DRESCHER, A. & MICHALOWSKI, R. L. (1984). *Géotechnique* **34**, No. 1, 1–10

Denisty variation in pseudo-steady plastic flow of granular media

A. DRESCHER* and R. L. MICHALOWSKI†

In the Paper two pseudo-steady problems of plane plastic flow of dilatant material are considered, and the distributions of densities within the deforming regions determined. The kinematic solutions discussed are characterized by dilation occurring exclusively across velocity discontinuity lines, that are treated in the analysis as shocks. The first problem is that of flow of granular material through a hopper, and the kinematic solution suggested by Drescher, Cousens & Bransby (1978) is used. It is shown that in the radial flow zone the density distribution is non-uniform. As a second problem, the indentation of a wedge into a ponderable half-space is considered, and a kinematic solution similar to that proposed by Hodge (1950) is adopted. It is found that regions with different density exist. An upper limit load for the process of wedge indentation is determined, and the solution for weightless material is compared with that given by Shield (1953). A comparison with some of the experimental results obtained by Butterfield & Andrawes (1972) is also given. Further, it is shown that the interpretation of a velocity discontinuity line as a shock in incipient flow problems yields an additional condition imposed on the kinematic admissibility of the solution.

Deux problèmes d'écoulement plastique pseudo-stationnaire sont étudiés en vue de déterminer la répartition de la densité au sein des regions deformées. Les solutions cinematiques sont caracterisées par l'existence d'une dilatance, présente uniquement le long des lignes de discontinuité de la vitesse. La particularité de cette étude est de traiter les discontinuités de la vitesse comme des chocs. Le premier problème est celui de l'écoulement d'un matériau granulairc dans un silo. Utilisant la solution cinématique de Drescher, Cousens & Bransby (1978), on montre que la répartition de la densité dans la zone d'écoulement radial n'est pas uniforme. Le second probleme traite de l'indentation d'un coin dans un demi-plan pesant. Adoptant une solution cinématique semblable a celle proposée par Hodge (1950), on trouve que des regions de densités différentes existent. La solution théorique est comparée avec les resultats expérimentaux obtenus par Butterfield & Andrawes (1972). On determine également une borne supérieure de la force nécessaire pour enfoncer le coin; pour un matériau non-pesant, la borne supérieure est comparée avec celle de Shield (1953). On demontre également que l'interprétation d'une ligne de discontinuité de la vitesse comme un choc, dans les problèmes d'initiation de l'écoulement plastique, conduit à une restriction sur l'admissibilité cinématique de la solution.

Discussion on this Paper closes on 1 July 1984. For further details see inside back cover.

* University of Minnesota.
† University of Minnesota, on leave from the Technical University of Poznan, Poland.

INTRODUCTION

This Paper deals with the variation of density of granular media treated as rigid-perfectly plastic solids, in some cases of quasi-static, pseudo-steady plane deformation.

The theory of plane plastic flow of granular media, as originated by the work of Drucker & Prager (1952) and Shield (1953) is virtually a straightforward extension of the theory of plastic deformation of ductile materials, such as metals. Accordingly, the theory is based on the concept of the yield condition and the potential flow rule. However, in the case of granular media, as opposed to metals, the yield condition depends on the mean stress. If the flow rule is postulated as being associated with this yield condition, the theory predicts dilation of the material under shear. Numerous experiments performed on dense soils, and other granular materials, indicate that indeed shear induces dilation, but less than that resulting from the theory. To account for the actual dilation, non-associated flow rules have been postulated (Hill, 1950; Jenike & Shield, 1959; Radenkovic, 1961; Mroz, 1974) with the plastic potential deviating from the yield condition. The extreme case, that of incompressibility, is enforced if the plastic potential does not depend on the mean stress.

If a granular material, although virtually discrete, is treated as a one-phase continuum, dilation is equivalent to the decrease of its representative density ρ, defined as

$$\rho = \sum_i \rho_i w_i \tag{1}$$

where ρ_i is the density of each constituent and w_i is its volumetric contribution; $\sum w_i = 1$.

In the solutions of plane plastic flow, the variation of the density within the deforming region is usually not considered, because most of the solutions apply to the so-called incipient flow problems. In this class of problems, the rate of dilation rather than the density can be determined. The density distribution can be obtained, however, for steady or pseudo-steady problems, if the kinematic solution is known.

This Paper determines the density distribution in a pseudo-steady plastic flow of a granular material through a plane hopper, and in a pseudo-steady process of indentation of a plane wedge into a granular half-space.

A kinematic solution for the flow in a plane hopper has been suggested by Drescher, Cousens & Bransby (1978). A static and kinematic solution of the problem of wedge indentation into a weightless granular medium obeying the associated flow rule was obtained by Shield (1953), whereas a similar problem for ponderable material and non-associated flow rule was considered by Drescher, Kwaszczynska & Mroz (1967). Valuable experimental results were also presented by Butterfield & Andrawes (1972). In the present Paper, simple kinematic solutions, similar to those suggested for metals by Hodge (1950), are discussed. These solutions, as well as the solution for the flow through hoppers, are characterized by dilation occurring exclusively across strong velocity discontinuity lines, separating regions of rigid-motion or incompressible deformation. Limitation to particular kinematic solutions was chosen deliberately to analyse in detail the density variation across discontinuity lines that are common in many plane strain boundary value problems. In the analysis, the discontinuity is treated as a 'shock', across which both the velocity and the density experience jump. It is shown that the density of the material discharged from the hopper is not uniform, as was assumed by Drescher *et al.* (1978). In the case of wedge indentation, besides evaluation of the densities within the flow domain, an upper limit load for ponderable material is given.

