Approximate Theory of Loads in Plane Asymmetrical Converging Hoppers

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SUMMARY

The slice method approach to the quasi-static problem of the wall pressure in an asymmetrical converging plane hopper is presented. Stresses acting on a single slice are determined from three linear, ordinary differential equations of equilibrium. Statical indeterminancy is retained by postulating a linear distribution of normal stress acting on the slice, and introducing two coefficients, similar to that of Janssen. These coefficients relate normal stress on the slice to the stress normal to the wall, on the left and right walls. An analytical solution for wall pressure distribution is obtained. An example is presented to illustrate the method.

INTRODUCTION

This paper deals with the problem of wall pressure in a converging, asymmetrical plane hopper. A plane hopper is one in which plane deformation takes place and the stress state is not a function of the co-ordinate perpendicular to the plane of deformation. This paper considers the pressure caused by the weight of the material at rest and during flow. A set of basic equations describing the problem has the same form in both considered cases since inertial terms in the equilibrium equations are neglected. However, the method used can involve inertial forces as shown by Mroz and Szymanski [1].

Two different approaches to the problem of load in bins and hoppers containing granular material can be distinguished. The first, more rigorous, requires equilibrium to be satisfied at each point. This approach is usually based on the assumption that a limit state occurs at every point of the material. Mohr–Coulomb limit condition and local equilibrium equations lead then to the set of hyperbolic-type equations that can be solved using the method of characteristics. This method was used by Sokolovski [2], Jenike [3] and others [4 - 6]. The second approach, an approximate one, is based on the concept of an element of the body being finite in one direction. Such an element is often called a slice and the method is called the method of slices or differential slice technique. Instead of local equilibrium, global equilibrium is required. This method was first suggested by Janssen [7] in 1895 in order to predict wall pressure in bins. It has found, however, much broader application in engineering [8 - 10].

It has been usually assumed by the authors that the geometry of the slice and the stress distribution along the finite dimension of the slice is symmetrical, Fig. 1(a), and the

(a) (b)

Fig. 1. Symmetrical slice with symmetrically distributed stresses (a), and asymmetrical slice (b).

problem was formulated in terms of averaged stresses [7, 11 - 14]. This results in identically satisfying equations for global moment equilibrium and force balance in the direction parallel to the slice. Hence, only one equilibrium equation becomes non-trivial. The objective of the present paper is to generalize the slice method for the case of a non-symmetrical plane slice, so that it can be applied to asymmetrical hoppers. This asymmetry produces

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three non-trivial equations describing static equilibrium.

To obtain a statically determinable system, additional relations between some of the stresses have to be known. For this reason, the method is usually applied to the problems where limit state can be assumed at some points of a slice. Subsequently, the method can be called collocational [13]. A yield function together with boundary conditions (such as the magnitude of the mobilized friction or given principal stress directions) give the required relations. Deformation quantities must be considered if the material does not behave in a rigid–plastic way. The slice method is also efficient if the relations mentioned above are found from experiments. Janssen [7] introduced a ratio K relating horizontal pressure to the uniformly distributed vertical one, for the problem of vertical bins. It is believed that the pressure acting on the wall of a hopper depends on the inclination of the wall. Hence, we shall define two ratios relating the stress normal to the wall to the vertical stress at the left and right walls:

$$K^{l} = \sigma_n^{l}/\sigma_z^{l} \qquad K^{r} = \sigma_n^{r}/\sigma_z^{r} \tag{1}$$

To determine the global vertical force acting on the horizontal plane of the slice, the distribution of the normal stress along the slice (Fig. 1(b)) must be known. Any form of the distribution can be assumed. In the present paper, this distribution is postulated to be linear.

BASIC EQUATIONS

Let us consider global equilibrium of a horizontal slice with sides inclined at the angles α^{l} and α^{r} to a vertical z-axis, Fig. 2(a). For the problem of a hopper, the sides of the slice are considered to be parallel to the walls of the hopper, Fig. 2(b). Stress vectors acting on the left and right walls are inclined to the outward unit normals at the angles ϕ_w^{l} and ϕ_w^{r} respectively. For the problem of flowing material, we can postulate these angles to be equal to the angle of the wall friction. In the case of material at rest, the angle of wall friction may not be fully mobilized and the angles ϕ_w^{l} and ϕ_w^{r} can vary anywhere between 0 and the value of the wall friction angle [11].

