}}}, \quad T_y' = \frac{T_y}{\mu_y} = N\frac{V_y}{(V_x^2+V_y^2)^{\frac{1}{2}}}. \tag{1.18}$$

Obviously, the relations (1.16)–(1.18) do not satisfy the normality rule in the sliding plane and do not follow from any physical model of the contact surface. On the other hand, the gradiental laws (1.10) and (1.13) can be generalized to any limiting conditions $f = f(T_x, T_y, N)$ and any case of anisotropy and thus can be accepted as general sliding rules. It remains an open question, however, if such normality rules as suggested by Moszyński are theoretically or experimentally substantiated. To investigate this question we consider a simple model of anisotropic asperities and show that potential flow rules in the sliding plane are not valid in general. Thus the non-associated velocity rules should be employed, using the potential function different from the limit friction condition. Moreover, the concavity of the limit surface occurs and it is related to a non-associated local sliding rule. In the next section we discuss two simple examples to illustrate this property. Next, the configuration hardening and softening effects will be briefly discussed and a phenomenological description of hardening or softening contact friction will be presented.

2. Example: a simple model of surface with unidirectional wedge asperities

The phenomena occurring at the contact surface and within the boundary layer are associated with large elastic and plastic deformation of randomly distributed asperities, thus resulting in the so-called Amonton's or Coulomb's laws of dry friction [14, 15]. In this section we shall discuss a model where, besides "small" isotropically distributed asperities, there exists a set of "large" unidirectionally orientated asperities on which the two surfaces slide, whereas the interaction of small asperities is simulated by the Amonton's law of friction. Thus the anisotropy of friction is simulated by the sliding mechanism on large asperities and the global velocity rules will be derived as a result of both local isotropic friction and sliding on inclined planes. Such a model has usually been used in simulating the relative motion of two layers of spheres or cylinders in order to derive the effective angle of friction (cf. Rowe [13]). It is believed, however, that it is also applicable in simulating anisotropy of machined metallic surfaces.

Let us consider a prototype model shown in Fig. 2. The rigid block of weight Q rests

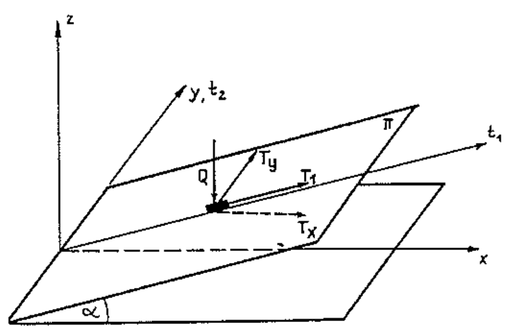

FIG. 2. The coordinate system and the corresponding forces acting on the block resting on the inclined plane.

on the plane π inclined at angle α to the horizontal plane. It is acted on by a horizontal force T whose components in the x, y coordinate system in the horizontal plane are T_x, T_y and in the local system t_1, t_2 in the inclined plane are T_1, T_2. Note that the t_2-axis coincides with the y-axis. The equilibrium conditions of the block require that

$$T_1 = T_x \cos\alpha - Q\sin\alpha, \quad N = T_x \sin\alpha + Q\cos\alpha \tag{2.1}$$

and the limit friction condition (1.4) is expressed as follows:

(2.2)
$$f(T_x, T_y, Q) = T_x^2(\cos^2\alpha - \mu^2\sin^2\alpha) - T_x Q\sin 2\alpha(1+\mu^2) + T_y^2 + Q^2(\sin^2\alpha - \mu^2\cos^2\alpha) = 0,$$

which is valid for $N = T_x\sin\alpha + Q\cos\alpha > 0$. The condition is represented by a conical surface in the space T_x, T_y, Q, whereas in the plane $Q = \text{const}$ we obtain an ellipse when $\alpha+\varrho < \frac{\pi}{2}$, a parabola when $\alpha+\varrho = \frac{\pi}{2}$ and a hyperbola for $\alpha+\varrho > \frac{\pi}{2}$. Here ϱ is the angle of friction, $\varrho = \operatorname{arctg}\mu$. Assuming that sliding occurs within the π-plane along the maximal tangential force, we obtain

$$V_1 = \dot{\lambda}\frac{T_x\cos\alpha - Q\sin\alpha}{[(T_x\cos\alpha - Q\sin\alpha)^2 + T_y^2]^{\frac{1}{2}}}, \quad V_2 = \dot{\lambda}\frac{T_y}{[(T_x\cos\alpha - Q\sin\alpha)^2 + T_y^2]^{\frac{1}{2}}} \tag{2.3}$$

and

$$\frac{V_x}{V_y} = \frac{(T_x\cos\alpha - Q\sin\alpha)\cos\alpha}{T_y}. \tag{2.4}$$

It is seen that this ratio is different from that following from the associated flow rule, namely,

$$\frac{V_x}{V_y} = \frac{T_x(\cos^2\alpha - \mu^2\sin^2\alpha) - Q\sin\alpha\cos\alpha(1+\mu)^2}{T_y} \tag{2.5}$$

and the velocity potential has the form

$$g(T_x, T_y, Q) = [(T_x\cos\alpha - Q\sin\alpha)^2 + T_y^2]^{\frac{1}{2}} - C = 0 \tag{2.6}$$

so that

$$V_x = \dot{\lambda}\frac{\partial g}{\partial T_x}, \quad V_y = \dot{\lambda}\frac{\partial g}{\partial T_y}, \quad V_z = -\dot{\lambda}\frac{\partial g}{\partial Q}. \tag{2.7}$$

