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AN ESTIMATE OF THE INFLUENCE OF SOIL WEIGHT ON BEARING CAPACITY USING LIMIT ANALYSIS

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ABSTRACT

An estimate of bearing capacity coefficient N_γ for a strip footing is made using the kinematical approach of limit analysis. This approach leads to an upper bound on the true limit load when the calculations of the three terms in the bearing capacity formula are consistent with one collapse mechanism. However, as accepted in other proposals, the estimate of the influence of the soil weight is calculated here separately from the terms dependent on the cohesion and the footing depth (overburden). It is conservative to take the minimum of each term in the bearing capacity formula, rather than minimum of the sum of the three terms. Coefficient N_γ from such calculations becomes increasingly conservative (underestimated) with the increase in the soil cohesion and the footing depth. In view of using the upper-bound approach, such "all-minimum" procedure is prudent and appropriate for design. Results are given for both rough and smooth footings. A substantial influence of the dilatancy angle on N_γ was found, and this is a reason for concern, since the non-associativity of plastic deformation of soils is usually not accounted for in design. Although calculations are numerical in nature, with optimization of the failure mechanism to obtain the best estimate, closed-form approximations are suggested for practical purposes.

Key words: bearing capacity, footings, limit analysis, plasticity analysis, self-weight, shallow foundations (**IGC**: E/3)

INTRODUCTION

The bearing capacity of shallow footings is a subject with a very long reference list. The basic structure of formulae used for calculations of bearing capacity today, however, is no different from that popularized by Terzaghi (1943). The first important contributions are due to Prandtl (1920) and Reissner (1924), who considered a punch over a weightless semi-infinite space, and Sokolovskii (1965), in regard to a ponderable soil, all under plane strain conditions. An axisymmetrical (circular) footing was considered by Cox et al. (1961). The solutions by Cox et al. and Sokolovskii can be regarded as rigorous within the framework of theory of plasticity, though not complete in the sense of Bishop (1953).

Terzaghi (1943) used an approximate approach where only a global equilibrium of rigid blocks was required, and concluded that the bearing capacity can be represented as a sum of three terms due to cohesion, overburden, and the weight of the soil. It is the third term (containing coefficient N_γ) that the designers and researchers agree upon the least. In most cases the calculations to determine the influence of the soil weight (N_γ) are not consistent with the adoption of classical Prandtl-Reissner coefficients N_c and N_q. Most suggestions for calculating N_γ fall below the magnitude of N_γ resulting from the original Prandtl mechanism. The reason for concern, however, is in neglecting to account for the non-associative nature of the plastic deformation of soils.

Issues related to estimating (testing for) the internal friction angle appropriate for calculations of bearing capacity are not discussed in this paper.

The early contributions are briefly reviewed in the next section, followed by a proposal to apply limit analysis in estimating the contribution of the soil weight to the bearing capacity. Calculations are performed using two schemes: (1) a procedure where the three terms in the bearing capacity formula are consistent with one failure mechanism, and (2) an "all-minimum" scheme where each term is taken at its minimum, but they are no longer consistent with the same mechanism (conservative estimate). The results are compared to those from the literature. The possible effect of the flow rule (associative or otherwise) is also discussed.

COMPLETE SOLUTION TO BEARING PRESSURE OF A WEIGHTLESS HALF-SPACE

Prandtl (1920) considered a rigid-perfectly plastic half space loaded by a strip punch. The failure criterion of the

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material was described by the Mohr-Coulomb function

$$f(\sigma_x, \sigma_z, \tau_{xz}) = (\sigma_x + \sigma_z)\sin\varphi - \sqrt{(\sigma_x - \sigma_z)^2 + 4\tau_{xz}^2} + 2c\cos\varphi = 0 \qquad (1)$$

where φ is the internal friction angle, and c is the cohesion. The punch-soil interface can be frictional or smooth, and the material is weightless. Eq. (1), along with two differential equations of equilibrium (plane deformation) lead to a set of hyperbolic-type differential equations, often referred to as Kötter equations (Kötter 1903), and these can be solved using the method of characteristics (slip-line method). Prandtl's stress boundary condition was zero traction on the surface of the half space, except for the strip punch where the pressure was unknown. The symmetrical half of the classical Prandtl mechanism is shown on the left-hand side of Fig. 1(a). A closed-form solution to the failure pressure, p', under the strip was found by Prandtl to be

$$p' = c\left(\tan^2\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)e^{\pi\tan\varphi} - 1\right)\cot\varphi \qquad (2)$$

Subsequently, Reissner (1924) considered a similar problem with two differences: the material was regarded as purely frictional ($c=0$), and the surface of the half-space (except for the strip) was loaded by a uniformly distributed pressure q. The solution of the hyperbolic-type equations for the new boundary conditions and no cohesion led Reissner to the following limit pressure

$$p'' = q\tan^2\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)e^{\pi\tan\varphi} \qquad (3)$$