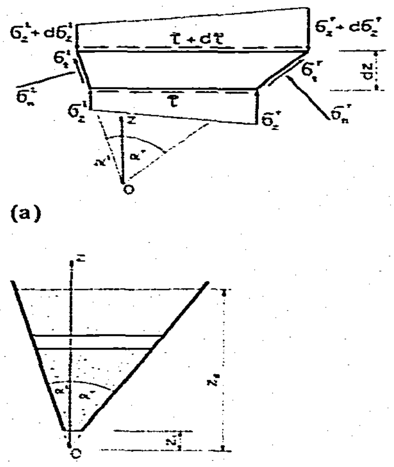


Fig. 2. Asymmetrical slice with linear distribution of normal vertical stress (a), location of a slice in a hopper (b).

An unknown distribution of the vertical stress acting on the horizontal plane (Fig. 1(b)) is replaced by an unknown linear one (Fig. 2(a)), with σ_z^{l} and σ_z^{r} denoting the values at the left and right wall. The type of distribution of the shear stress on the horizontal plane does not influence the global equilibrium equations. Hence, we introduce the mean shear stress $\bar{\tau}$ defined as the stress averaged over the width of the slice.

Vertical force balance and moment equilibrium with respect to the center point of the slice lead to the following equations (after using relations (1) and neglecting second-order terms):

$$\begin{aligned} &\frac{d\sigma_z^{l}}{dz} + \frac{d\sigma_z^{r}}{dz} + \frac{\sigma_z^{l}}{z}(A^{l} + B^{l}) + \\ &\quad + \frac{\sigma_z^{r}}{z}(A^{r} + B^{r}) = -2\gamma \\ &\frac{d\sigma_z^{l}}{dz} - \frac{d\sigma_z^{r}}{dz} - \frac{\sigma_z^{l}}{z}(B^{l} - A^{l}) + \\ &\quad + \frac{\sigma_z^{r}}{z}(B^{r} - A^{r}) - 12\bar{\tau}/a = 0 \end{aligned} \tag{2}$$

where γ is the specific weight of a material and

$$A^{l} = [3 \tan \alpha^{l} - 4K^{l}(\tan \phi_w^{l} + \tan \alpha^{l})]/a$$

$$A^{r} = [3 \tan \alpha^{r} - 4K^{r}(\tan \phi_w^{r} + \tan \alpha^{r})]/a$$

$$B^l = [\tan\alpha^r - 2\tan\alpha^l + 2K^l(\tan\phi_w^l + \tan\alpha^l)]/a$$

$$B^r = [\tan\alpha^l - 2\tan\alpha^r + 2K^r(\tan\phi_w^r + \tan\alpha^r)]/a$$

$$a = \tan\alpha^l + \tan\alpha^r$$

Using the horizontal force equilibrium equation and transforming eqns. (2), the final set of equations for the considered problem can be written in the form of the following three linear, ordinary differential, non-homogeneous equations:

$$z\frac{d\sigma_z^l}{dz} + \sigma_z^l A^l + \sigma_z^r B^r - 6\bar{\tau}/a = -\gamma z$$

$$z\frac{d\sigma_z^r}{dz} + \sigma_z^l B^l + \sigma_z^r A^r + 6\bar{\tau}/a = -\gamma z \qquad (3)$$

$$z\frac{d\bar{\tau}}{dz} - \sigma_z^l D^l + \sigma_z^r D^r + \bar{\tau} = 0$$

where

$$D^l = K^l(\tan\alpha^l \tan\phi_w^l - 1)/a$$

$$D^r = K^r(\tan\alpha^r \tan\phi_w^r - 1)/a$$

Stresses σ_z^l, σ_z^r and $\bar{\tau}$ are the unknown functions of z, whereas K^l, K^r, ϕ_w^l and ϕ_w^r are considered to be constant.

