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SHORT COMMUNICATIONS

A DIFFERENTIAL SLICE APPROACH TO THE PROBLEM OF RETAINING WALL LOADING

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SUMMARY

A differential slice technique is applied in order to obtain the solution for the pressure distribution on rigid retaining walls. The technique is based on a statical analysis of a plane differential slice, for which only the global equilibrium is required. An analytical solution to the problem is found for a non-cohesive material and a linear slip surface. For more complicated cases the numerical solution is required.

INTRODUCTION

Two different approaches to the problem of loading on rigid retaining walls can be distinguished. The first approach, introduced by Coulomb,[1] and later developed by others, is based on the equilibrium analysis of a rigid wedge. The wedge is limited by a wall on one side, and a potential slip surface on the other. The wall friction angle is mobilized, and the frictional Coulomb yield condition is satisfied along the slip surface. Global equilibrium of the wedge is required. The location of the slip surface produces the maximum total force acting on the wall: The external loading on the top surface can be included in the analysis. For non-cohesive soils an analytical solution can be found. The same method can be used in order to find the passive loading; however, this time the inclination of the slip is found from the condition of minimum total force on the wall. Various graphical methods were developed in order to obtain the solution based on the assumptions suggested first by Coulomb.

The second approach to the problem of retaining wall loading is based on the assumption that the limit equilibrium state, described by the Mohr–Coulomb yield condition, is satisfied at each point of the backfill. Local equilibrium equations and the Mohr–Coulomb yield condition lead to a set of hyperbolic type equations, and the solution can be obtained using the method of characteristics.[2] For a vertical smooth wall and a horizontal surface of the backfill, the method of characteristics reduces to the case known in the literature as the Rankine theory of lateral earth pressure.

The method presented in this paper is based on the statical analysis of a plane element with a finite dimension in one direction (differential slice). Such a method was first suggested by Janssen[3] in order to predict wall pressure caused by the granular materials in vertical bins. Janssen required only the global equilibrium of the slice to be satisfied. The statics of the problem shown by Janssen[3] was finally described by one equilibrium equation, since, owing to the symmetry of the problem, the moment equilibrium equation and the equation of the

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horizontal force balance become trivial. A similar method was used later by other authors in order to analyse the forging processes of metals,[4] and wall pressures in converging bunkers,[5] also to analyse some effects in triaxial tests on sand,[6] etc. All these problems involve symmetrical geometry. A similar method is applied in this paper in order to predict loading caused by the soil media on rigid retaining walls. In this case, however, the geometry of a slice cannot be considered as symmetrical. Here the method is called the differential slice technique (after Hancock and Nedderman[7]), as opposed to the slice method used for the purpose of the slope stability analysis,[8] which is based on different assumptions and formulated in terms of the factor of safety. It is known to the author that the effort to adapt the slice method to the problem of retaining walls was also made.[9]

The idea of the application of the differential slice technique to the problem of retaining wall loading is shown in Figure 1. Line AO is a potential slip surface, similar to the one in Coulomb's theory. The shape of this surface has to be chosen first, and the location can be optimized for maximum total force on the wall (in the active state). Global equilibrium of an arbitrarily chosen element (slice), parallel to the top surface and limited by the slip surface on one side and the wall on the other is required.

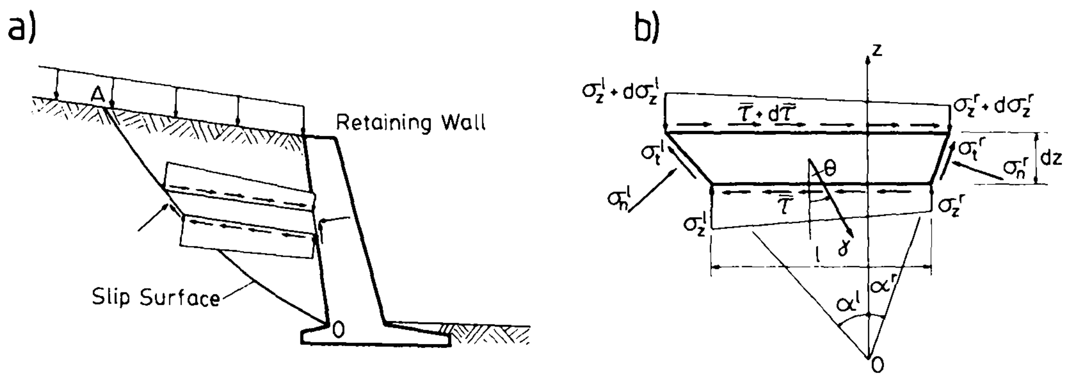

Figure 1(a). Retaining wall scheme; (b) Differential slice

In the next section the global statical equilibrium of a non-symmetrical differential slice will be analysed. The assumptions will be outlined and a set of basic equations derived. Similar equations for a non-cohesive material were used by the author to predict wall pressure distribution in plane asymmetrical hoppers.[10]