The velocity potential is represented by the ellipse on the T_x, T_y — plane. Figures 3a and 3b show the elliptical and the parabolic yield conditions as well as the velocity potential. Connecting the typical point P on the limit surface $f = 0$ with the centre of the ellipse $g = 0$, we determine the associated point R on the velocity potential and the horizontal velocity vector is directed along the gradient vector at R. Note that the potential (2.6) generates also the vertical velocity V_z and the "dilatation" angle is defined by the relation

$$\sin\nu = \frac{V_z}{|V|} = \frac{(T_x\cos\alpha - Q\sin\alpha)\sin\alpha}{[(T_x\cos\alpha - Q\sin\alpha)^2 + T_y^2]^{\frac{1}{2}}}. \tag{2.8}$$

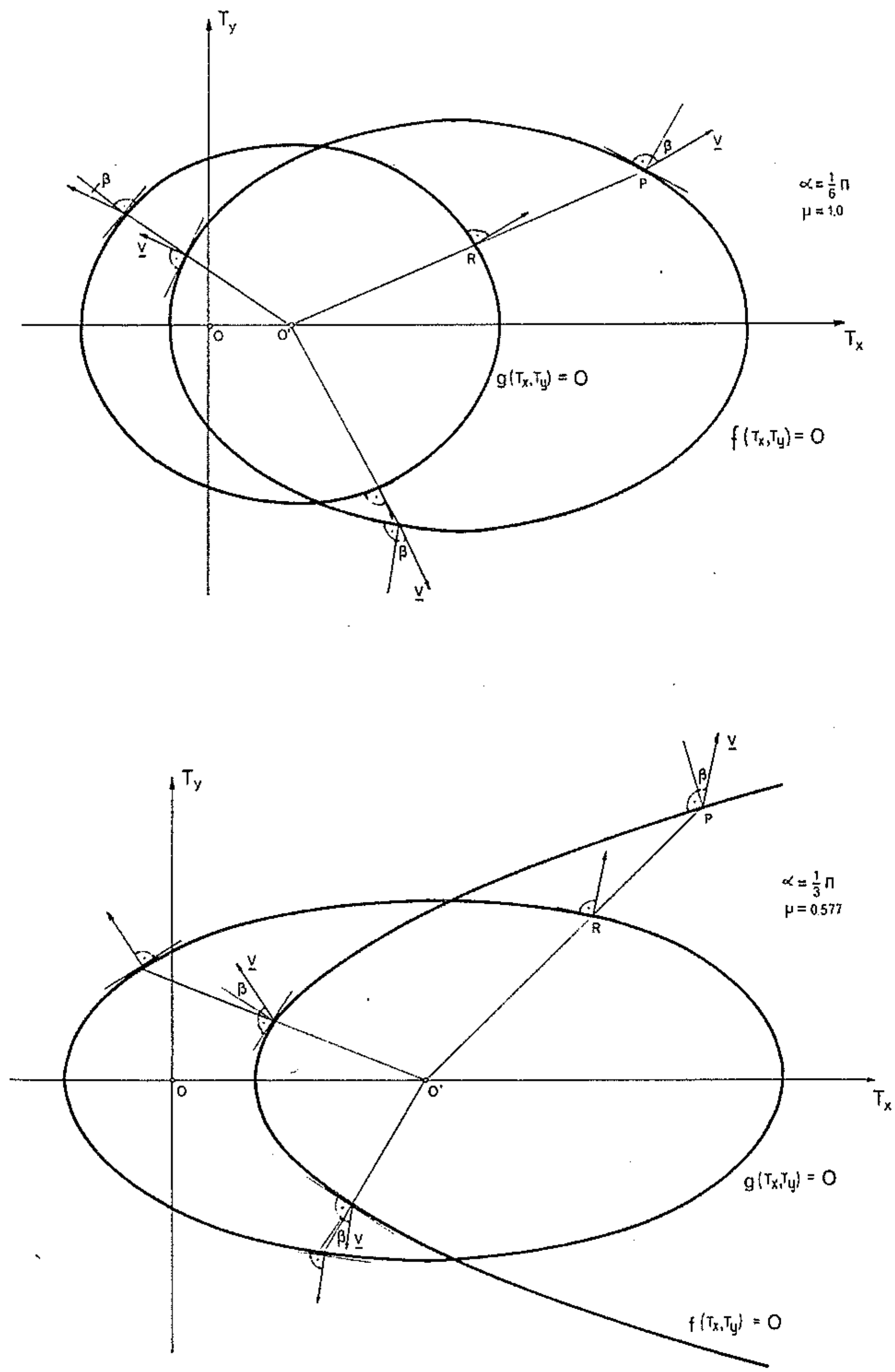

FIG. 3. a) Elliptical yield condition $f = 0$ and the velocity potential $g = 0$; b) parabolic yield condition $f = 0$ and the velocity potential $g = 0$.

Consider now the model of two rigid plates shown in Figs. 4a and b. The upper plate is acted on by the horizontal force inclined at the angle φ to the y-axis and the wedge asperities at the contact are aligned along the y-axis. The three mechanisms of motion may occur depending on the angle φ:

i) motion with active contact on the left flank of the asperity, $N_1 > 0$, $N_2 = 0$, ii) motion with active contact on the right flank, $N_2 > 0$, $N_1 = 0$, iii) sliding along the y-axis