The characteristic equation (see the Appendix) of the set of equations (3), for positive and reasonable values of K^l, K^r, ϕ_w^l and ϕ_w^r, has one real and two complex roots:

$$r_1 \qquad r_2 = \alpha + i\beta \qquad r_3 = \alpha - i\beta \qquad (4)$$

The final solution of the set of equations (3) is real if two of the integration constants are assumed to be conjugate complex numbers. The solution can be written in the following explicit form:

$$\sigma_z^l = \gamma R^l z + C_1 M z r_1 + 2(\nu C_2 - \delta C_3) z^\alpha \cos(\beta \ln z) - 2(\delta C_2 + \nu C_3) z^\alpha \sin(\beta \ln z)$$

$$\sigma_z^r = \gamma R^r z + C_1 N z r_1 + 2(\eta C_2 - \mu C_3) z^\alpha \cos(\beta \ln z) - 2(\mu C_2 + \eta C_3) z^\alpha \sin(\beta \ln z) \qquad (5)$$

$$\bar{\tau} = \gamma R^t z + C_1 z r_1 + 2C_2 z^\alpha \cos(\beta \ln z) - 2C_3 z^\alpha \sin(\beta \ln z)$$

where C_1, C_2 and C_3 are constants to be determined from the boundary conditions at $z = z_0$. r_1, α and β are the functions of coefficients K^l, K^r, geometry of a hopper and mobilized friction, and follow from the solution (4) of the characteristic equation. Quantities M, N, ν, δ, η and μ are defined as follows:

$$M(r_1) = M \qquad M(r_2) = \nu + i\delta \qquad M(r_3) = \nu - i\delta$$
$$N(r_1) = N \qquad N(r_2) = \eta + i\mu \qquad N(r_3) = \eta - i\mu \qquad (6)$$

where $M(r)$ and $N(r)$ are functions of the roots (4) of the characteristic equation and are given in the Appendix together with the expressions for R^l, R^r, R^t.

For the boundary values σ_0^l, σ_0^r, τ_0 at $z = z_0$, the following expressions for constants C_1, C_2 and C_3 can be written

$$C_1 = [F_1\mu - F_2\delta + F_3(\delta\eta - \nu\mu)]/Hz_0^{r_1}$$

$$C_2 = \{(MF_3 - F_1)[\mu\cos(\beta \ln z_0) + \eta\sin(\beta \ln z_0)] + \nu\sin(\beta \ln z_0) + (F_2 - NF_3)[\delta\cos(\beta \ln z_0) + \sin(\beta \ln z_0)(NF_1 - MF_2)\}/2Hz_0^\alpha$$

$$C_3 = \{(MF_3 - F_1)[\eta\cos(\beta \ln z_0) - \mu\sin(\beta \ln z_0)] + (F_2 - NF_3)[\nu\cos(\beta \ln z_0) - \delta\sin(\beta \ln z_0)] + \cos(\beta \ln z_0)(NF_1 - MF_2)\}/2Hz_0^\alpha$$

where

$$F_1 = \sigma_0^l - \gamma R^l z_0$$

$$F_2 = \sigma_0^r - \gamma R^r z_0$$

$$F_3 = \bar{\tau}_0 - \gamma R^t z_0$$

and H is given in the Appendix. Normal and tangential stresses on the wall can be computed, after using eqn. (5), directly from the definitions of K^l and K^r (eqn. (1)) and from known mobilized angle of friction on the wall.

Equations (5) do not hold for the symmetric case, *i.e.* when $\alpha^l = \alpha^r$ and $K^l = K^r$. To solve this case, new functions σ and s are introduced, defined by

$$\sigma = \sigma_z^l + \sigma_z^r \qquad s = \sigma_z^l - \sigma_z^r \qquad (7)$$

From eqn. (2) and the third equation of (3) it follows that

$$z\frac{d\sigma}{dz} + (A + B)\sigma = -2\gamma z$$

$$z\frac{ds}{dz} - (B - A)s - 12\bar{\tau}/a = 0 \qquad (8)$$

$$z\frac{d\bar{\tau}}{dz} - Ds + \bar{\tau} = 0$$

where $A = A^l = A^r$, $B = B^l = B^r$ and $D = D^l = D^r$. It is clear that the first equation of (8) can be solved independently. The solution can be written as follows:

$$\sigma = -2\gamma z/(A + B + 1) + {} + [\sigma_0 + 2\gamma z_0/(A + B + 1)](z_0/z)^{A+B} \qquad (9)$$