DENSITY VARIATION ACROSS STRONG VELOCITY DISCONTINUITY LINE

The kinematics of plane-strain boundary-value problems for a rigid–perfectly plastic isotropic material is formulated in terms of velocities relative to some boundary velocity (cf Hill, 1950). As the equations governing velocities are hyperbolic, strong velocity discontinuities are admissible. For a dilating granular material, obeying the potential flow rule and the Mohr–Coulomb yield condition, the jump in velocity across a discontinuity line $[V]$ satisfies the equation

$$[V^n] = [V^t] \tan \nu \tag{2}$$

where $[V^n]$ and $[V^t]$ are the normal and tangential components of the velocity jump $[V]$ respectively, and ν is the angle of dilation if the non-associated flow rule is assumed. For the associated flow rule $\nu = \phi$, where ϕ is the angle of internal friction (cf Shield, 1953; Davis, 1968).

In plasticity of granular media, the velocity discontinuity line is usually interpreted as a mathematical approximation of a thin, but finite, material layer, across which the velocities experience rapid change and the layer dilates. The concept of a thin layer is introduced to preserve finite dilation of the material in incipient flow problems; for zero thickness of a material layer, the jump in the normal component of the velocity would produce separation or an infinite rate of dilation. In steady or pseudo-steady problems, however, the discontinuity line cannot be a material layer, since with elapsing time the particles travel across the discontinuity. It is unnecessary, therefore, to assign any thickness to the layer, and the velocity discontinuity can be regarded as a shock in the dilating material. The relation between the jump in velocity and in density results directly from the principle of mass conservation applied to a strong velocity discontinuity or shock. For a stationary discontinuity line, the well-known continuity condition for the flux of matter is (cf Mises, 1958; Thomas, 1961)

$$\rho_1 V_1{}^n = \rho_2 V_2{}^n \tag{3}$$

or

$$[\rho V^n] = 0 \tag{4}$$

where $V_1{}^n$ and $V_2{}^n$ are the normal components of the velocities V_1 and V_2 (Fig. 1(a)) of the material on both sides of the discontinuity, ρ_1 and ρ_2 are the corresponding densities, and the square brackets in equation (4) denote the jump (Fig. 1(b)). Thus, any normal jump in the velocity is associated with a jump in density; for a positive jump $[V^n]$ in the direction of flow, the density decreases, and the material dilates. Equation (3) holds for non-steady flow also, with the velocities in equation (3) measured with respect to the moving discontinuity. Denoting the speed of propagation of the discontinuity line by V_p, equation (3) becomes

$$\rho_1(V_1{}^n - V_p{}^n) = \rho_2(V_2{}^n - V_p{}^n) \tag{5}$$

where $V_p{}^n$ is the normal component of V_p, and the velocities are measured with respect to a fixed reference system (Figs 1(c) and 1(d)).

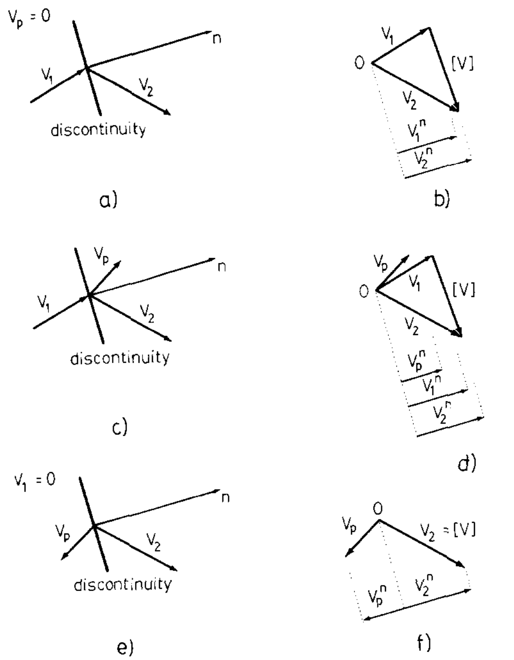

Fig. 1. (a) and (b) Stationary velocity discontinuity; (c) and (d) velocity discontinuity as a shock in the velocity field; (e) and (f) velocity discontinuity as a shock propagating into the material at rest

Equation (5) was used by Cowin & Comfort (1982) in the analysis of propagation of a one-dimensional rarefaction wave and condensation shock in a granular material flowing gravitationally through a vertical channel. Combining equation (5) with the equation of momentum conservation across the discontinuity, Cowin & Comfort obtained an expression relating the speed of propagation of the discontinuity to the jump in normal stresses and in densities of the material. In an earlier paper by Cowin & Nunziato (1981), a theory of dilatant waves in granular media was suggested. In the present Paper, however, equation (5) is considered solely as a condition relating the velocity and the density jump across a velocity discontinuity, which otherwise results from the theory of quasi-static plastic flow of granular media. In particular, equation (2) for the velocity jump still holds.

If the kinematic solution for the velocities V_1 and V_2 on both sides of the discontinuity is known, and the speed of propagation V_p is given, equation (5) can be used to determine one of the densities, ρ_1 or ρ_2. Alternatively, the normal component of the speed of propagation $V_p{}^n$ can be found if ρ_1 and ρ_2 are known. In incompressible flow, where $\rho_1=\rho_2$ and $V_1{}^n=V_2{}^n$, the normal component of the speed of propagation is not related to the density, and is governed solely by the boundary conditions of the problem. For a dilatant material

$$V_p{}^n=\frac{V_1{}^n-V_2{}^n\rho_2/\rho_1}{1-\rho_2/\rho_1} \qquad (6)$$

and the magnitude and the direction of $V_p{}^n$ depends on the magnitude of ρ_1, ρ_2, $V_1{}^n$ and $V_2{}^n$.