EQUILIBRIUM EQUATIONS FOR A DIFFERENTIAL SLICE

The differential slice technique presented in this paper is based on the statical analysis of the slice shown in Figure 1(b). The z-axis is perpendicular to the finite dimension of the slice. The sides of the slice are inclined to the z-axis at the angles α^{l} and α^{r}, respectively, and the gravity force at the angle θ. As opposed to the case where the geometry of the slice is symmetrical, all three equations of global equilibrium in the case shown in Figure 1(b) are non-trivial. However, the problem remains statically indeterminable. The first assumption made in order to remove this indeterminancy requires that the distribution of the stress σ_z along the slice is known. Any distribution can be taken into consideration. In this paper this distribution is postulated to be linear. Distribution of the shear stress along the slice does not influence any of the global equilibrium equations; thus the mean shear stress $\bar{\tau}$, defined as the stress averaged over the length of the slice, is introduced here.

The second important requirement which is made here in order to obtain a statically determined system assumes that relations between some of the stress state components at both ends of the slice are known. The first two relations describe the inclination of the stress vectors at both ends of the slice† (ρ and c are given parameters);

$$\sigma_t^{l} = \sigma_n^{l} \tan \rho^{l} + c^{l}, \qquad \sigma_t^{r} = \sigma^{r} \tan \rho^{r} + c^{r} \tag{1}$$

The second group of relations is assumed to be as follows:

$$\sigma_n^{l} = K^{l} \sigma_z^{l} + L^{l}, \qquad \sigma_n^{r} = K^{r} \sigma_z^{r} + L^{r} \tag{2}$$

The values of parameters K and L in (2) follow from the analysis of the stress state at both ends of the slice. It will be seen later that the assumed forms of relations (1) and (2) are sufficient for the problem outlined in the title of this paper.

Relations similar to (1) and (2) were adopted by Janssen[3] and other authors who used the differential slice technique in its symmetrical form.[4–6] However, these relations had homogeneous forms and, owing to the symmetry, had the same parameters for the left and right ends of the slice.

The final set of equations for global statical equilibrium of the slice shown in Figure 1(b) can be written in the form of the following three linear, ordinary differential, non-homogeneous equations (after using (1) and (2), and neglecting second order terms):

$$\begin{aligned} z\frac{d\sigma_z^{l}}{dz} + \sigma_z^{l}A^{l} + \sigma_z^{r}B^{r} - 6\bar{\tau}/a &= B - z\gamma \cos\theta \\ z\frac{d\sigma_z^{r}}{dz} + \sigma_z^{l}B^{l} + \sigma_z^{r}A^{r} + 6\bar{\tau}/a &= A - z\gamma \cos\theta \\ z\frac{d\tau}{dz} - \sigma_z^{l}D^{l} + \sigma_z^{r}D^{r} + \bar{\tau} &= D - z\gamma \cos\theta \end{aligned} \tag{3}$$

where γ is the specific weight of the material. The stresses σ_z^{l}, σ_z^{r} and $\bar{\tau}$ are the unknown functions of z. The coefficients A^{l}, A^{r}, B^{l}, B^{r}, D^{l}, D^{r}, A, B, D and a are given in the Appendix.

The coefficients in the basic set of equations (3) are constant if the inclination of the sides of the slice is constant and the parameters in relations (1) and (2) are constant. An effort to obtain an analytical solution for such a case is made in the next section.