Solving the equations simultaneously for a frictional-cohesive material and boundary condition q, one obtains exactly the same slip-line field, and the result can be written as

$$p = cN_c + qN_q \qquad (4)$$

where

$$N_c = (N_q - 1)\cot\varphi, \quad N_q = \tan^2\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)e^{\pi\tan\varphi} \qquad (5)$$

The solution by the method of characteristics, however, is restricted to finding a stress state which is both in equilibrium and satisfies the condition in Eq. (1) only in the neighborhood of the footing (Fig. 1(a)). In order to demonstrate that this solution is complete (Bishop, 1953), one needs to prove that a statically admissible extension of the stress field into the entire half-space exists, and that a kinematically admissible mechanism can be found, consistent with the stress solution. This was done by Shield (1953), who proved that the formula in Eq. (4) with coefficients in (5) is, indeed, the exact solution to the average limit pressure distributed on a strip over a weightless Mohr-Coulomb half-space.

This solution, with bearing capacity factors in Eq. (5), is adopted today in most bearing capacity formulae.

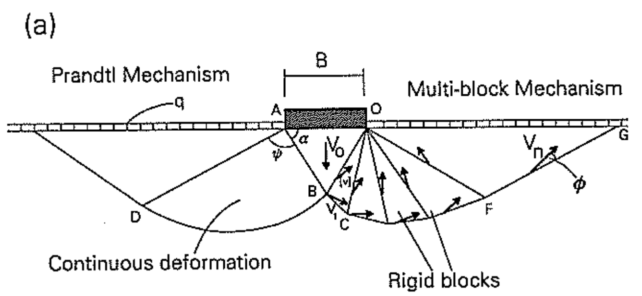
Hill (1950) proposed a different mechanism for a punch-indentation problem, Fig. 2. If a smooth foundation and a weightless soil is considered, the coefficients N_c and N_q based on the Hill mechanism are identical to those in Eq. (5). The extent of this mechanism, however, is considerably smaller, which substantially affects the influence of the self-weight for ponderable soils.

Sokolovskii (1965) used the method of characteristics for ponderable soils (it was used earlier by Lundgren and Mortensen, 1953, to estimate the influence of γ on bearing capacity). The equations of characteristics for the case where $\gamma > 0$ are identical to those for a weightless soil, but the relations along slip lines differ. Consequently, the solution cannot be obtained in a closed form, and the contributions related to cohesion, boundary condition q, and the soil weight γ cannot be separated.

The bearing capacity is commonly described as a sum of the two terms in Eq. (4) plus another term dependent on specific weight of the soil γ

$$p = cN_c + qN_q + \frac{1}{2}\gamma BN_\gamma \qquad (6)$$

where B is the footing width. The third term, however, is usually arrived at without assuring that the failure mechanism be consistent with that associated with the



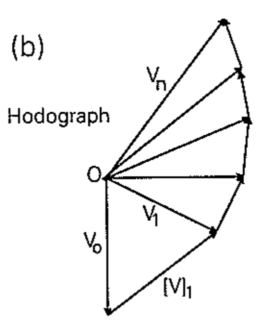


Fig. 1. **Prandtl mechanism; (a) original mechanism with continual deformation and the rigid-block collapse pattern, and (b) hodograph**

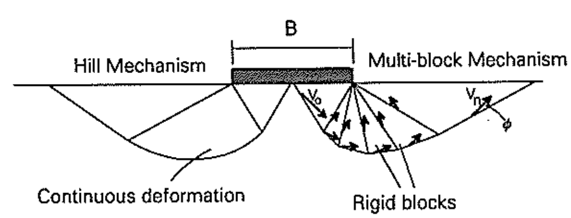


Fig. 2. **Hill mechanism of collapse under a smooth strip punch**

first two terms. Whereas such considerations may be useful for design purposes, they are inherently approximate. To make a reasonable and tractable judgement on the magnitude of the third term in Eq. (6), we employ the kinematical theorem of limit analysis. The advantage of this method is in a clear distribution of the total bearing capacity into cohesion, overburden (depth), and soil weight components.