Although the above consideration was devoted primarily to evaluation of densities across the velocity discontinuity lines in steady and pseudo-steady flow problems, it may be interesting to extend the analysis to incipient flow problems. In many incipient flow problems, the plastically deforming region is separated by a strong velocity discontinuity line from the region at rest. Examples are the punch indentation problem (bearing capacity problem) and motion of a rigid wall into a granular mass (retaining wall problem). Dilation of the deforming material results in $\rho_2/\rho_1<1$, where ρ_1 is the initial density of the material. Since the velocity of the region at rest is zero, $V_1=0$, and the velocity jump is directed into the plastic region, equation (6) indicates that the velocity discontinuity line should propagate in the direction opposite to $V_2{}^n$, i.e. propagate into the material at rest (Figs 1(e) and 1(f)). Propagation is physically possible provided the material outside the plastified region is not rigid. Thus, the interpretation of the velocity discontinuity in incipient flow problems as a shock imposes an additional condition on the kinematic admissibility of the solution. In view of this condition, some of the solutions where the velocity discontinuity becomes tangent to a rigid wall or base would be kinematically inadmissible (e.g. Mandel & Salencon, 1972). Obviously, this condition does not have to be met if the concept of a thin layer is retained.

In the following sections, equation (5) is used to determine the variation of the density in two pseudo-steady, plane-strain problems.

PSEUDO-STEADY FLOW THROUGH A PLANE HOPPER

Most of the experimental and theoretical work on flow of granular media through hoppers and bins refers to the advanced stage of discharge and steady state of flow (cf. Pariseau, 1970; Bransby, Blair-Fish & James 1973; Perry, Rothwell & Woodfin, 1976; Brennen & Pearce 1978; Spink & Nedderman, 1978; Nedderman & Tüzün, 1979; Tüzün & Nedderman; 1982). Steady state can be assumed if the bin section of

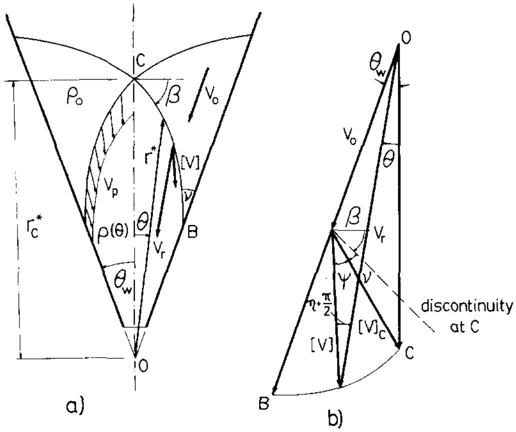

Fig. 2. (a) Geometry of flow pattern in a plane hopper; (b) hodograph

the bunker is tall, or the bunker or hopper is short but the material is continuously supplied to compensate for the material discharged through outlet.

Advanced, non-steady flow through a plane, mass-flow hopper was studied by Drescher *et al.* (1978). Experimental observations by means of X-ray technique revealed a pseudo-steady pattern of flow. In the upper portion of the flowing mass, two almost rigid regions form, with velocities in each region nearly constant and parallel to the hopper walls, and with uniform density ρ_0. A distinct curvilinear boundary separates these regions from the zone below, where the velocities of particles are approximately radial and the density is lower than ρ_0 (see Fig. 2(a)). Although both dense and loose zones shrink with progressive discharge, the separating boundary preserves radial geometrical similarity. As the directions of the velocities also remain virtually unchanged during discharge, the pattern of flow can be regarded as geometrically self-similar, and the process of flow as pseudo-steady. To describe mathematically the observed flow pattern a kinematic solution was suggested, based on a rigid–perfectly plastic model of the material. The boundary between the dense and loose zones was treated as a strong velocity discontinuity line, across which the velocities of particles change direction and the material dilates. The corresponding hodograph for the particles crossing the discontinuity line is shown in Fig. 2(b). In the radial flow zone, incompressibility was assumed, resulting in the velocity field

$$V_r = A(\theta)/r \tag{7}$$

where $A(\theta)$ is a function of θ only.

This solution was treated by Drescher *et al.* (1978) as a kinematic solution for unsteady, geometrically self-similar flow in hoppers. Accordingly, the density of the material in the radial flow zone was computed under the assumption of uniformity. This interpretation, and the magnitude of the density obtained, are in fact erroneous, because the propagation of the velocity discontinuity line was disregarded.

To derive an expression for the density distribution in the radial zone, besides knowing the velocity jump resulting from the hodograph, the speed of propagation V_p of the discontinuity line has to be known. Geometrical similarity requires the speed V_p to be radial pointing towards the apex, and so distributed along the discontinuity that at any given time t the geometry of the discontinuity is proportional to the characteristic radius r_c^* (see Fig. 2(a)). The equation for the discontinuity line is

$$r^* = r_c^* \exp \int_0^\theta \tan(\eta + \nu)\, d\theta \tag{8}$$

where ν is the angle of dilation of the material and the angle η is shown in Fig. 2(b). Thus V_p is given by

$$V_p = V_p^c \exp \int_0^\theta \tan(\eta + \nu)\, d\theta \tag{9}$$

where V_p^c is the speed of propagation at point C. The components normal to the discontinuity line of the velocities above and below the discontinuity, V_0^n and V_r^{*n}, and the normal component of the speed of propagation V_p^n, are

$$\begin{aligned} V_0^n &= V_0 \sin(\theta_w + \beta + \psi) \\ V_r^{*n} &= V_r^* \sin(\theta + \beta + \psi) \\ V_p^n &= V_p \sin(\theta + \beta + \psi) \end{aligned} \tag{10}$$

Using the expression for V_r^* derived by Drescher *et al.* (1978), and equations (5), (9) and (10), the density ρ below the discontinuity can be expressed as equation (11) shown in Fig. 3, where β and ψ are shown in Fig. 2(b).