SOLUTION OF THE BASIC SET OF EQUATIONS

The characteristic equation of the set of equations (3) can be written in the form:

$$\begin{vmatrix} A^{l}-r & B^{r} & 6/a \\ B^{l} & A^{r}-r & -6/a \\ -D^{l} & D^{r} & (1/a)-r \end{vmatrix} = 0 \tag{4}$$

Equation (4) has usually one real and two complex roots (for the parameters K, L, ρ estimated as reasonable values for a practical case of a retaining wall):

$$r_1, r_2 = \alpha + i\beta, \qquad r_3 = \alpha - i\beta \tag{5}$$

†Superscripts l and r denote the values at the left and right ends of the slice, respectively.

However, the solution of (3) can be found in a real form if two of the integration constants are postulated to be conjugate complex numbers. The final solution can be written in the following form:

$$\begin{aligned} \sigma_z^l &= T^l+\gamma R^l z+C_1 M z^{r_1}+2(\nu C_2-\delta C_3)z^\alpha \cos(\beta \ln z)-2(\delta C_2+\nu C_3)z^\alpha \sin(\beta \ln z)\\ \sigma_z^r &= T^r+\gamma R^r z+C_1 N z^{r_1}+2(\eta C_2-\mu C_3)z^\alpha \cos(\beta \ln z)-2(\mu C_2+\eta C_3)z^\alpha \sin(\beta \ln z)\\ \bar{\tau} &= T^t+\gamma R^t z+C_1 z^{r_1}+2C_2 z^\alpha \cos(\beta \ln z)-2C_3 z^\alpha \sin(\beta (\ln z) \end{aligned} \tag{6}$$

where C_1, C_2 and C_3 are constants to be determined from the boundary conditions at $z=z_0$, and r_1, α and β follow from the solution (5) of the characteristic equation M, N, ν, δ, η and μ are defined as follows:

$$\begin{aligned} M(r_1)&=M, \qquad M(r_2)=\nu+\mathrm{i}\delta, \qquad M(r_3)=\nu-\mathrm{i}\delta\\ N(r_1)&=N, \qquad N(r_2)=\eta+\mathrm{i}\mu, \qquad N(r_3)=\eta-\mathrm{i}\mu \end{aligned} \tag{7}$$

where $M(r)$ and $N(r)$ are functions of the roots (5) of the characteristic equation and are given in the Appendix together with the expressions for T^l, T^r, T^t and R^l, R^r, R^t.

The analytical solution (6) is valid only for constant parameters in relations (1) and (2) and constant angles α^l and α^r (see Figure 1(b)). In many practical cases such assumptions may be too far from the actual situation. In such cases a numerical method of solving the set of equations (3) is recommended.

EXAMPLES

Three examples are given in this section: the first two for the case of active pressure caused by a non-cohesive and a cohesive material, and the third example for the passive state (non-cohesive material). For all three examples the computations are made for a linear and cylindrical shape of the slip surface. The inclination of the linear slip surface and the centre of the circular slip surface are found for maximum total force in the active state, and minimum in the passive state. The wall stress distribution obtained from the differential slice technique is compared to the one obtained from the method of characteristics.

The parameters in relations (1) and (2) are found under the assumption that the limit stress state, described by the Mohr–Coulomb yield condition, occurs along the slip surface and along the wall. It is assumed in the examples that there is no adhesion between the wall and the soil, and that the wall friction angle ϕ_w is mobilized (i.e. the stress vectors on the wall are inclined to the outward unit normal at the angle ϕ_w). Denoting the internal friction angle by ϕ and the cohesion of the soil by c, the parameters in (1) are as follows: $\rho^l=\phi$, $c^l=c$, $\rho^r=\phi_w$, $c^r=0$. The expressions for parameters K^l, L^l, K^r and L^r in (2) can be derived from the above assumptions. They can be written in the following form:

$$\begin{aligned} \mathrm{K}^l&=\cos^2\phi/[1\pm\sin\phi \sin(2\alpha^l\pm\phi)]\\ L^l&=c\cot\phi(K^l-1) \end{aligned} \tag{8}$$

and

$$\begin{aligned} K^r&=[1-\sin\phi\cos(2\psi+2\alpha^r)]/(1+\sin\phi\cos 2\psi)\\ L^r&=c\cot\phi(K^r-1) \end{aligned} \tag{9}$$

where

$$2\psi = \arcsin[(p - c\cot\phi)\sin\phi_w/p\sin\phi] - 2\alpha^r - \phi \qquad \text{active}$$

$$2\psi = \arcsin[(p - c\cot\phi)\sin\phi_w/p\sin\phi] - 2\alpha^r + \phi_w - \pi \qquad \text{passive}$$

and

$$p = -(\sigma_x + \sigma_z)/2 + c\cot\phi$$

The upper signs in (8) provide the value of K^1 for the active state and the lower signs for the passive case.