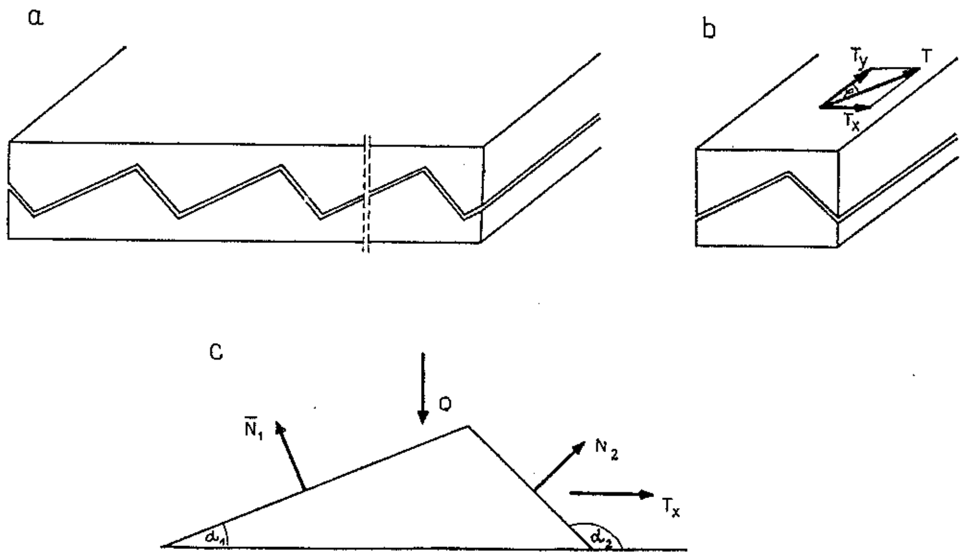

FIG. 4. a) ; b) A simple model of parallel asperities; c) the forces acting on the single asperity.

with the two active asperity flanks, $N_1 > 0$, $N_2 > 0$. The first two cases correspond to the previous example and the limit condition takes the form

$$f_i(T_x, T_y, Q) = T_x^2(\cos^2\alpha_i - \mu^2\sin^2\alpha_i) - T_x Q \sin 2\alpha_i (1+\mu^2) + T_y^2 + Q^2(\sin^2\alpha_i - \mu^2\cos^2\alpha_i) = 0, \tag{2.9}$$

where $i = 1$ when $N_2 = 0$ and $i = 2$ when $N_1 = 0$. In order to investigate the third mechanism, consider the plate portion with single asperity shown in Fig. 4b. The equilibrium equations for this portion are

$$T_x = N_1 \sin\alpha_1 - N_2 \sin\alpha_2, \quad T_y = \mu(N_1 + N_2), \quad Q = N_1\cos\alpha_1 - N_2\cos\alpha_2 \tag{2.10}$$

and since $T_x = T\sin\varphi$, $T_y = T\cos\varphi$, from Eqs. (2.10) we obtain

$$\begin{aligned} N_1 &= T\left(\frac{\cos\varphi}{\mu} + \frac{\sin\varphi}{\sin\alpha_2}\right)\frac{\sin\alpha_2}{\sin\alpha_1 + \sin\alpha_2}, \\ N_2 &= T\left(\frac{\cos\varphi}{\mu} + \frac{\sin\varphi}{\sin\alpha_1}\right)\frac{\sin\alpha_1}{\sin\alpha_1 + \sin\alpha_2} \end{aligned} \tag{2.11}$$

and the limit condition can be expressed in the form

$$f_3(T_x, T_y, Q) = T_x(\cos\alpha_1 + \cos\alpha_2)\mu + T_y \sin(\alpha_2 - \alpha_1) - Q(\sin\alpha_1 + \sin\alpha_2)\mu = 0. \tag{2.12}$$

The condition (2.12) applies when $N_1 > 0$ and $N_2 > 0$, that is, when

$$-\frac{\sin\alpha_2}{\mu} < \operatorname{tg}\varphi < \frac{\sin\alpha_1}{\mu}. \tag{2.13}$$

The velocity potentials now are

$$g_i(T_x, T_y, Q) = [(T_x\cos\alpha_i - Q\sin\alpha_i)^2 + T_y^2]^{\frac{1}{2}} - C = 0, \quad i = 1, 2 \tag{2.14}$$

and

$$g_3(T_x, T_y, Q) = T_y \mp C = 0. \tag{2.15}$$

Figure 5 presents the limit curve $f = 0$ and the velocity potential $g = 0$ on the plane T_x, T_y (Q = const) for $\alpha_1 = \frac{1}{3}\pi$, $\alpha_2 = \frac{5}{6}\pi$ and $\mu = 1.0$, and Figs. 6a and b show the limit and potential surfaces in the space T_x, T_y, Q. The velocity potential in the T_x, T_y

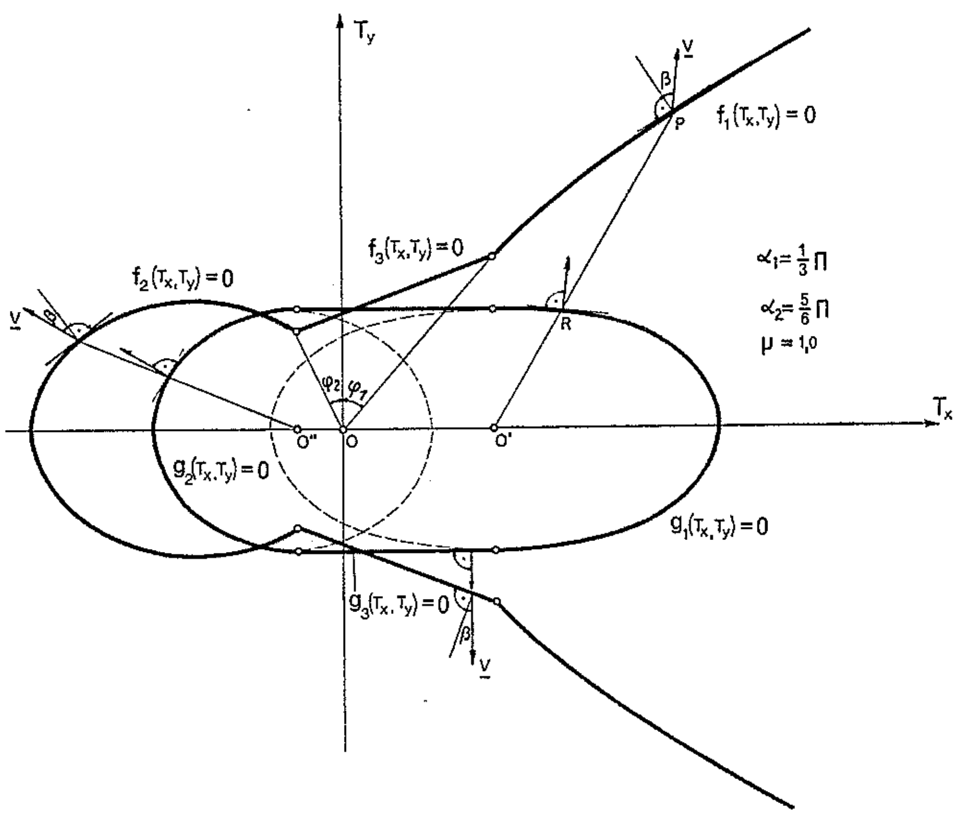

FIG. 5. The limit condition $f = 0$ and the velocity potential $g = 0$ for single asperity.