Equation (11) indicates that the density ρ below the discontinuity line is not constant, but varies with the angle θ. As in the radial flow zone, incompressibility is assumed, the density variation with θ must remain unchanged down to the outlet, i.e. the density distribution in the radial flow zone is non-uniform, although it depends on θ only.

The variation of ρ with θ depends on the unknown ratio V_0/V_p^c. It is possible, however, to compare the density variation in the steady

$$\frac{\rho}{\rho_0}=\frac{\dfrac{V_0\sin(\theta_w+\beta+\psi)}{V_p^{\,c}\sin(\theta+\beta+\psi)\exp\int_0^\theta\tan(\eta+\nu)\,d\theta}-1}{\dfrac{V_0[\cos(\theta_w-\theta)\cos(\beta+\nu)-\sin\theta_w\sin\eta\exp(\psi\tan\nu)]}{V_p^{\,c}\cos(\beta+\nu)\exp\int_0^\theta\tan(\eta+\nu)\,d\theta)}-1} \qquad (11)$$

Fig. 3. Equation (11)

state of flow, $V_p=0$, and in the pseudo-steady flow by assuming $V_p^{\,c}$ within the range $V_r^{*c}>V_p^{\,c}>0$. Fig. 4 shows the distribution of the dimensionless density ρ/ρ_0 for $V_p=0$ and $V_0/V_p^{\,c}=1{\cdot}2$, for a hopper with the included angle $2\theta_w=20°$ and $\nu=20°$. It was assumed that in the radial flow zone, the velocities are the same for both steady and pseudo-steady flow. The results show that during pseudo-steady flow the densities are lower than during steady flow, and in neither case are constant, as was assumed by Drescher *et al.* (1978). The density they computed equals the weighted mean density $(\rho/\rho_0)_m$ for steady state flow (see Fig. 4).

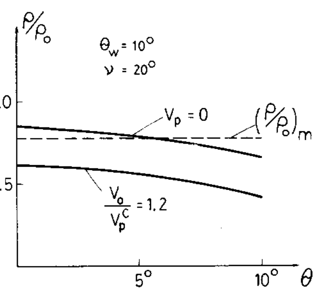

Fig. 4. Density variation in a hopper with half-angle $\theta_w=10°$ and dilation angle $\nu=20°$

The theoretical solution for the densities in the radial zone predicts a relatively high dilation of the material. Although no accurate measurements of the density distribution during flow are available, the computed densities may be unrealistic. It is possible, however, to modify the solution by allowing for the variation of the angle of dilation ν along the discontinuity, which would result in a much smaller density change.

WEDGE INDENTATION PROBLEM

Indentation of a wedge into a rigid–perfectly plastic half-space is a classical problem in the theory of plasticity. Statics and kinematics of the problem for a frictionless wedge and Tresca-type material of the half-space was first given by Hill, Lee & Tupper (1947). Since the problem was treated as pseudo-steady, the geometry of the flow domain is proportional to the depth of indentation for any stage of the process. An approximate solution, with two rigid-motion regions separated by a velocity discontinuity line, was shown later by Hodge (1950). The solution to a similar problem but for a dilational and weightless material, described by the Mohr–Coulomb yield condition and the associated flow rule, was given by Shield (1953). A non-steady, static and kinematic solution for an incompressible, frictional and ponderable material was suggested by Drescher *et al.* (1967), and compared with some experimental results. An extensive experimental study of indentation of a wedge into a sand was performed by Butterfield & Andrawes (1972).

The solution of the wedge indentation problem presented here is similar to that given by Hodge (1950) for metals, and is not complete in the sense of plastic flow theory (i.e. no stress field is suggested), but a kinematically admissible velocity field is constructed. It is imposed that the material displaced above the original surface of the half-space forms straight-line lips at both sides of the wedge and the flow mechanism is geometrically similar at each stage of indentation. Thus, the pseudo-steady character of the process is enforced. It is also assumed that the wedge is rigid, with frictionless flanks, and the half-space is ponderable with uniform initial density ρ_0 (gravitational field g). The material is rigid–perfectly plastic, and obeys the Mohr–Coulomb yield condition and the associated or non-associated flow rule.

As the geometrical self-similarity of deformation is assumed in the solution, any stage of indentation can be selected for the analysis of the kinematic field. The simplest kinematically admissible mechanism of indentation is shown in Fig. 5. The line BD is a velocity discontinuity line, inclined at an angle α_1 to the horizon. The region ABD moves as a rigid body with the velocity V_1. The vector V_1 is inclined at the dilation angle ν to the line BD and is related to the velocity of indentation by

$$V_1=V_0\sin\beta/\cos(\alpha_1+\beta+\nu) \qquad (12)$$

To derive an expression for the unknown angle η of inclination of the lip AB to the horizontal surface, the geometrical relations in Fig. 5 can be used. The distance d between the vertex D