The first example (active state, non-cohesive material) is shown in Figure 2. The wall is 5 m high and inclined at an angle 5° to the vertical direction. The top surface is inclined at an angle 10° to the horizontal direction and it is loaded with uniformly distributed external loading $q = 20\ \mathrm{kN/m^2}$. The specific weight of the material $\gamma = 18\ \mathrm{kN/m^3}$, the internal friction angle $\phi = 32°$, and the wall friction angle $\phi_w = 15°$. For a linear slip surface (the broken line in Figure 2) the solution was obtained analytically. The corresponding wall pressure distribution is shown by the broken line diagram. The 'disturbances' in the wall stress distribution in the neighbourhood of the toe (point 0) result from the trigonometric functions present in (6). At the toe, this distribution rapidly approaches 0 (r_1 and α in (6) are positive, and T^r becomes 0 for non-cohesive material).

The dotted diagram in Figure 2 shows the normal wall pressure distribution (obtained numerically) for a circular slip surface marked by the dotted line. Point 0 on the z-axis was defined as the point of intersection of the extended directions of the sides of a single slice (Figure 1(b)). For a varying angle α^1 the location of this point on z-axis is not constant. Therefore z was replaced in (3) by the expression $l\sqrt{(\tan\alpha^1 + \tan\alpha^r)}$, where l is the length of

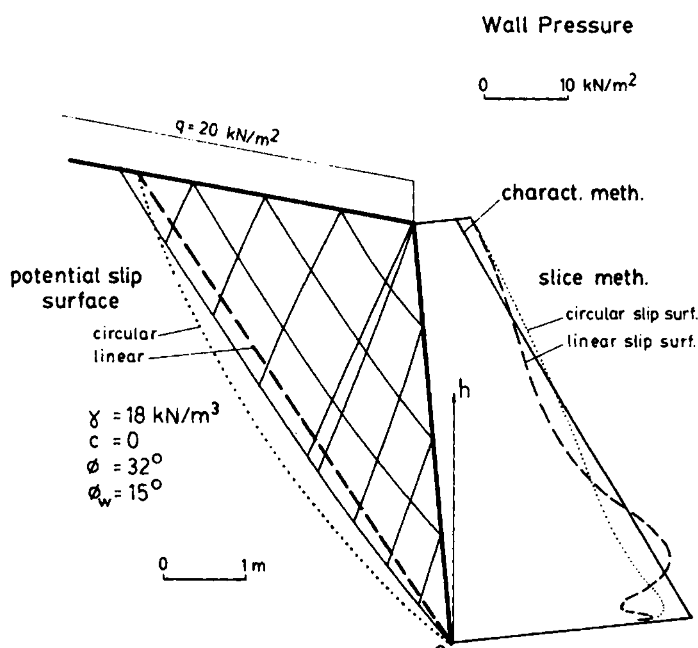

Figure 2. Active pressure on wall (example I)

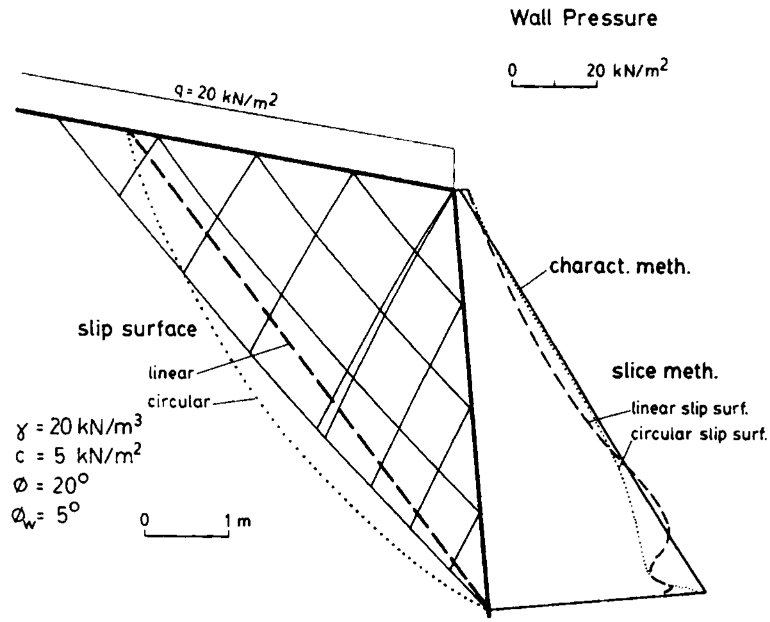

Figure 3. Active pressure on wall (example II)