LIMIT ANALYSIS OF SHALLOW FOOTINGS

The approach used here is based on the kinematical theorem of limit analysis (Drucker et al. 1952). We assume, for now, that the soil deformation conforms to the normality rule

$$\dot{\varepsilon}_{ij}=\dot{\lambda}\frac{\partial f(\sigma_{ij})}{\partial\sigma_{ij}},\quad \dot{\lambda}\geq 0,\quad i,j=1,2,3 \tag{7}$$

where $\dot{\varepsilon}_{ij}$ is the strain rate tensor, σ_{ij} is the stress tensor, $\dot{\lambda}$ is a non-negative multiplier, and $f(\sigma_{ij})$ is the failure criterion. The kinematical (or upper-bound) theorem states that *the rate of work done by the external forces (surface tractions and material weight) is less than or equal to the rate of energy dissipation in any kinematically admissible velocity field*. This can be written as

$$\int_V \sigma_{ij}^*\dot{\varepsilon}_{ij}^* dV \geq \int_S T_i v_i^* dS+\int_V X_i v_i^* dV \quad i,j=1,2,3 \tag{8}$$

where $\dot{\varepsilon}_{ij}^*$ is the strain rate in the kinematically admissible velocity field, $\dot{\varepsilon}_{ij}^*=-(v_{i,j}^*+v_{j,i}^*)/2$, σ_{ij}^* is the stress tensor associated with $\dot{\varepsilon}_{ij}^*$, X_i is the vector of distributed forces (weight), and S and V are the loaded boundary (surface) and the volume, respectively. T_i is a stress vector on boundary S. Thus, equating the work dissipation rate to the rate of work of the external forces in any admissible failure mechanism will lead to a limit load which is not lower than the true limit load. A kinematically admissible mechanism is one where the deformation conforms to the flow rule in Eq. (7) and satisfies the kinematical boundary conditions.

The term on the left-hand side of Eq. (8) represents the energy dissipation rate, which, for the case of the Mohr-Coulomb material with a failure stress state satisfying Eq. (1), is always a homogeneous function of order one of cohesion. The first term on the right-hand side can be split into two terms

$$\int_S T_i v_i^* dS=\int_{S_v} p_i v_i dS_v+\int_{S_q} q_i v_i^* dS_q \quad i,j=1,2,3 \tag{9}$$

where S_v is the boundary where the velocity is prescribed (v_i) and the load (p_i)is unknown, and S_q is the boundary where the traction is prescribed (q_i). For Eq. (8) to yield an upper bound, the unknown load needs to be active ($\int p_i v_i dS_v>0$ on boundary S_v), and such calculation is possible when v_i is constant on S_v.

Consequently, the ultimate bearing pressure calculated from (8) can be written in the form as in Eq. (6), with functions N_c, N_q, and N_γ dependent on the geometry of the failure mechanism and on the material properties of the soil. Since theorem (8) leads to the upper bound for the average bearing pressure, a reasonable estimate of the bearing capacity can be made if the geometry of the failure mechanism is optimized to yield the minimum of the bearing pressure.

A reasonable collapse mechanism is one with rigid blocks, as on the right-hand side of Fig. 1(a), where all deformation occurs along velocity discontinuities separating the blocks. Kinematical admissibility requires that the velocity "jump" vectors be inclined at φ to the discontinuity surfaces. Velocities of the blocks and velocity discontinuity vectors can be calculated from the geometrical relations in the hodograph (Fig. 1(b)). V_0 is the velocity boundary condition (velocity of the footing), and both the footing and the triangular block beneath the footing (OAB) move as one rigid block with velocity V_0. The subsequent block (OBC) moves with velocity V_1 inclined at angle φ to discontinuity BC. For the mechanism to be kinematically admissible surface OB must now be a velocity discontinuity with velocity jump vector $[V]_1$, (inclined at φ to OB) such that the sum of vectors V_0 and $[V]_1$ equals V_1. This is depicted in the hodograph (Fig. 1(b)). The velocities of the subsequent blocks in the mechanism and the velocity discontinuity vectors between them can be determined in a similar manner, as shown in the hodograph.

The rigid-block mechanism differs from the classical Prandtl mechanism and other mechanisms presented earlier (e.g., Chen 1975) in that it does not contain any regions with a continuous deformation. A region with continuous deformation for the Prandtl mechanism is contained in area ABD of Fig. 1(a).

It is interesting to notice that the geometry of the multi-rigid-block mechanism is less restrictive than that in the original Prandtl mechanism, or other mechanisms which include continuous deformation fields. This is because the continuing deformation regions are limited by well-defined log-spiral surfaces (such as BD on the left-hand side of Fig. 1(a)), while the multi-block mechanism with a large number of blocks allows for an arbitrary shape of that surface. Therefore, the estimate of coefficient N_γ is likely to be better for the rigid-block mechanism. It can also be noticed that when the slip-line method is used to solve for bearing capacity of the soil with self-weight (ponderable soil), the slip-line separating the deformation region from the soil at rest is not strictly a log-spiral.

If the bearing capacity of a weightless soil is calculated using a large number of blocks, the result becomes identical to that in Eqs. (4) and (5), and the optimized mechanism follows the classical Prandtl mechanism ($\alpha=\pi/4+\varphi/2$, $\psi=\pi/2$) with a continual deformation field in the region enclosed by the log spiral, Fig. 1(a) (when $\varphi=32°$ and 10 blocks are used, the difference between the exact solution and the numerical upper bound calculated is less than 0.7%, for 20 blocks-less than 0.2%, and for 50 blocks-less than 0.02%).