plane is composed of two ellipses and two horizontal lines. Thus, sliding on one wedge flank is represented by the elliptical potentials and motion along asperity by linear potentials.

It is interesting to note that the horizontal sliding along the y-axis, is also described by the non-associated rule when $\alpha_2 \neq 180-\alpha_1$, that is, when the wedge flanks are inclined at different angles to the horizontal plane. The vertical component of velocity is given by Eqs. (2.7) and the dilatation angle for mechanisms i) and ii) is determined from Eq. (2.8). In the case of the associated local velocity rule (1.6), the limit conditions (2.9) remain unchanged. It takes, however, a different form when simultaneous sliding on both planes occurs. The equilibrium equations for $N_1 > 0$, $N_2 > 0$ can now be derived from

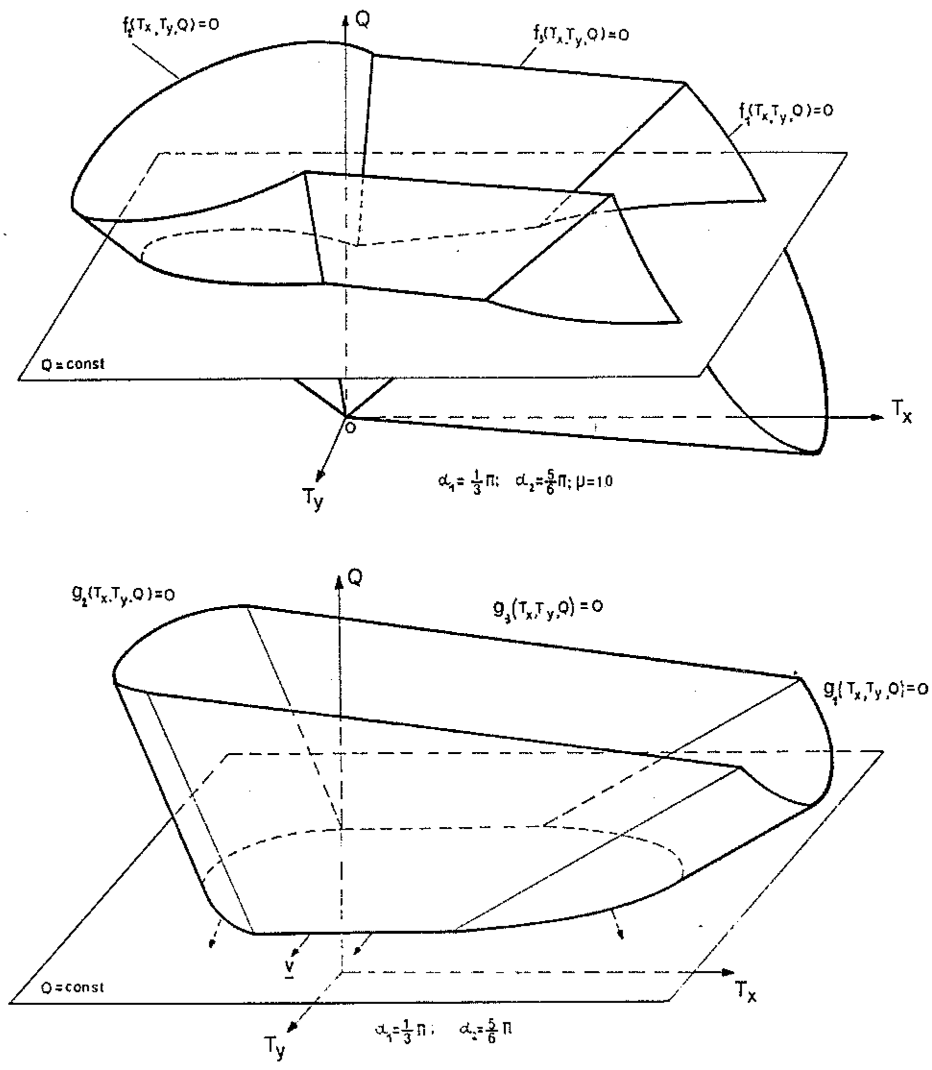

FIG. 6. a) Limit surface $f = 0$ in the space T_x, T_y, Q; b) velocity potential $g = 0$ in the space T_x, T_y, Q.