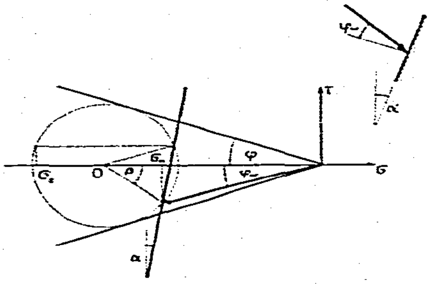
where σ_0 is the value of σ on the boundary $z = z_0$. For condition $\sigma_0 \geqslant 0$, $s_0 = \bar{\tau}_0 = 0$, the solution of the remaining equations of (8) becomes: $s = 0$, $\bar{\tau} = 0$. From eqn. (7) we obtain $\sigma_z^l = \sigma_z^r = \sigma/2$. Hence, the solution is equivalent to the one obtained by Dabrowski [14] and later by Walker [12].

EXAMPLE

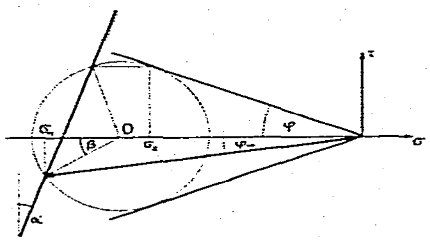
An example is given in this section in order to illustrate the method.

Wall pressure in vertical bins has been a concern of many authors, although the nature of the coefficient K relating the vertical stress to the horizontal one is not quite clear. In more recent works [11, 15], this coefficient is defined as the ratio of vertical stress averaged over a cross-sectional area of the container to the horizontal stress averaged over the perimeter of the cross-section. However, it is still questionable whether it is a function of wall friction [15] or depends only on the material parameters of the media. A short historical survey of this problem was given by Sundaram and Cowin [15].

It was assumed in this paper that the distribution of the vertical stress along the slice is linear and K^l and K^r are the 'local' coefficients relating the vertical stress to the normal wall pressure, at the left and right walls. It was also assumed that K^l and K^r do not depend on the co-ordinate z. When the coefficients K^l and K^r are functions of z or the walls are not linear, a numerical method of solving of the set of basic eqns. (3) should be used. In the following example, it is assumed that the limit state described by the Mohr–Coulomb yield function occurs in the neighborhood of both walls and the formulae for K^l and K^r are derived from the geometrical relations in Fig. 3. Active state (Fig. 3(a)) is assumed to provide static pressure and passive state (Fig. 3(b)) pressure during flow. The following expression results:



(a)



(b)

Fig. 3. Mohr circles for stress state in the neighborhood of the wall for active (a) and passive (b) state.

$$K = (1 \mp \sin\phi\cos\beta)/[1 \pm \sin\phi\cos(\beta \mp 2\alpha)] \qquad (10)$$

where

$\beta = -$, $+ \arcsin(\sin\phi_w/\sin\phi)$ active

$\beta = \phi_w + \arcsin(\sin\phi_w/\sin\phi)$ passive

and ϕ being the angle of internal friction of a material, ϕ_w the angle of wall friction, and α the angle of inclination of the wall to the vertical axes. The upper signs in eqn. (10) provide the value of K for active state and the lower ones for the passive state. Equation (10) gives the values of K^l and K^r by substituting ϕ_w and α for the left and right walls respectively. Such estimation of coefficients K^l and K^r may raise some controversies, particularly in the active state, in which the wall friction does not have to be fully mobilized and the