CALCULATIONS OF N_γ

Whereas the best upper bound to the bearing capacity for $\gamma=0$ yields the exact solution (Eqs. (4) and (5)), for $\gamma>0$ the rigid-block mechanism associated with the best upper bound no longer follows the exact shape of the Prandtl mechanism. This implies that if the bearing capacity for $\gamma>0$ is calculated according to (8), the first two terms in Eq. (6) will no longer contain bearing capacity coefficients equal to those in Eq. (5). The shape of the failure mechanism is still dependent on the internal friction angle of the soil, but, in addition, it also depends on other material parameters and the footing size (more concisely, it depends on dimensionless coefficients $c/\gamma B$ and $q/\gamma B$). Similarly, the slip-line field for the problem (Sokolovskii, 1965) also becomes dependent on coefficients $c/\gamma B$ and $q/\gamma B$.

First, calculations are performed in a consistent manner where all coefficients in Eq. (6) are calculated based on one failure mechanism. The geometry of this mechanism is optimized to assure that the minimum of the bearing pressure is obtained (the best upper bound). For this reason all independent angles which describe the shape of the blocks (in the multi-block mechanism) are considered variable in an optimization routine where the objective function is the bearing capacity of the footing. The number of variables increases, of course, with an increase in the number of blocks. Kinematical admissibility of the mechanism introduces a number of constraints on the variables. A simple gradiental optimization method was found to be quite effective, where all independent parameters (angles) were varied by a prescribed angle increment (step) in a sequential manner within one loop of the procedure, and the loop was repeated until the minimum was found. Once the minimum was found, the angle increment was dropped down and the procedure repeated. The smallest step used in calculations was 0.05°.

After the bearing capacity was determined (for given parameters $c/\gamma B$, $q/\gamma B$), it was decomposed into three components dependent on cohesion (c), surcharge load (q) and the self-weight (γ). This is possible because Eq. (8) used in calculations leads to bearing capacity represented as the sum of three terms resulting from the energy dissipation rate (term dependent on c), work-rate of the surcharge (term dependent on q), and the rate of work of the weight of the soil (γ). Coefficients N_c, N_q and N_γ were then extracted from the respective terms.

The bearing capacity coefficients are, in general, dependent on $c/\gamma B$ and $q/\gamma B$. Calculations were performed for $q/\gamma B$ equal to 0 and 2, for varied $c/\gamma B$. The results are shown in Tables 1 and 2. The discussion of results follows in the next section.

Introducing N_c and N_q as functions of φ, $c/\gamma B$, and $q/\gamma B$ initiates complexities undesired in the design process. It is proposed then that, in accordance with other existing proposals, we adopt the first two coefficients as in Eq. (5), and find the coefficient N_γ from the kinematical approach of limit analysis of the strip punch-indentation problem where $c=0$ and $q=0$. Under such circumstances N_γ becomes a function of φ only, but the geometry of the mechanism for which N_γ reaches its minimum is no longer consistent with that associated with the adopted coefficients N_c and N_q (Eq. (5)).

Table 1. Bearing capacity coefficients, $\varphi=15°$ (Best upper bound)

$\frac{q}{\gamma B}$	$\frac{c}{\gamma B}$	N_c	N_q	N_γ
0	0	—	—	1.938
	1	11.062	—	2.885
	2	11.006	—	3.036
	5	10.982	—	3.175
2	0	—	3.998	2.699
	1	11.021	3.953	2.987
	2	10.995	3.946	3.084
	5	10.981	3.942	3.187

Table 2. Bearing capacity coefficients, $\varphi=35°$ (Best upper bound)

$\frac{q}{\gamma B}$	$\frac{c}{\gamma B}$	N_c	N_q	N_γ
0	0	—	—	48.681
	1	48.256	—	54.118
	2	47.099	—	57.380
	5	46.420	—	61.482
2	0	—	34.338	55.647
	1	46.897	33.838	58.271
	2	46.629	33.649	59.799
	5	46.313	33.429	62.646

From a theoretical standpoint, such consideration is inconsistent. This inconsistency is common to all known techniques for bearing capacity calculations based on Eq. (6). However, this is an accepted design procedure. From a mathematical standpoint this inconsistency comes from the fact that the minimum of each term in Eq. (6) is sought, rather than, more correctly, the minimum of their sum. It can be argued that, in view of using the upper-bound approach, such "all-minimum" procedure is prudent and appropriate for design. The results of calculations based on the symmetrical multi-rigid-block mechanism are shown in column 2 of Table 3 for varied φ.