the condition that the velocity vectors on both planes are parallel and the angle of inclination of these vectors to the planes is $\varrho = \operatorname{arctg}\mu$. The limit condition (for $N_1 > 0$, $N_2 > 0$) can be expressed in the form

$$f_3'(T_x, T_y, Q) = T_x(A+B)m\mu + T_y(AC-BD) - Q(C+D)m\mu = 0 \tag{2.16}$$

which applies when

$$-\frac{C}{m\mu} < \operatorname{tg}\varphi' < \frac{D}{m\mu}, \tag{2.17}$$

where

$$\begin{aligned} A &= \cos\alpha_1 - n_1\mu, & k_1 &= k\cos\xi + n\sin\xi, \\ B &= \cos\alpha_2 + n_2\mu, & n_1 &= -k\sin\xi + n\cos\xi, \\ C &= \sin\alpha_2 - k_2\mu, & k_2 &= -k\cos\xi + n\sin\xi, \\ D &= \sin\alpha_1 + k_1\mu, & n_2 &= k\sin\xi + n\cos\xi, \end{aligned}$$

and

$$k = \sin\delta \operatorname{tg}\varrho, \quad m = \arcsin(\operatorname{tg}\delta \operatorname{tg}\varrho), \quad n = (1-k^2-m^2)^{\frac{1}{2}},$$

$$\delta = \frac{1}{2}(\pi+\alpha_1-\alpha_2), \quad \xi = \frac{1}{2}(\pi-\alpha_1-\alpha_2), \quad \varrho = \operatorname{arc}\operatorname{tg}\mu.$$

The limit condition computed for the associated local sliding rule (1.6) is shown in Fig. 7 where the limit surface corresponding to the non-associated velocity rule and defined by Eqs. (2.9) and (2.12) is presented by the broken line.

It is seen that the concavity of the limit surface, Eqs. (2.9) and (2.12), *results from the non-associated local velocity rule*. Thus, whereas local convexity and normality result

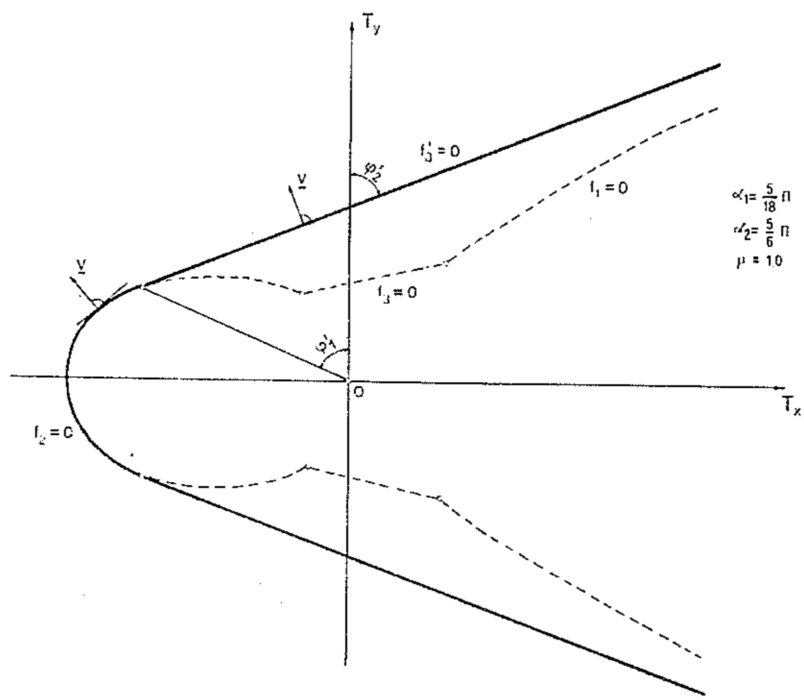

FIG. 7. Limit condition $f = 0$ for associated and non-associated sliding rule (broken line).

also in global convexity and normality in the force space, the departure from the normality rule in local contact sliding results in the concavity of the global limit surface and the non-associated sliding rule for the whole system, both in the sliding plane and in normal direction.

3. Sliding of cylinders and spheres: contact "softening"

Figure 8 illustrates the mechanism of sliding of a cylinder on the layer of cylinders of the same diameter. The coordinate axes are used to represent the displacement components u_x, u_y, whereas the force components T_x, T_y are represented in the local system translating along the displacement trajectory. Points 1–6 correspond to origins of the

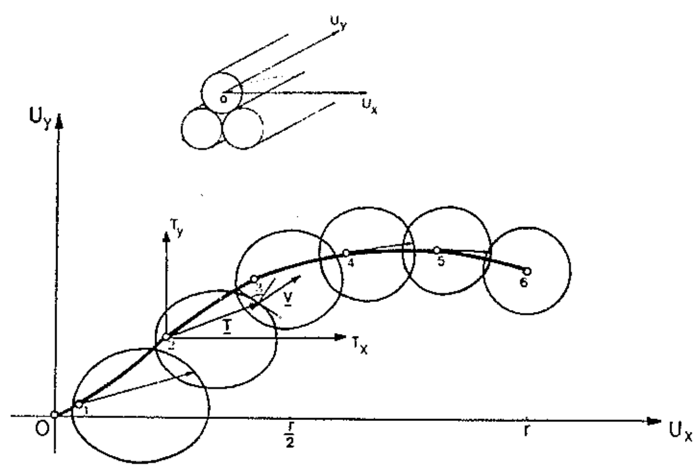

FIG. 8. Evolution of limit conditions in the displacement plane for a cylinder sliding on the layer of cylinders.

force reference system. It is seen that the limiting friction lines which are described by Eq. (2.2) with varying α and constant Q undergo modification in the course of motion, exhibiting the effect of contact "softening" due to varying α. Thus the force T acting on the cylinder will be a decreasing function of the displacement $|\mathbf{u}|$.

Figure 9 shows the solution of a similar problem for a sphere sliding on a layer of spheres of the same diameter (the combined rotation and sliding is not considered here). Now the modification of the yield surface is more complex, and both its translation and rotation are observed.

For a specified translation trajectory in the x, y-plane, the solution is obtained by solving incrementally the sliding rules (2.7) and accounting for the varying limit condition due to the variation of the sliding plane tangential to both spheres.