the slice (Figure 1(b)). l and α^1 are known for a given geometry of the slip surface, and for convenience in calculating they can be expressed as the functions of a fixed axis h (Figure 2).

The third diagram in Figure 2 (the continuous line) shows the pressure distribution obtained by the method of characteristics.

The second example (active state, cohesive material) is shown in Figure 3. The inclination of the wall and the top surface is the same as in the first example. The material parameters are: $\gamma = 20\ \mathrm{kN/m^3}$, $\phi = 20°$, $c = 5\ \mathrm{kN/m^2}$ and $\phi_w = 5°$. The coefficients K^r and L^r in (2) for a cohesive material are functions of the stress state (see (9)), hence solutions for both linear and circular slip surface were obtained numerically.

The third example (passive state, non-cohesive material) is shown in Figure 4. The wall is vertical, 2 m high. The top surface is inclined to the horizontal direction at an angle 5°. The following material parameters were selected: $\gamma = 18\ \mathrm{kN/m^3}$, $\phi = 32°$, $\phi_w = 15°$. The solution for the linear slip surface was obtained analytically, and for the cylindrical slip surface numerically. Solution of the characteristic equation of the basic set of equations (3) for the passive case gives a negative value of r_1 (see (5)), thus the stresses at the toe of the wall ($z = 0$) following from (6) approach infinity. Infinite pressure on the wall cannot be conceived in a practical case, hence the application of the differential slice technique to the problem of passive pressure is questionable. However, the integral of the wall pressure is finite, and the technique can be used for prediction of the total force.

Total normal forces on the wall for all three examples are shown in Table I.

CONCLUDING REMARKS

The prediction of the wall pressure on rigid retaining walls presented in this paper is based on assumptions similar to those in Coulomb's theory. However, the equilibrium of the slices

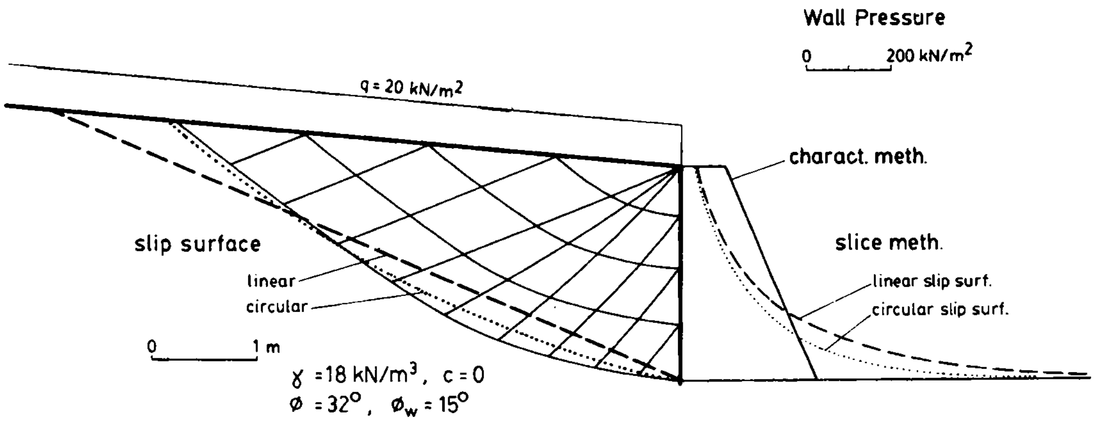

Figure 4. Passive pressure on wall (example III)

parallel to the top surface rather than the global equilibrium of the wedge acting on the wall, is required. Hence, not only the total force but also the distribution of the wall pressure can be computed.

The basic set of equations follows from the statical equilibrium analysis of a differential slice. The problem is statically indeterminable, so the following additional assumptions were made: (1) the distribution of the normal stress along the slice is linear, (2) relations between some of the stress state components at the ends of the slice are given.