Calculations also have been performed using the Hill-type mechanism (Fig. 2). Although the calculations based on the Prandtl-type mechanism yield the upper bound to the bearing capacity for both rough and smooth footings, the Hill-type mechanism gives a better estimate for smooth footings over ponderable soils. A kinematically admissible Hill-type mechanism includes sliding at the footing-soil interface as the displacement of the soil immediately beneath the footing has an outward horizontal component. Therefore, for a rough footing, an additional energy dissipation rate needs to be accounted for at the footing-soil interface when the Hill-type mechanism is used. Here, this mechanism is used to estimate the bearing capacity of smooth footings (zero dissipation at footing-soil interface). The Hill-type rigid-

Table 3. Coefficient N_γ from limit analysis

φ (°) (1)	$N_{\gamma(\text{rough})}$* (2)	$N_{\gamma(\text{smooth})}$* (3)	$N_{\gamma(\text{Prandtl})}$** (4)	$N_{\gamma(\text{cont1})}$*** (5)	$N_{\gamma(\text{cont2})}$**** (6)
0	0	0	0	0	0
5	0.181	0.127	0.495	0.261	0.382
10	0.706	0.423	1.446	0.921	1.164
15	1.938	1.050	3.282	2.324	2.725
20	4.468	2.332	6.904	5.236	5.871
25	9.765	5.020	14.327	11.389	12.409
30	21.394	10.918	30.381	24.983	26.702
35	48.681	24.749	67.739	57.112	60.236
40	118.827	60.215	163.500	140.479	146.765
45	322.835	164.308	442.750	385.963	400.476
50	1025.98	519.089	1412.69	1245.42	1285.73

* Based on multi-block symmetrical mechanisms.
** Original Prandtl mechanism (Prandtl 1920).
*** Continuous one-sided Prandtl-type mechanism.
**** Continuous symmetrical Prandtl-type mechanism.

block mechanism (Fig. 2) was optimized to yield the best (minimum) upper bound to the bearing pressure ($c=0$, $q=0$), and the results are included in column 3 of Table 3.

RESULTS

Coefficients calculated using a consistent kinematical limit analysis approach are presented in Tables 1 and 2, for selected parameters $c/\gamma B$ and $q/\gamma B$. Though for practical purposes N_γ needs to be estimated with only one digit past the decimal point, for reasons of comparison to other methods all results in the tables are truncated to three digits. Bearing capacity calculations based on these coefficients provide the best upper bound to the true limit load. The rigid-block mechanism in Fig. 1(a) was used in calculations with 50 rigid blocks. This mechanism then was optimized so that the minimum bearing capacity was reached. For the case where $\gamma=0$ this mechanism yields the exact solution. For $\gamma>0$ no exact solution is known, but the "flexibility" of the mechanism is believed to assure that the solution is very close to the true limit bearing pressure. This is supported by similar calculations where very close upper and lower bounds were found, with the upper bound calculated using similar mechanisms (Michalowski and Shi, 1993).

The variability of coefficients N_c and N_q is rather small with the change in $c/\gamma B$ and $q/\gamma B$, and adoption of the least values in Eq. (5), as generally accepted, is justified. Coefficient N_γ, however, increases significantly with an increase in both $c/\gamma B$ and $q/\gamma B$. For design purposes it is convenient to represent this coefficient as a function of φ only. A prudent choice is, of course, its minimum value, which is obtained when $c=0$ and $q=0$. Coefficient N_γ so calculated becomes increasingly conservative (underestimated) with the increase in $q/\gamma B$ and $c/\gamma B$. For a wide range of φ this coefficient is given in Table 3 (column 2) along with that from calculations for the smooth footing based on the Hill-type mechanism (column 3). The best upper bound to N_γ for a smooth footing is roughly half of that for a rough one. For realistic field conditions, a smooth footing gives an overconservative estimate of N_γ.

The proposed coefficient N_γ for a strip footing can be represented by the function

$$N_\gamma = e^{a+b\tan\varphi}\tan\varphi \tag{10}$$

The least-squares routine was used to arrive at coefficients $a=0.6605$ and $b=5.1163$ (rough footing), and for practical purposes they can be truncated, and the function N_γ can be given as

$$N_\gamma = e^{0.66+5.11\tan\varphi}\tan\varphi \tag{11}$$

The expression in Eq. (11) predicts N_γ within 1% of that in column 2 of Table 3 for range of φ from 25° to 50°. For very low φ the (relative) accuracy is not as good, but it is insignificant since for such soils the term containing N_γ is negligible altogether.

For comparison, results of calculations based on the original Prandtl, and continuous Prandtl-type mechanisms, are given in the last three columns of Table 3. The results in column 4 are for the original Prandtl mechanism (Prandtl, 1920), and these results are the only consistent ones with coefficients N_c and N_q in Eq. (5).