Mathematically, the problem can be formulated in a similar way as incremental relations in the theory of plasticity with account for hardening and softening [2]. Now the limit friction condition can be expressed as follows:

$$f(\mathbf{T}, \psi, \alpha) = 0, \tag{3.1}$$

and the velocity potential is

$$g(\mathbf{T}, \psi, \alpha) = 0, \tag{3.2}$$

where α and ψ are the Euler angles of the vector normal to the plane tangential to both spheres at the contact point with respect to the z and x axes. Note that α can be identified with the angle of inclination of the tangential plane with the x, y-plane. In the previous

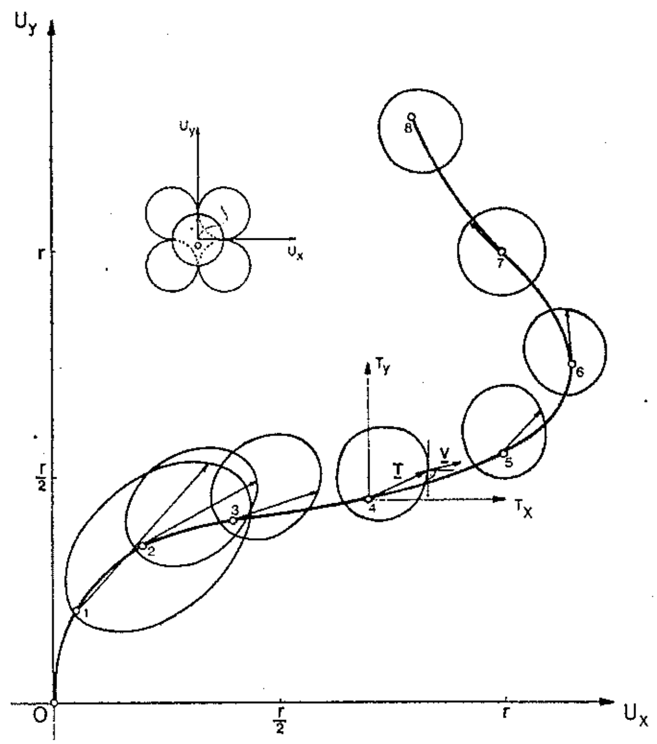

FIG. 9. Evolution of limit conditions in the displacement plane for a sphere sliding on the layer of spheres.

example, both ψ and α were fixed; however, α now varies in the case of a sliding cylinder and both ψ and α vary for the case of a sliding sphere. Hence we have

$$V_x = \dot{\lambda}\frac{\partial g}{\partial T_x}, \quad V_y = \dot{\lambda}\frac{\partial g}{\partial T_y}, \quad V_z = -\dot{\lambda}\frac{\partial g}{\partial Q} \tag{3.3}$$

and

$$\begin{aligned} \dot{\psi} &= AV_x + BV_y, \\ \dot{\alpha} &= CV_x + DV_y, \end{aligned} \tag{3.4}$$

where A, B, C, D are functions of $\mathbf{T}$, α, and ψ. The consistency relation

$$\frac{\partial f}{\partial \mathbf{T}}\dot{\mathbf{T}} + \frac{\partial f}{\partial \psi}\dot{\psi} + \frac{\partial f}{\partial \alpha}\dot{\alpha} = 0 \tag{3.5}$$

together with the relations (2.2), (2.6) and (3.3) form the system of equations which enables to determine $\dot{\lambda}$. Since instantaneous motion occurs in the tangential plane, the proper use of Eqs. (2.2) and (2.6) is justified. In the case of a cylinder we have

$$\dot{\psi} = 0, \quad \dot{\alpha} = -\frac{V_x}{R\cos\alpha} \tag{3.6}$$

and

$$\dot{\lambda} = \frac{R\cos\alpha}{\frac{\partial f}{\partial\alpha}\frac{\partial g}{\partial T_x}}\dot{f}_T, \tag{3.7}$$

where

$$\dot{f}_T = \frac{\partial f}{\partial T_x}\dot{T}_x + \frac{\partial f}{\partial T_y}\dot{T}_y \tag{3.8}$$

and R is the diameter of the cylinder. The functions $f = 0$ and $g = 0$ are identical with Eqs. (2.2) and (2.6) with varying α. The rate $\dot{\alpha}$ can be expressed as follows:

$$\dot{\alpha} = -\frac{\dot{f}_T}{\partial f/\partial\alpha} \tag{3.9}$$

or, after using Eq. (2.2), in the explicit form

$$\dot{\alpha} = -\frac{[2T_x(\cos^2\alpha-\mu^2\sin^2\alpha)-Q\sin 2\alpha(1+\mu^2)]\dot{T}_x+2T_y\dot{T}_y}{(1+\mu^2)[\sin 2\alpha(Q^2-T_x^2)-2T_xQ\cos 2\alpha]}. \tag{3.10}$$

The translation and shrinkage of the yield surface can now be computed for the given increments $dT_x = \dot{T}_x dt$, $dT_y = \dot{T}_y dt$ when the active process is performed (see Fig. 8), or increments of forces can be found for prescribed displacement increments $du_x = V_x dt$, $du_y = V_y dt$(1). In the case of a sphere, the yield condition (2.2) must be written in new coordinates T'_x, T'_y where

$$T'_x = T_x\cos\psi - T_y\sin\psi, \quad T'_y = T_x\sin\psi + T_y\cos\psi \tag{3.11}$$

and ψ is the angle between the vertical plane, perpendicular to the plane tangential to both spheres, and the x-axis. Now we have

$$\dot{\psi} = \frac{|V'|\cos(\psi+\delta)}{R\cos\alpha}, \quad \dot{\alpha} = \frac{|V'|\sin(\psi+\delta)}{R\cos\alpha}, \tag{3.12}$$

where

$$|V|' = (V_x'^2 + V_y'^2)^{\frac{1}{2}}, \quad \delta = \operatorname{arctg}\frac{V'_x}{V'_y},$$

and R is the diameter of the sphere.