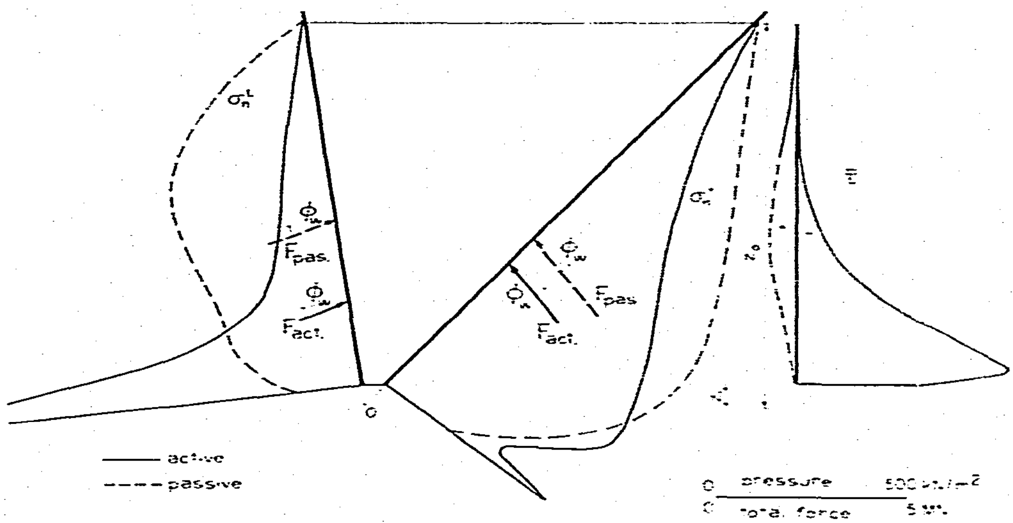


Fig. 4. Distribution of stresses on the walls of a hopper.

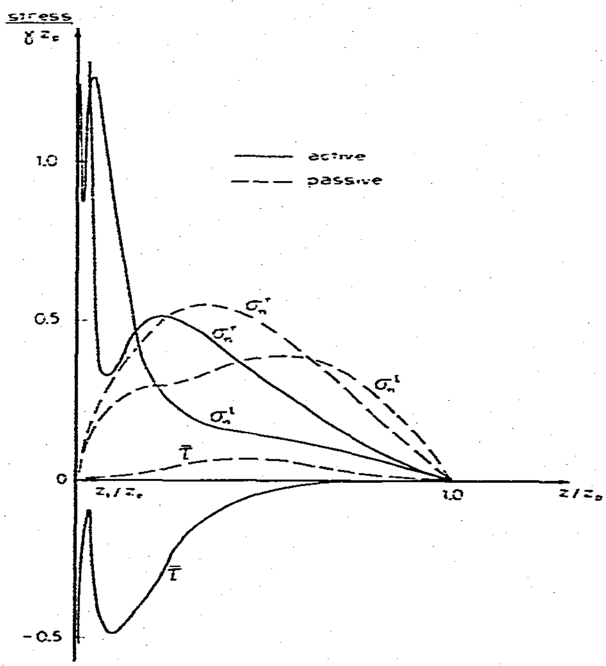


Fig. 5. Dimensionless diagram of stress distribution.

appearance of the limit state is questionable. It is not, however, the objective of this paper to discuss whether the values of K^1 and K^r should be estimated from such a simple stress analysis; it seems that empirical research is most suitable for the estimation of these values. Coefficients K^1 and K^r appear in the solution as constant parameters and the obtained solution is valid for any reasonable value of K^1 and K^r.

Diagrams of the wall pressure and mean shear stress, and the total reactions of the walls for a hopper, with zero surcharge at $z = z_0$, are shown in Fig. 4. The following data were selected: $\alpha^1 = 10°$, $\alpha^r = 40°$, $z_0 = 6.0$ m, $\phi = 30°$, $\phi_w^1 = \phi_w^r = 15°$, $\gamma = 18$ kN/m^3. Dimensionless diagrams are shown in Fig. 5.

It is seen from Figs. 4 and 5 that the stress distribution on walls for the active state has some disturbances, particularly in the neighborhood of the apex of the hopper. These disturbances become more regular oscillations as z approaches 0. An oscillatory pressure distribution results from the trigonometric functions present in the eqns. (5). These oscillations are much less visible for the passive state since in this case $\alpha > 0$ (see eqn. (4)) and the pressure tends to zero at the apex while in the active state, where $\alpha < 0$, it tends to infinity. The solution of the set of eqns. (3) is not of the oscillatory type if all three roots of the characteristic equation are real numbers.