The results in column 5 are based on the mechanism with a continuous deformation region, but with one-sided collapse process, as presented in Fig. 3. Angles α and ψ in this mechanism were varied in the analysis to obtain minimum value of coefficient N_γ. The same optimization procedure was used as described earlier. Finally, the results from analysis based on the continuous symmetrical Prandtl-type mechanism are given in the last column of Table 3. Again, angles α and ψ were variable in optimization procedure. It is interesting to notice that when $\gamma=0$, coefficients N_c and N_q are identical whether symmetrical or non-symmetrical mechanism is considered. Coefficients N_γ are also identical but only if the original

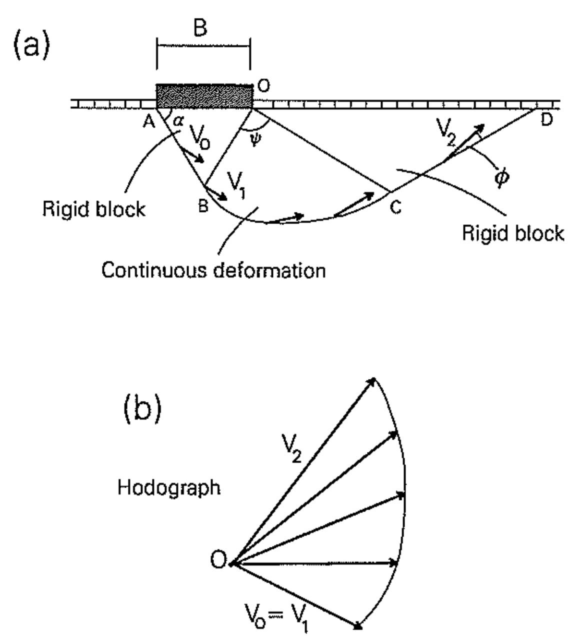


Fig. 3. One-sided collapse pattern (based on Prandtl mechanism)

Prandtl mechanism is used where $\alpha=\pi/4+\varphi/2$ and $\psi=\pi/2$ (column 4, Table 3). When the symmetrical and nonsymmetrical mechanisms are subjected to optimization procedure, however, the nonsymmetrical one yields a lower value of N_γ.

Coefficients in the last column of Table 3 are the same as those given by Chen (1975) in Table 4.5 ("Prandtl 1"). At first it is surprising that the multi-block mechanism without a continuous region yields the best (the least) upper bound to coefficient N_γ. This should not be unexpected, however, since the multi-block mechanism is the only one that does not introduce any restrictions on the shape of the velocity discontinuity separating the deforming soil from the stationary one. All other mechanisms (considered here and in Chen, 1975) require that at least a segment of that discontinuity conforms to the log-spiral shape.

The calculated coefficient N_γ is compared to those suggested by Meyerhof (1963) [$N_\gamma=(N_q-1)\tan(1.4\varphi)$], Hansen (1970) [$N_\gamma=1.5(N_q-1)\tan\varphi$], and Vesic (1973) [$N_\gamma=2(N_q+1)\tan\varphi$]. The plot of N_γ as a function of φ is shown in Fig. 4. The calculations consistent with the classical Prandtl mechanism yield the highest value of coefficient N_γ and the bearing capacity calculated using this coefficient is always a strict upper bound to the true limit load. The function proposed by Vesic (1973) seems to be closest to the "all-minimum" calculations from the kinematical approach of limit analysis. Vesic proposed this function as a conservative approximation of numerical results given earlier by Caquot and Kerisel (1953). This function slightly overestimates N_γ proposed here for $\varphi<35°$, and it underestimates it for larger angles φ. Proposals both by Meyerhof (1963) and Hansen (1970), which coincide for $\varphi<30°$, both significantly underestimate the solution proposed here. Chen (1975) approximated N_γ obtained from the Prandtl continuous mechanism (but with optimized geometry) using a closed-form function [$N_\gamma=2(N_q+1)\tan\varphi\tan(\pi/4+\varphi/5)$], which runs somewhere between the upper bound solution based on the classical Prandtl mechanism and the one in Eq. (11) (Chen's function is not shown in Fig. 4).

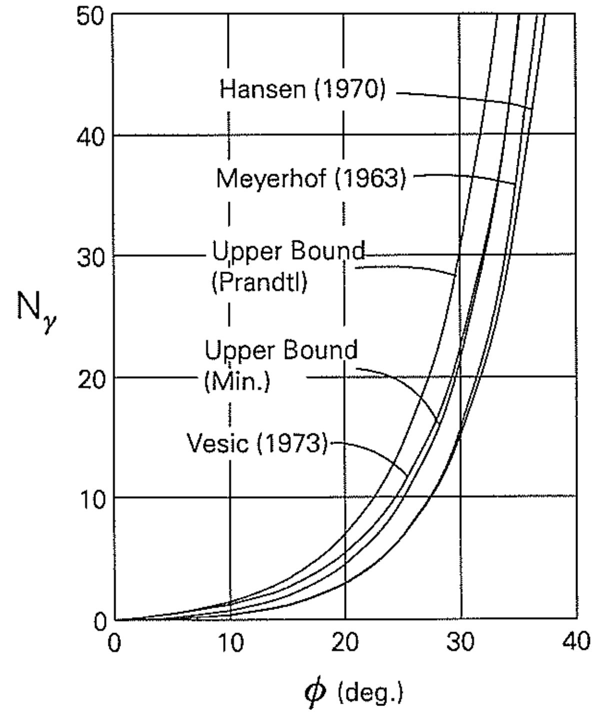


Fig. 4. Coefficient N_γ as function of φ (rough footing)