Using the consistency relation

$$\frac{\partial f}{\partial \mathbf{T}'}\dot{\mathbf{T}}' + \frac{\partial f}{\partial\alpha}\dot{\alpha} + \frac{\partial f}{\partial\psi}\dot{\psi} = 0$$

(1) The rate formulation can be transformed into an incremental formulation by selecting any time scale. Then $dT_x = \dot{T}_x dt$, $dT_y = \dot{T}_y dt$, $du_x = V_x dt$, $du_y = V_y dt$, $du_z = V_z dt$.

together with Eqs. (2.2), (2.6) and (3.3), we can determine $\dot{\lambda}$ in the form

$$
\dot{\lambda} = \frac{-\left(A\frac{\partial T_x}{\partial T'_x}+2T_y\frac{\partial T_y}{\partial T'_x}\right)\dot{T}'_x-\left(A\frac{\partial T_x}{\partial T'_y}+2T_y\frac{\partial T_y}{\partial T'_y}\right)\dot{T}'_y}{H\left[\sin(\psi+\delta)\frac{\partial f}{\partial \alpha}+\cos(\psi+\delta)\left(A\frac{\partial T_x}{\partial \psi}+2T_y\frac{\partial T_y}{\partial \psi}\right)\right]} R\cos\alpha, \tag{3.13}
$$

where T_x, T_y are derived from Eqs. (3.11) and

$$
A = 2T_x(\cos^2\alpha-\mu^2\sin^2\alpha)-Q\sin 2\alpha(1+\mu^2),
$$

$$
H=\left\{\frac{B^2\left[\left(\frac{\partial T_x}{\partial T'_x}\right)^2+\left(\frac{\partial T_x}{\partial T'_y}\right)^2\right]+2BT_y\left(\frac{\partial T_x}{\partial T'_x}\frac{\partial T_y}{\partial T'_x}+\frac{\partial T_x}{\partial T'_y}\frac{\partial T_y}{\partial T'_y}\right)+T_y^2\left[\left(\frac{\partial T_y}{\partial T'_x}\right)^2+\left(\frac{\partial T_y}{\partial T'_y}\right)^2\right]}{\left(\frac{B}{\cos\alpha}\right)^2+T_y^2}\right\}^{\frac{1}{2}},
$$

$$
B = (T_x\cos\alpha-Q\sin\alpha)\cos\alpha
$$

and

$$
\operatorname{tg}\delta = \frac{B\frac{\partial T_x}{\partial T'_x}+T_y\frac{\partial T_y}{\partial T'_x}}{B\frac{\partial T_x}{\partial T'_y}+T_y\frac{\partial T_y}{\partial T'_y}}.
$$

Now, using Eqs. (3.12) and the non-associated velocity rule (3.3) the modification of the yield surface can be computed (see Fig. 9 where the solution is presented for the specified trajectory of motion in the x, y-plane).

4. Elastic-plastic constitutive laws for contact sliding

The preceding discussion of sliding rules for rigid materials can be generalized by assuming that both elastic and plastic deformations occur at contact asperities together with a sliding mechanism on "large" asperities. Assume that normal and tangential rates of displacement are decomposed into reversible (elastic) and irreversible (plastic) components

$$
\dot{\delta}_n = \dot{\delta}^e_n+\dot{\delta}^p_n, \qquad \dot{\delta}_t = \dot{\delta}^e_t+\dot{\delta}^p_t \tag{4.1}
$$

with the corresponding constitutive laws

$$
\dot{\delta}^e_n = \frac{1}{K_n}\dot{N}, \qquad \dot{\delta}^e_{t1} = \frac{1}{K_t}\dot{T}_1, \qquad \dot{\delta}^e_{t2} = \frac{1}{K_t}\dot{T}_2, \tag{4.2}
$$

$$
\dot{\delta}^p_n = \dot{\lambda}g_n, \qquad \dot{\delta}^p_{t1} = \dot{\lambda}g_{t1}, \qquad \dot{\delta}^p_{t2} = \dot{\lambda}g_{t2}, \tag{4.3}
$$

where δ_{t1}, δ_{t2} are tangential displacement components in the orthogonal system 1, 2 and δ_n denotes the normal displacement component; K_n and K_t are the elastic stiffness moduli of the contact layer. The contact yield condition has the form

$$
f(N, T_1, T_2, \varkappa, \beta) = 0, \tag{4.4}
$$

where the hardening parameter $\varkappa$ describes the contact hardening due to tangential sliding and β describes the hardening and softening due to variation of the effective contact area, $A_{\text{cont.}} = \beta A_{\text{total}}$. Let us postulate that

$$\dot{\varkappa} = [(\dot{\delta}^p_{t1})^2+(\dot{\delta}^p_{t2})^2]^{\frac{1}{2}}, \quad \dot{\beta} = B(\beta)\,\dot{\delta}^p_n, \tag{4.5}$$

where $B(\beta) > 0$. Differentiating the relation (4.4), we obtain

$$\dot{f}_T + \frac{\partial f}{\partial \varkappa}\dot{\lambda}(g^2_{t1}+g^2_{t2})^{\frac{1}{2}} + \frac{\partial f}{\partial \beta}B(\beta)\,\dot{\lambda} g_n = 0, \tag{4.6}$$

where

$$\dot{f}_T = \frac{\partial f}{\partial N}\dot{N} + \frac{\partial f}{\partial T_1}\dot{T}_1 + \frac{\partial f}{\partial T_2}\dot{T}_2.$$

From (4.6), we have

$$\dot{\lambda} = \frac{\dot{f}_T}{-\left[\dfrac{\partial f}{\partial \varkappa}(g^2_{t1}+g^2_{t2})^{\frac{1}{2}} + \dfrac{\partial f}{\partial \beta}B(\beta)g_n\right]} = \frac{\dot{f}_T}{H}, \tag{4.7}$$

and the *friction hardening modulus* H is expressed by the denominator of Eq. (4.7). The value of this modulus is thus governed by two processes: hardening due to tangential sliding on asperities with associated plastic deformations which are localized in the surface layer, and variation of the effective contact area manifested macroscopically by the normal displacement component with associated hardening and softening.