CONCLUSIONS

Results obtained in this paper allow for prediction of distribution of the wall stresses in a converging asymmetrical plane hopper. The method presented in the paper requires the distribution of the normal stress along the slice to be known. In order to obtain a statically determinable system, two ratios, relating the stress normal to the wall to the vertical stress at the left (K^l) and right (K^r) walls, were assumed to be known. The solution was obtained for linear distribution of the vertical stress along the slice, and it has an analytical form for linear walls and coefficients K^l and K^r constant along the walls.

The method presented in the paper is based on the statical analysis of a differential slice and the properties of the material do not appear explicitly in the set of basic equations. They can, however, influence the coefficients K^l and K^r. If a rigid–ideal plastic model of the granular material is considered, the stresses following from the solution should not violate the Mohr–Coulomb yield condition. Therefore the values of K^l and K^r have to be contained in the admissible range of which the lower and upper limits follow from the analysis of stress state at the left and right walls. It was assumed in the example presented in the previous section that the vertical stress at the left and right walls obtained from the solution and the components of the stress vectors acting on the walls produce the stress tensors which describe the stress state at the walls; *i.e.* no stress discontinuity along the walls appears. Hence, for this assumption, the values of the coefficients K^l and K^r estimated in the example can be considered as the lower (active state) and upper (passive state) limit values.

The method is based on the global equilibrium of the slice, so the distribution of the shear stress along the slice does not follow from the solution. Thus, the correctness of the obtained solution in terms of not exceeding the yield condition can be proved only in terms of the global force acting on the slice. The inclination of the global force acting on top of the slice to the outward unit normal should not exceed the value of the internal friction angle, *i.e.* the inequality

$$|\bar{\tau}| \leqslant \tan\phi(\sigma_z^l + \sigma_z^r)/2$$

has to be satisfied for every z throughout the hopper.

The problem of loads in a parallel plane bin with unequally mobilized friction on the left and right wall was also solved by the author. However, engineering significance of such a solution is far smaller.

The method used in the paper is not new; however, the formulation in terms of non-symmetrical geometry and non-symmetrically distributed stresses is different from the traditional one. The formulation of the slice method presented in this paper can have far greater application than the problem of wall stress distribution in hoppers. The method seems to be particularly well suited for geotechnical predictions, such as slope stability, retaining wall loading, *etc.*

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APPENDIX

The characteristic equation of the set of eqns. (3) can be written in the form

$$\begin{vmatrix} A^l - r & B^r & 6/a \\ B^l & A^r - r & -6/a \\ -D^l & D^r & 1/a - r \end{vmatrix} = 0$$

The following expressions for functions $M(r_i)$ and $N(r_i)$ in eqn. (6) were derived:

$$M(r_i) = \frac{6(A^r + B^r + r_i)/a}{A^l A^r - B^l B^r + (A^l + A^r) r_i + r_i^2},$$

$$N(r_i) = [D^l M(r_i) - r_i - 1]/D^r$$

where r_i ($i = 1, 2, 3$) are the roots of the characteristic equation. Constants R^l, R^r and R^t appearing in eqns. (5) can be expressed as follows:

$$R^l = 2(\nu\xi - \delta\lambda) + M(\delta - \mu)/[H(1 - r_1)]$$

$$R^r = 2(\eta\xi - \mu\lambda) + N(\delta - \mu)/[H(1 - r_1)] \quad (11)$$

$$R^t = 2\xi + (\delta - \mu)/[H(1 - r_1)]$$

where

$$H = \mu M - \delta N + \delta\eta - \mu\nu$$

and ξ and λ can be determined from the equality

$$\frac{1}{2}\,\frac{N - M + \nu - \eta + i(\mu - \delta)}{\beta H + i(1 - \alpha)H} = \xi + i\lambda$$

Expressions (11) however, have a different form when $r_1 = 1$:

$$R^l = 2(\nu\xi - \delta\lambda) + M(\delta - \mu)\ln z/H$$

$$R^r = 2(\eta\xi - \mu\lambda) + N(\delta - \mu)\ln z/H$$

$$R^t = 2\xi + (\delta - \mu)\ln z/H$$