Table 4. N_γ for rough footing (nonassociative flow rule)

φ (°)	N_γ $\nu=0.5\varphi$	N_γ $\nu=0.25°$	N_γ $\nu=0$
0	0	0	0
5	0.180	0.180	0.180
10	0.700	0.693	0.683
15	1.894	1.842	1.773
20	4.261	4.028	3.734
25	8.931	8.064	7.058
30	18.309	15.442	12.457
35	37.606	28.798	20.849
40	78.649	52.772	33.306
45	169.436	95.428	50.924
50	379.868	170.386	74.614

Table 5. N_γ for smooth footing (nonassociative flow rule)

φ (°)	N_γ $\nu=0.5\varphi$	N_γ $\nu=0.25°$	N_γ $\nu=0$
0	0	0	0
5	0.127	0.127	0.126
10	0.419	0.415	0.413
15	1.027	1.001	0.966
20	2.225	2.107	1.962
25	4.597	4.156	3.646
30	9.353	7.899	6.385
35	19.134	14.667	10.642
40	39.937	26.834	16.958
45	86.045	48.385	25.886
50	192.660	86.400	37.876

The solution to N_γ for smooth footings obtained here (based on a Hill-type mechanism) can be approximated with the function

$$N_\gamma=e^{5.1\tan\varphi}\tan\varphi \tag{12}$$

INFLUENCE OF THE FLOW RULE ON BEARING CAPACITY COEFFICIENTS

The associative flow rule as applied to frictional soil is often debated. Experiments indicate a much smaller dilatancy than that predicted by the normality rule in Eq. (7). The use of the associative flow rule is standard in rigorous limit analysis, since only for the normality rule can the upper- and lower-bound theorems be proven true. It is possible, however, to include the non-associativity of the soil deformation using a technique recently devised by Drescher and Detournay (1993).

The discontinuities in a collapse mechanism are the characteristics of the hyperbolic-type equations describing the plastic velocity field. For the associative flow rule, these characteristics coincide with the stress characteristics (often called the slip-lines). For a coaxial but non-associative material, velocity characteristics are different from the stress characteristics (coaxiality: principal directions of the stress tensor coincide with the principal directions of the strain-rate tensor). Assuming ν is the dilatancy angle, velocity characteristics in the non-associative field are inclined to stress characteristics at $(\varphi-\nu)/2$. A traction vector on stress characteristics can be decomposed into a shear stress vector (tangent to the characteristic) and the remaining part, which is inclined at angle φ to the normal to the slip line. From the geometrical relations in a limit Mohr circle diagram, one can show that the stress vector (traction) on the non-associative velocity characteristic (or velocity discontinuity) has a c^* component (tangent to the characteristic) whose magnitude is

$$c^*=c\frac{\cos\nu\cos\varphi}{1-\sin\nu\sin\varphi} \tag{13}$$

where c is the actual cohesion, and the remaining part of the traction is inclined to the normal to the discontinuity at φ^*

$$\tan\varphi^*=\frac{\cos\nu\sin\varphi}{1-\sin\nu\sin\varphi} \tag{14}$$

The limit load for a non-associative soil can be calculated then using the same technique as described in previous sections, but with the velocity vectors inclined to the discontinuities at φ^* and with the cohesion replaced by c^* in calculations of the energy dissipation rate (for a more elaborate explanation *see* Drescher and Detournay, 1993, or Michalowski and Shi, 1996).

Consequently, the bearing capacity still can be calculated from Eq. (6), with c replaced by c^* (Eq. (13)), and with coefficients N_c and N_q calculated from (5) with φ replaced by φ^* (Eq. (14)).

Coefficient N_γ was obtained for the non-associative flow rule again from the kinematical approach of limit analysis (multi-block mechanism). Results are given in Tables 4 and 5, for the rough and smooth footings, respectively, and for different dilation angles (given as a fraction of φ). The non-associativity has a negligible effect on N_γ for an internal friction angle below 25°, but the effect becomes very significant for larger φ. A graphical comparison of the results to those calculated using the normality rule ($\nu=\varphi$) is given in Fig. 5.