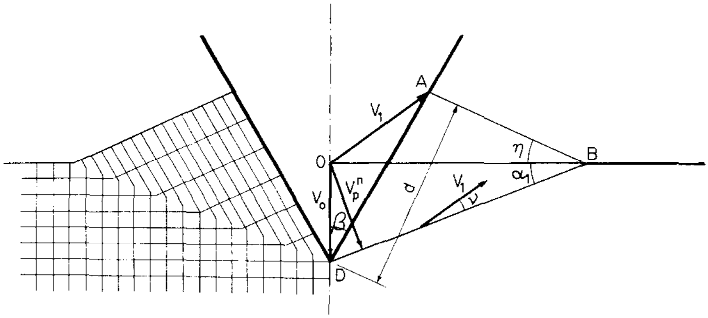

Fig. 5. Simple kinematically admissible mechanism of wedge indentation

and the lip AB can be expressed as

$$d = \mathrm{OD}\left(\cos\eta + \frac{\sin\eta}{\tan\alpha_1}\right) \tag{13}$$

where OD is equal to $V_0 t$. The rate of increase of the distance d is

$$V_0 \cos\eta + V_1 \sin(\eta + \alpha_1 + \nu)$$

Hence, after time t the distance d is equal to

$$d = t[V_0 \cos\eta + V_1 \sin(\eta + \alpha_1 + \nu)] \tag{14}$$

By combining equations (13) and (14), the angle η can be expressed as

$$\tan\eta = \frac{\sin(\alpha_1 + \nu)}{\dfrac{\cos(\alpha_1 + \beta + \nu)}{\sin\beta \tan\alpha_1} - \cos(\alpha_1 + \nu)} \tag{15}$$

In the problem considered, the normal speed of propagation of the line BD is uniquely related to the velocity V_0 of the wedge (Fig. 5) and is equal to $V_0 \cos\alpha_1$. The material dilates passing through the discontinuity BD, and its density ρ_1, computed using equation (5), is

$$\frac{\rho_0}{\rho_1} = 1 + \frac{\sin\beta \sin\nu}{\cos\alpha_1 \cos(\alpha_1 + \beta + \nu)} \tag{16}$$

The density of the material in the region ABD is uniform and constant at any instance of indentation. The deformation of the originally square grid is shown on the left-hand side of Fig. 5. It can be shown that the formula for the angle η (equation (15)) may be derived directly from the principle of mass conservation, which requires the mass originally occupying the triangle OBD, $m = \rho_0 S^{\mathrm{OBD}}$, to be equal to the mass of the deformed region ABD, $m = \rho_1 S^{\mathrm{ABD}}$, where ρ_1 is given by equation (16) and S stands for the area of the triangles.

A kinematically admissible mechanism with two rigid-motion regions separated by a discontinuity line is shown in Fig. 6. The velocity V_1 of the region ACD is described by equation (12); the velocity V_2 of the region CBA and the velocity jump $[V]$ along the line AC are

$$V_2 = V_1 \frac{\cos(\alpha_3 - \alpha_1 - \beta - 2\nu)}{\cos(\alpha_2 - \alpha_3 + \beta + 2\nu)} \tag{17}$$

$$[V] = V_1 \frac{\sin(\alpha_2 - \alpha_1)}{\cos(\alpha_2 - \alpha_3 + \beta + 2\nu)} \tag{18}$$

The expression for the angle η cannot now be obtained explicitly. Defining angle ξ as shown in Fig. 6, the distance d between the vertex D and

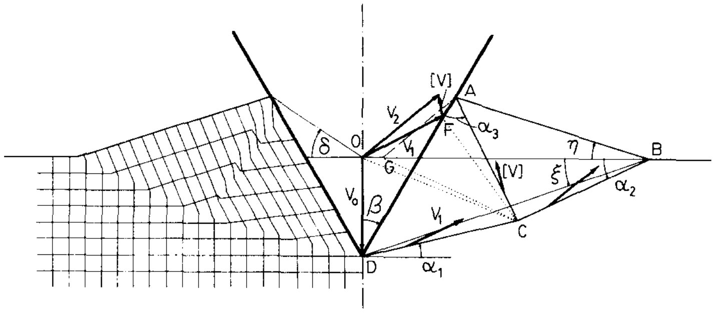

Fig. 6. Kinematically admissible mechanism for wedge indentation, consisting of two rigid-motion regions separated by a shock

the lip is

$$d = \mathrm{OD}\left(\cos\eta + \frac{\sin\eta}{\tan\xi}\right)$$

and the rate of increase of d is

$$V_0 \cos\eta + V_2 \sin(\eta + \alpha_2 + \nu)$$

The angle η has to be determined from the equation

$$V_0 \frac{\sin\eta}{\tan\xi} - V_2 \sin(\eta + \alpha_2 + \nu) = 0 \qquad (19)$$

where V_2 can be expressed as a function of V_0 using equations (17) and (12), and ξ is described by

$$\tan\xi = (\cos\beta - B\sin\eta)/(B\cos\eta + \sin\beta)$$
$$B = \frac{\cos(\beta + \alpha_1)\cos(\alpha_2 + \beta - \alpha_3)}{\sin(\alpha_2 + \eta)\cos(\alpha_3 - \alpha_1 - \beta)}$$