A more particular form of the relations (4.3) is obtained by postulating that the sliding is governed by the normality rule, that is, normal and tangential rates of translation are proportional to components of a vector normal to the surface $f = 0$, that is,

$$g_n = \frac{\partial f}{\partial N}, \quad g_{t1} = \frac{\partial f}{\partial T_1}, \quad g_{t2} = \frac{\partial f}{\partial T_2}. \tag{4.8}$$

These sliding rules correspond to the isotropic contact surface. Whereas $\varkappa$ increases with translation, the parameter β may increase or decrease depending on the rate of change of the normal component $\dot{\delta}^p_n$ which represents plastic deformation of asperities during compression.

Figure 10a presents the yield surface in the space T_1, T_2, N. In particular, it may be assumed that this surface is a rotational ellipsoid whose expansion or contraction and translation along the N-axis depend on one parameter, for instance,

$$f(N, T_1, T_2, \varkappa, \beta) = (N-A)^2 + \frac{T_1^2}{m^2} + \frac{T_2^2}{m^2} - A^2 = 0 \tag{4.9}$$

with the constant ratio of semi-axes $m = c/a = b/a = \operatorname{tg}\varrho$. The parameter A may depend on both $\varkappa$ and β, and this dependence is schematically shown in Fig. 10b.

Using Eqs. (4.8), the relations (4.3) and (4.7) take the form

$$\dot{\delta}^p_n = \frac{1}{H}2(N-A)\dot{f}_T, \quad \dot{\delta}^p_{t1} = \frac{1}{H}\frac{2T_1}{m^2}\dot{f}_T, \quad \dot{\delta}^p_{t2} = \frac{1}{H}\frac{2T_2}{m^2}\dot{f}_T \tag{4.10}$$

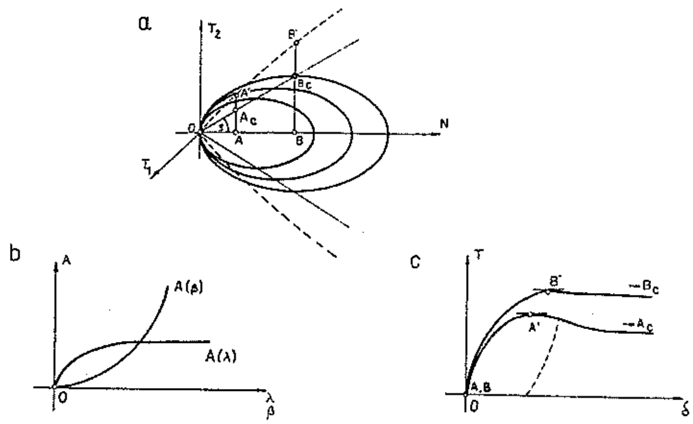

FIG. 10 a) Proposed yield surface in the space T_1, T_2, N; b) variation of characteristic contact function $A(\beta, \varkappa)$ with β and $\varkappa$; c) tangential force — displacement curves for fixed values of N.

where, according to Eq. (4.7),

$$H = 2\left[\frac{\partial f}{\partial \beta}(A-N)B(\beta) - \frac{\partial f}{\partial \varkappa}\left(\frac{T_1^2+T_2^2}{m^4}\right)^{1/2}\right]. \tag{4.11}$$

Figure 10c presents schematically the relation between T and δ for two fixed values of N corresponding to points A and B in Fig. 10a. In the first case, we observe contact dilatation with corresponding instability after reaching the maximum value at A'; in the second case, T monotonically increase to B' and, next, slowly diminishes to the asymptotic value at B_c. The lines connecting the asymptotic points A_c, B_c and the lines corresponding to the maximal points are shown in Fig. 10a. Depending on particular forms of $A(\varkappa, \beta)$, the details of such behaviour may vary, although the general feature of contact behaviour remains unchanged. A similar yield condition depending only on one parameter β was discussed by CALLADINE [16] and applied in the derivation of incremental constitutive laws for clays.

The anisotropy may be described similarly as in the previous section by expressing the yield condition in terms of external forces Q, T_x, T_y, that is,

$$F(Q, T_x, T_y, \varkappa_i, \beta_i, \alpha_i, \psi_i) = 0, \tag{4.12}$$

where α_i and ψ_i define the tangential plane to the contact i ($i = 1, 2, \ldots, n$) with respect to the global coordinate system x, y, z. Now, local contact properties are described by the parameters $\varkappa_i, \beta_i$ whereas the configuration contact hardening and softening is described by rate equations for α_i and ψ_i. An application of such flow rules to the analysis of the motion of granular materials will be discussed in subsequent papers.

5. Concluding remarks

The present paper has an introductory character and applies the concept of plasticity in the description of sliding rules of rigid or elastic-plastic bodies with frictional contacts. Due to the non-associated character of a local sliding rule, the rigid body motion on wedge asperities does not obey the normality rule in the horizontal plane and, similarly, the rate of dilatation is predicted by the velocity potential different from the limit condition. Thus there is no reason to expect that sliding rules for anisotropic contacts will obey the normality rule and this case calls for further study, both theoretical and experimental. The theoretically obtained concavity of the limit surface *results from the non-associated local flow rule*. Further, by introducing a more accurate description of the contact layer based on two state parameters $\varkappa$ and β, the velocity rules can be derived similarly as flow rules in hardening plasticity. The problem of multi-body contact sliding can be solved, in principle, by using an incremental procedure and tracing the history of $\varkappa_i, \beta_i, \alpha_i, \psi_i$ for each contact and satisfying simultaneously equilibrium and compatibility conditions. Simple examples of contact softening are discussed in Sect. 3. It is believed that further development of multi-body unilateral contact mechanics with proper sliding rules will constitute a uniform formulation for a systematic study of the behaviour of granular materials and other frictional systems.

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Received June 27, 1977.