Bolton (1986) suggested a relation between the dilatancy angle ν and angle φ for granular soils. According to this relation, a likely value of ν can be estimated as $\nu=1.25\ (\varphi-\varphi_{\mathrm{crit}})$, where φ_{crit} is the internal friction angle at the critical state. The dotted line in Fig. 5 indicates N_γ where the dilatancy angle varies according to Bolton's suggestion ($\varphi_{\mathrm{crit}}=33°$ was taken in calculations).

Numerical solutions to coefficient N_γ for any dilation angle ν ($0\leq\nu\leq\varphi$) can be approximated for the rough footing as

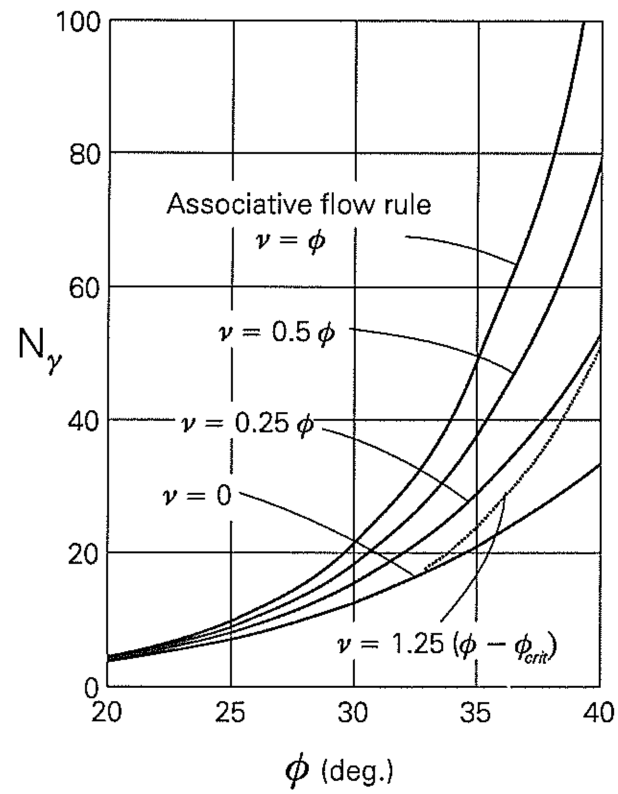


Fig. 5. The effect of dilation angle on coefficient N_γ (rough footing)

$$N_\gamma=e^{0.66+5.11\tan\varphi^*}\tan\varphi^* \tag{15}$$

and for smooth footings

$$N_\gamma=e^{5.1\tan\varphi^*}\tan\varphi^* \tag{16}$$

where $\tan\varphi^*$ is given in Eq. (14).

FINAL REMARKS

Bearing capacity factors can be calculated effectively using the kinematical approach of limit analysis. Consistent calculations show a dependence of all coefficients on dimensionless parameters $c/\gamma B$ and $q/\gamma B$ in addition to φ. The traditional techniques adopt coefficients N_c and N_q as functions of φ, after Prandtl (1920) and Reissner (1924), and coefficient N_γ is derived from independent, often semi-empirical, considerations. This results in a bearing capacity formula where the term dependent on the soil weight is not consistent with the two other terms. It can be shown that if all terms are consistent with one collapse mechanism, coefficient N_γ assumes higher values than those suggested by any widely used method.

Consistent calculations show that neglecting the dependence of N_c and N_q on $c/\gamma B$ and $q/\gamma B$ is certainly acceptable, as it leads to a rather small conservative approximation with respect to the strict least upper bound (but is the exact solution when $\gamma=0$). Whereas N_γ depends on φ and both $c/\gamma B$ and $q/\gamma B$, its minimum is reached when $c/\gamma B=0$ and $q/\gamma B=0$. A closed-form approximation of the minimum solution can be used for practical purposes

(Eq. (11)). N_γ so calculated is a reasonable estimate of the soil weight influence, but it becomes increasingly conservative with an increase in the soil cohesion $(c/\gamma B)$ and the footing depth (or surcharge load at the footing level, $q/\gamma B$). The conservatism in this approach stems from the fact that each term in Eq. (6) is taken at its minimum, rather than terms related to the minimum of their sum.

The estimate of coefficient N_γ arrived at in this paper appears to coincide with the suggestion of Vesic (1973), whereas recommendations by Meyerhof (1963) and Hansen (1970) underestimate the solution considerably. Given the approximate nature of estimated c and φ for soils, all these suggestions are acceptable in practice, though for large φ the last two may be overconservative.

Significant differences in estimations of bearing capacity factors originate from accounting for or not accounting for the non-associativity of the plastic soil deformation. These differences become very substantial for large φ. For a given φ, coefficient N_γ drops significantly with a decrease in the dilation angle. The influence of the non-associative flow rule for soils is typically not considered in design, and it is a reason for concern.

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