In order to determine the density variation within the deformed material, the speeds of propagation of the discontinuity lines have to be known first. Fig. 7 shows the expansion of the flow region related to the time increment Δt. Using geometrical relations from the hodograph in Fig. 7, the normal components of the speeds of propagation of the discontinuity lines DC, CB and CA can be expressed as

$$\left.\begin{aligned} V_{DC}{}^{n} &= V_0 \cos\alpha_1 \\ V_{CB}{}^{n} &= V_0 \sin\alpha_2/\tan\xi \\ V_{CA}{}^{n} &= V_0 \sin\beta\cos(\beta + \delta - \alpha_3)/\cos(\beta + \delta) \end{aligned}\right\} \qquad (20)$$

where

$$\tan\delta = D\sin\eta/(1 - D\cos\eta)$$
$$D = \cos(\beta + \xi)/[\cos\xi\cos(\eta - \beta)]$$

Three regions with different densities can be distinguished in the flow domain ABCD (Fig. 6): region DCF with the density ρ_1, region CBA with the density ρ_2 and region ACF with ρ_3. The material in the triangle DCF originally occupied the area DCO. The dilation of this material occurred across the propagating line DC. Similarly, dilation of the material in the region CBA, which originally occupied the triangle CBG, occurred exclusively across the line CB. Material in the narrow area ACF, however, first dilated along the line CB and then was subjected to additional dilation while passing line AC (it is clear from the hodograph that the normal component of the speed of propagation of the line AC is larger than the component of V_2 normal to AC). The densities ρ_1, ρ_2 and ρ_3 can be obtained from equations (5) and (20). The

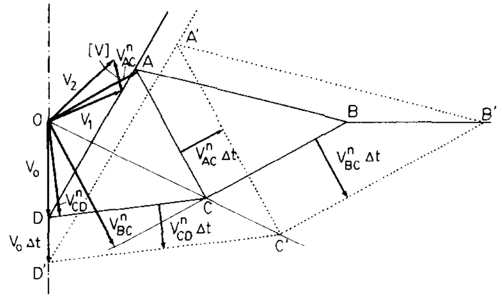

Fig. 7. Expansion of the flow regions related to the time increment Δt

density ρ_1 in the region DCF is described by equation (16); the density ρ_2 in CBA is given by

$$\frac{\rho_0}{\rho_2} = 1 + \frac{\sin\beta\sin\nu\tan\xi\cos(\alpha_3 - \alpha_1 - \beta - 2\nu)}{\sin\alpha_2\cos(\alpha_1 + \beta + \nu)\cos(\alpha_2 - \alpha_3 + \beta + 2\nu)} \qquad (21)$$

and ρ_3 in the area CFA by

$$\frac{\rho_3}{\rho_2} = \frac{A - \cos(\beta + \delta - \alpha_3)\cos(\alpha_1 + \beta + \nu)}{\sin\alpha_3\sin(\alpha_1 + \nu - \delta)} \qquad (22)$$

where

$$A = \cos(\beta + \delta) \times \frac{\cos(\alpha_1 - \alpha_3 + \beta + 2\nu)\cos(\alpha_2 - \alpha_3 + \beta + \nu)}{\cos(\alpha_2 - \alpha_3 + \beta + 2\nu)} \qquad (22a)$$

The deformed grid shown on the left-hand side of Fig. 6 was obtained for a material with the angle of dilation $\nu = 15°$. For the initial density of the material $\rho_0 = 1{\cdot}8 \times 10^3$ kg/m^3, the resulting densities in the regions DCF, CBA and ACF are $\rho_1 = 1{\cdot}43 \times 10^3$, $\rho_3 = 1{\cdot}47 \times 10^3$ and $\rho_3 = 0{\cdot}96 \times 10^3$ kg/m^3. The value of ρ_3 seems to be unrealistic. Assuming, however, that the dilation angle of the material within the flowing region is lower than that of the virgin material, say $\nu_2 = 5^0$, the density ρ_1 remains unchanged, whereas $\rho_2 = 1{\cdot}48 \times 10^3$ kg/m^3 and $\rho_2 = 1{\cdot}29 \times 10^3$ kg/m^3. In the derived formule in such cases the angle ν should be substituted by ν_1 (dilation angle of the virgin material) and in equations (17), (18), (21) and (22(a)) 2ν should be substituted by $\nu_1 + \nu_2$.

These mechanisms may serve to estimate the upper bound of the limit load of the wedge. In such cases, however, it is imperative that the potential in the flow rule is identified with the yield condition ($\nu = \phi$). The following analysis is restricted to the second mechanism only. The total rate of external work consists of the rate of

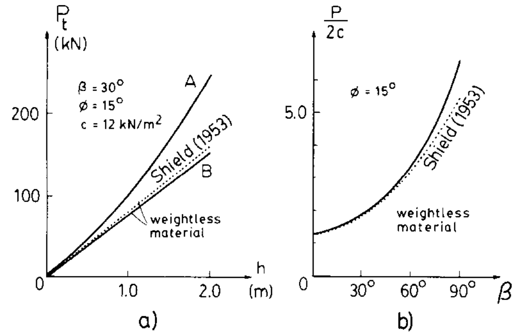

Fig. 8. (a) Upper limit force versus depth of indentation of a wedge into a ponderable (line A) and weightless material (line B), and the solution obtained by Shield (1953) using the method of characteristics; (b) dimensionless mean pressure on a wedge flank versus half-angle β for a kinematically admissible mechanism in which the moving boundary was postulated to be identical to that from the solution of Shield (1953)