The first assumption is very crucial for the differential slice technique. A linear approximation of the normal stress along the slice seems to be reasonable if compared with the distribution obtained by the method of characteristics. This distribution is constant in the Rankine zone, changes rapidly across the fan of characteristics, and remains close to linear, with low gradient, between the fan and the wall. A linear approximation seems to be particularly satisfactory in the active case, in which the fan of characteristics is spread in a narrow area. For some active state cases, the stress field resulting from the method of characteristics may be discontinuous. However, the method of characteristics cannot be considered here as a criterion for the differential slice technique, since it is based on a questionable assumption that the limit equilibrium state appears at every point of the backfill.

The given relations mentioned above in the second assumption include the material properties of the soil. The parameters in these relations follow from the assumption that the limit equilibrium state, described by the Mohr–Coulomb yield condition, occurs along the slip surface and along the wall. For a cohesive material and small external loading on the top surface,

Table I. Total normal force acting on the wall (kN)

	Example I	Example II	Example III
Method of characteristics	85·88	132·60	442·73
Differential slice technique linear slip surface	87·60	135·77	522·75
Differential slice technique cylindrical slip surface	87·89	133·74	419·44

tensile stresses may result. To avoid the tensile forces between the wall and the material, the top layer of the backfill can be substituted by an artificial loading.[2]

The correctness of the obtained solution (in terms of not exceeding the yield condition) can be proved only in terms of the global force acting on the slice, since the distribution of the shear stress τ does not follow from the solution. The inequality

$$\bar{\tau} \leqslant \tan \phi(\sigma_z^{l} + \sigma_z^{r}) + c$$

has to be satisfied for every z throughout the analysed backfill.

The wall pressure distribution obtained in the paper is significantly different from the distribution following from the method of characteristics only in the passive case. It is clear that the integration constants C_1, C_2 and C_3 in (6) are the functions of the total height of the wall. Thus, the pressure at a certain constant depth below the top surface of the soil will change if the total height of the wall is changed. In such sense the total height of the wall affects the global distribution of the wall pressure. If the method of characteristics is used, however, the distribution is not sensitive to the total height of the wall, i.e. the pressure distribution along the wall of height h is identical to the distribution along the segment h of any wall higher than h. The total forces on the wall remain relatively close for both the differential slice and characteristics method (see Table I).

The advantage of the differential slice technique over traditional methods based on Coulomb's assumptions is that it can be applied for any reasonable shape of the slip surface, and gives not only the total force, but also the distribution of the wall pressure. It can also be applied when the backfill is non-homogeneous or the friction varies along the wall. The advantage over the method of characteristics is that it does not require the limit stress state to occur at every point of the backfill.

The differential slice technique cannot be considered as a lower bound approach in limit analysis since it requires only global equilibrium of the differential slice.

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APPENDIX

The coefficients in (3) are

$$A^{l(r)} = [3 \tan \alpha^{l(r)} - 4(\tan \rho^{l(r)} + \tan \alpha^{l(r)})K^{l(r)}]/a$$

$$B^{l(r)} = [\tan \alpha^{r(l)} - 2 \tan \alpha^{l(r)} + 2 (\tan \rho^{l(r)} + \tan \alpha^{l(r)})K^{l(r)}]/a$$

$$D^{l(r)} = (\tan \alpha^{l(r)} \tan \rho^{l(r)} - 1)K^{l(r)}/a$$

$$A = [-2L^{l}(\tan \rho^{l} + \tan \alpha^{l}) + 4L^{r}(\tan \rho^{r} + \tan \alpha^{r}) - 2c^{l} + 4c^{r}]/a$$

$$B = [-2L^{r}(\tan \rho^{r} + \tan \alpha^{r}) + 4L^{l}(\tan \rho^{l} + \tan \alpha^{l}) - 2c^{r} + 4c^{l}]/a$$

$$D = [(L^{l} \tan \alpha^{l} + c^{l}) \tan \alpha^{l} - (L^{r} \tan \rho^{r} + c^{r}) \tan \alpha^{r} - L^{l} + L^{r}]/a$$

$$a = \tan \alpha^{l} + \tan \alpha^{r}$$

The following expressions for functions $M(r_i)$ and $N(r_i)$ in (7) were derived:

$$M(r_i)=[6(A^{r}+B^{r}+r_i)/a]/[A^{l}A^{r}-B^{l}B^{r}+(A^{l}+A^{r})r_i+r_i^2]$$

$$N(r_i)=[D^{l}M(r_i)-r_i-1]/D^{r}$$

where r_i $(i=1,2,3)$ are the roots (5) of the characteristic equation (4). Constants R^{l}, R^{r}, R^{t} and T^{l}, T^{r}, T^{t} appearing in the solution (6) can be expressed as follows:

$$\begin{aligned}R^{l}&=2(\nu\xi-\delta\lambda)+ME/[H(1-r)]\\R^{r}&=2(\eta\xi-\mu\lambda)+NE/[H(1-r_1)]\\R^{t}&=2\xi+E/[H(1-r_1)]\end{aligned}\tag{10}$$

$$\begin{aligned}T^{l}&=2(\omega\nu-\varepsilon\delta)-MT/(Hr_1)\\T^{r}&=2(\omega\eta-\varepsilon\mu)-NT/(Hr_1)\\T^{t}&=2\omega-T/(Hr_1)\end{aligned}\tag{11}$$

where

$$E=(\delta-\mu)\cos\theta-(\delta\eta-\mu\nu)\sin\theta$$

$$T=\mu B-\delta A+(\delta\eta-\mu\nu)D, \qquad H=\mu M-\delta N-\mu\nu+\delta\eta$$

and ξ, λ, ω and ε can be determined from the equalities:

$$\xi+\mathrm{i}\lambda=(P+\mathrm{i}Q)/2[H\beta+\mathrm{i}H(1-\alpha)]$$

$$\omega+\mathrm{i}\varepsilon=(F+\mathrm{i}G)/2(H\beta-\mathrm{i}H\alpha)$$

where

$$P=\cos\theta(N-M+\nu-\eta)-\sin\theta(N\nu-M\eta)$$

$$Q=\cos\theta(\mu-\delta)-\sin\theta(M\mu-N\delta)$$

$$F=B(\eta-N)+A(M-\nu)+D(N\nu-M\eta)$$

$$G=A\delta-B\mu+D(M\mu-N\delta)$$

Expressions (10) have a different form when $r_1=1$:

$$R^{l}=2(\nu\xi-\delta\lambda)+ME\ln z/H$$

$$R^{r}=2(\eta\xi-\mu\lambda)+NE\ln z/H, \qquad R^{t}=2\xi+E\ln z/H$$

and (11) when $r_1=0$:

$$T^{l}=2(\omega\nu-\varepsilon\delta)-MT\ln z/H$$

$$T^{r}=2(\omega\eta-\varepsilon\mu)-NT\ln z/H, \qquad T^{t}=2\omega-T\ln z/H$$

REFERENCES

1. J. Heyman, *Coulomb's Memoir on Statics*, Cambridge University Press, 1972.
2. V. V. Sokolovski, *Statics of Granular Media*, Pergamon Press, Oxford 1965.
3. H. A. Janssen, 'Versuche uber Getreidedruck in Silozellen', *Zeitschrift Verein Deutscher Ingenieure*, **39**, 1045–1049 (1895).

4. H. Lippmann, 'Elementary methods for the analysis of certain forging processes', *Int. J. Mech. Sci.* **1**, 109–120 (1960).
5. D. M. Walker, 'An approximate theory for pressures and arching in hoppers', *Chem. Eng. Sci.* **21**, 975–997 (1966).
6. A. Drescher and J. Vardoulakis, 'Geometric softening in the triaxial test on granular material', to be published in *Geotechnique.*
7. A. W. Hancock and R. M. Nedderman, 'Prediction of stresses on vertical bunker walls', *Trans. Instn. Chem. Engrs,* **52**, 170–179 (1974).
8. N. R. Morgenstern and V. E. Price, 'The analysis of the stability of general slip surface', *Geotechnique,* **15**, 79–93 (1965).
9. R. F. Craig, *Fifth European Conference on Soil Mechanics and Foundation Engineering,* Madrid 1972, Session II: Structures Subjected to Lateral Forces, Discussion, pp. 155–157.
10. R. L. Michalowski, 'Approximate theory of loads in plane asymmetrical converging hoppers', submitted for publication in *Powder Technology.*