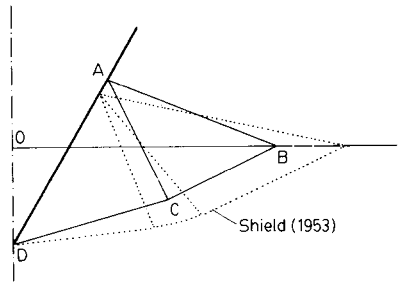

Fig. 9. Geometry of the postulated kinematically admissible mechanism (ABCD), and the mechanism of Shield (1953)

work of the indentation limit load and the rate of work of body forces. Since the densities in the regions DCF, CBA and ACF are different, the body forces will vary accordingly. The rate of dissipated work consists of the rate of work along three discontinuity lines: DC, CB and CA (Fig. 6). The total limit load P_t can thus be expressed as

$$\begin{aligned} P_t V_0/2 = {} & g[\rho_1 S^{DCF} + \rho_3 S^{ACF})]V_1 \sin(\alpha_1 + \phi) \\ & + g\rho_2 S^{CBA} V_2 \sin(\alpha_2 + \phi) \\ & + DCcV_1 \cos\phi + CBcV_2 \cos\phi \\ & + CAc[V] \cos\phi \end{aligned} \tag{23}$$

where c is the cohesion of the material, $g = 9{\cdot}81$ m/s², S stands for the area, and DC, CB and CA are the lengths of the discontinuities. Velocities V_1, V_2 and $[V]$ in equation (23) can be expressed as functions of V_0 using equations (12), (17) and (18), and the areas of the regions with different densities, S^{DCF}, S^{ACF} and S^{CBA}, and the lengths DC, CB and CA can be derived from the geometrical relations in Fig. 6.

In Fig. 8(a) some computational results are presented. Line A shows the total force P_t versus the depth of indentation of the wedge below the original surface of the half-space. The following parameters were selected: the wedge angle $2\beta = 60°$, initial density of the material $\rho_0 = 1{\cdot}8 \times 10^3$ kg/m³, $\phi = 15°$ and $c = 12$ kN/m². The angles α_1, α_2 and α_3 were optimized in order to obtain the minimum value of the force P_t (the least upper limit load). The values of α_i ($i = 1, 2, 3$) that correspond to $(P_t)^{min}$ for a ponderable material vary with the depth of indentation; wedge indentation in a ponderable material is not a pseudo-steady problem (cf. Drescher *et al.*, 1967). The mechanism, however, was assumed to be geometrically similar. Thus, in the example presented in Fig. 8(a) the values of α_i were fixed and obtained for $(P_t)^{min}$ for an indentation depth of 1 m ($\alpha_1 = 14°$, $\alpha_2 = 25°$ and $\alpha_3 = 58°$). Accordingly, line A does not represent the least upper limit load. It was found, however, that the reference depth for $(P_t)^{min}$ does not affect the results substantially.

Line B in Fig. 8(a) shows the total limit load for a weightless material with $\phi = 15°$ and $c = 12$ kN/m². In this case the problem is strictly self-similar, and one set of α_i, independent of the depth of indentation, is obtained for $(P_t)^{min}$($\alpha_1 = 16°$, $\alpha_2 = 26°$ and $\alpha_3 = 57°$), and line B represents the least upper limit load. For a weightless material the total force obtained from the upper bound approach is lower than that obtained by Shield (1953) using the method of characteristics. Such a result seems to be incorrect. However, there is no contradiction with the upper bound theorem of limit analysis because the slope of the moving boundary in both solutions is different (Fig. 9). Fig. 8(b) shows the dimensionless mean pressure on the wedge flank as a function of the wedge half-angle β for a weightless material. The dotted line corresponds to the solution obtained by Shield, and the solid line to the kinematically admissible mechanism in which the boundary of the deformed material was assumed to be the same as in Shield's solution.

It is easy to extend this solution to a wedge with rough flanks. In this case the rate of work dissipated on friction along the wedge flank should be added to the total rate of work.

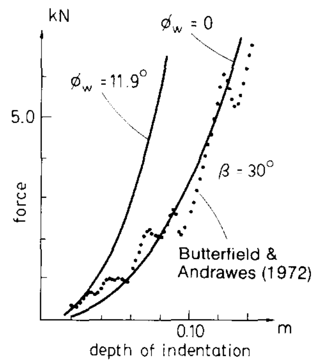

Fig. 10. Upper limit force for the postulated flow mechanism versus depth of indentation, and the experimental results after Butterfield & Andrawes (1972)

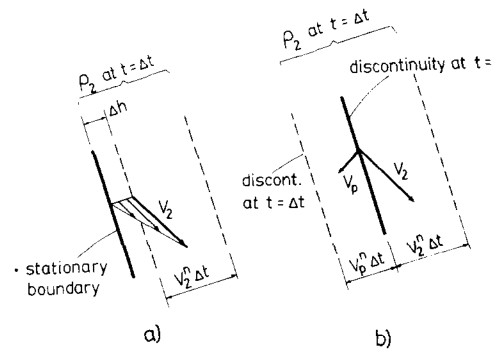

Fig. 11. Velocity discontinuity: (a) as a thin layer of material; (b) as a shock

Denoting the angle of the Coulomb surface friction by ϕ_w, the total limit load P_t^f can be expressed as

$$P_t^f = P_t \Big/ \left[1 - \frac{\sin \phi_w \cos(\alpha_1 + \phi)}{\sin(\beta + \phi_w)\cos(\phi + \alpha_1 + \beta)}\right] \quad (24)$$

where P_t is the limit force for a frictionless wedge (equation (23)).

A comparison of the limit indentation force obtained from this solution, and the force experimentally measured by Butterfield & Andrawes (Fig. 4(a), 1972) is shown in Fig. 10. In the computations the material parameters of the dense sand measured by Butterfield & Andrawes (1972) were used, and the associated flow rule was postulated. The curves in Fig. 10 represent the results of computations for a frictionless wedge and the wedge which was described by Butterfield & Andrawes as smooth ($\phi_w = 11{\cdot}9°$). Theoretical results for the latter differ substantially from the experimental measurements. However, the associated flow rule leads to an unrealistic dilation of the material in the flowing regions.

CONCLUSIONS

In the solutions of steady and pseudo-steady processes of plane plastic flow of dilatant granular media, a strong velocity discontinuity cannot be treated as a material layer. Thus, it is unnecessary to assign any thickness to the discontinuity, which in turn, can be regarded as a shock. In the incipient flow problems, however, two interpretations of the velocity discontinuity are admissible. In the first, the discontinuity is identified with a thin material layer undergoing dilation. In the second, the discontinuity is treated as a propagating shock. The latter interpretation imposes an additional requirement on the kinematic admissibility of the solutions: physical capability of propagation of the discontinuity.

These remarks refer to theoretical solutions of plane plastic flow, and it is beyond the scope of this Paper to furnish physical justification in favour of one of the interpretations. It should be noted that the physical justification based on experimentally observed zones of concentrated dilation (rupture surfaces and shear bands) is not straightforward. In fact, both interpretations lead to a finite thickness of a dilated zone after some time (Fig. 11). The only difference is shown by a different expansion of the zone: either in the direction of the plastified region (Fig. 11(a)) or in the direction of the region at rest (Fig. 11(b)).

In pseudo-steady flow, the neglect of propagation of the discontinuity may result in erroneous calculation of the density variation, as was demonstrated in the problem of flow of granular material through a plane hopper.

A kinematically admissible solution was constructed for the problem of pseudo-steady indentation of a wedge into a ponderable, dilating soil mass. This solution indicates that for computing the upper limit load, the density distribution has to be known at any instance of indentation. It also demonstrates that in problems with an unknown boundary of the deforming region, a simple kinematic solution may give a lower limit load than does the method of characteristics.

REFERENCES

Bransby, P. L., Blair-Fish, P. M. & James, R. G. (1973). An investigation of the flow of granular materials. *Powder Technol.* **8,** 197–206.

Brennen, C. & Pearce, J. C. (1978). Granular material flow in two-dimensional hoppers. *J. Appl. Mech.* **45,** 43–50.

Butterfield, R. & Andrawes, K. Z. (1972). An investigation of a plane strain continuous penetration problem. *Géotechnique* **22,** No. 4, 597–617.

Cowin, S. C. & Comfort, III, W. J. (1982). Gravity induced density discontinuity waves in sand columns. *J. Appl. Mech.* **49,** 497–500.

Cowin, S. C. & Nunziato, J. W. (1981). Waves of dilatancy in granular material with incompressible granules. *Int. J. Engng Sci.* **19,** 993–1008.

Davis, E. H. (1968). Theories of plasticity and the failure of soil masses. *Soil mechanics selected topics.* (ed. I. K. Lee). London: Butterworths.

Drescher, A., Cousens, T. W. & Bransby, P. L. (1978). Kinematics of the mass flow of granular material through a plane hopper. *Géotechnique* **28,** No. 1, 27–42.

Drescher, A., Kwaszczynska, K. & Mroz, Z. (1967). Statics and kinematics of the granular medium in the case of wedge indentation. *Archwm Mech. Stosow.* **17,** 157–165.

Drucker, D. C. & Prager, W. (1952). Soil mechanics and plastic analysis or limit design. *Q. Appl. Math.* **10,** 157–165.

Hill, R. (1950). *The mathematical theory of plasticity.* Oxford: Oxford University Press.

Hill, R., Lee, E. H. & Tupper, S. J. (1947). The theory of wedge indentation of ductile materials. *Proc. R. Soc.*, Series (A) **188,** 273–289.

Hodge, P. G. (1950). Approximate solutions of problems of plane plastic flow. *J. Appl. Mech.* **17,** 257–264.

Jenike, A. W. & Shield, R. T. (1959). On the plastic flow of Coulomb solids beyond original failure. *J. Appl. Mech.* **27,** 599–602.

Mandel, J. & Salencon, J. (1972). Force portante d'un sol sur une assise rigide. *Géotechnique* **22,** No. 1, 79–93.

Mises, von, R. (1958). *Mathematical theory of compressible fluid flow.* New York: Academic Press.

Mroz, Z. (1974). *Non-associated flow rules in description of plastic flow of granular material.* Udine: CISM.

Nedderman, R. M. & Tüzün, U. (1979). A kinematic model for the flow of granular materials. *Powder Technol.* **22,** 243–253.

Pariseau, W. G. (1970). Discontinuity velocity fields in gravity flows of granular materials through slots. *Powder Technol.* **3,** 218–226.

Perry, M. G., Rothwell, E. & Woodfin, W. T. (1976). Model studies of mass-flow bunkers. II. Velocity distribution in the discharge of solids from mass-flow bunkers. *Powder Technol.* **14,** 81–92.

Radenkovic, D. (1961). *Theorie de charge limites, extension à la mechanique de sols.* Paris: Ecole Polytechnique.

Shield, R. T. (1953). Mixed boundary value-problems in soil mechanics. *Q. Appl. Maths.* **11,** 61–75.

Spink, C. D. & Nedderman, R. M. (1978). Gravity discharge rate of fine particles from hoppers. *Powder Technol.* **21,** 245–261.

Thomas, T. Y. (1961). *Plastic flow and fracture of solids.* New York: Academic Press.

Tüzün, U. & Nedderman, R. M. (1982). An investigation of the flow boundary during steady-state discharge from a funnel-flow bunker. *Powder Technol.* **31,** 27–43.