Jacobian of the Cross Product

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The following was derived to ease a proof in Math 658 F10 regarding the Jacobi Identity.

1 Claim

Given two vector valued function $\vec{A}$ and $\vec{B}$ on $\mathbb{R}^3$:

$$\begin{aligned} \vec{A} &= \begin{bmatrix} A_1(x,y,z) & A_2(x,y,z) & A_3(x,y,z) \end{bmatrix}^{\mathrm{T}} \\ \vec{B} &= \begin{bmatrix} B_1(x,y,z) & B_2(x,y,z) & B_3(x,y,z) \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

and the matrix form of the cross product:

$$\vec{\xi}^{\,\times} = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}, \tag{1}$$

then the Jacobian ($J(\bullet)$) of a cross product of $\vec{A}$ and $\vec{B}$ is

$$\boxed{J(\vec{A} \times \vec{B}) = \vec{A}^{\,\times} J(\vec{B}) - \vec{B}^{\,\times} J(\vec{A}).} \tag{2}$$

2 Derivation

Given that

$$\vec{A} \times \vec{B} = \begin{bmatrix} A_2B_3 - A_3B_2 & A_3B_1 - A_1B_3 & A_1B_2 - A_2B_1 \end{bmatrix}^{\mathrm{T}},$$

then

$$J(A \times B) = \begin{bmatrix} \frac{\partial}{\partial x}(A_2B_3 - A_3B_2) & \frac{\partial}{\partial y}(A_2B_3 - A_3B_2) & \frac{\partial}{\partial z}(A_2B_3 - A_3B_2) \\ \frac{\partial}{\partial x}(A_3B_1 - A_1B_3) & \frac{\partial}{\partial y}(A_3B_1 - A_1B_3) & \frac{\partial}{\partial z}(A_3B_1 - A_1B_3) \\ \frac{\partial}{\partial x}(A_1B_2 - A_2B_1) & \frac{\partial}{\partial y}(A_1B_2 - A_2B_1) & \frac{\partial}{\partial z}(A_1B_2 - A_2B_1) \end{bmatrix}$$

$$\frac{\partial}{\partial q^k}(A_iB_j - A_jB_i) = \frac{\partial}{\partial q^k}(A_i)B_j + A_i\frac{\partial}{\partial q^k}(B_j) - \frac{\partial}{\partial q^k}(A_j)B_i - A_j\frac{\partial}{\partial q^k}(B_i)$$

Separating the partials of A and partials of B

$$\frac{\partial}{\partial q^k}(A_iB_j - A_jB_i) = \left(\frac{\partial}{\partial q^k}(A_i)B_j - \frac{\partial}{\partial q^k}(A_j)B_i\right) + \left(A_i\frac{\partial}{\partial q^k}(B_j) - A_j\frac{\partial}{\partial q^k}(B_i)\right)$$

then

$$\begin{aligned}
J(A\times B) = & \begin{bmatrix} A_2\frac{\partial}{\partial x}(B_3) - A_3\frac{\partial}{\partial x}(B_2) & A_2\frac{\partial}{\partial y}(B_3) - A_3\frac{\partial}{\partial y}(B_2) & A_2\frac{\partial}{\partial z}(B_3) - A_3\frac{\partial}{\partial z}(B_2) \\ A_3\frac{\partial}{\partial x}(B_1) - A_1\frac{\partial}{\partial x}(B_3) & A_3\frac{\partial}{\partial y}(B_1) - A_1\frac{\partial}{\partial y}(B_3) & A_3\frac{\partial}{\partial z}(B_1) - A_1\frac{\partial}{\partial z}(B_3) \\ A_1\frac{\partial}{\partial x}(B_2) - A_2\frac{\partial}{\partial x}(B_1) & A_1\frac{\partial}{\partial y}(B_2) - A_2\frac{\partial}{\partial y}(B_1) & A_1\frac{\partial}{\partial z}(B_2) - A_2\frac{\partial}{\partial z}(B_1) \end{bmatrix} \\
& + \begin{bmatrix} \frac{\partial}{\partial x}(A_2)B_3 - \frac{\partial}{\partial x}(A_3)B_2 & \frac{\partial}{\partial y}(A_2)B_3 - \frac{\partial}{\partial y}(A_3)B_2 & \frac{\partial}{\partial z}(A_2)B_3 - \frac{\partial}{\partial z}(A_3)B_2 \\ \frac{\partial}{\partial x}(A_3)B_1 - \frac{\partial}{\partial x}(A_1)B_3 & \frac{\partial}{\partial y}(A_3)B_1 - \frac{\partial}{\partial y}(A_1)B_3 & \frac{\partial}{\partial z}(A_3)B_1 - \frac{\partial}{\partial z}(A_1)B_3 \\ \frac{\partial}{\partial x}(A_1)B_2 - \frac{\partial}{\partial x}(A_2)B_1 & \frac{\partial}{\partial y}(A_1)B_2 - \frac{\partial}{\partial y}(A_2)B_1 & \frac{\partial}{\partial z}(A_1)B_2 - \frac{\partial}{\partial z}(A_2)B_1 \end{bmatrix} \\
= & \begin{bmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x}(B_1) & \frac{\partial}{\partial y}(B_1) & \frac{\partial}{\partial z}(B_1) \\ \frac{\partial}{\partial x}(B_2) & \frac{\partial}{\partial y}(B_2) & \frac{\partial}{\partial z}(B_2) \\ \frac{\partial}{\partial x}(B_3) & \frac{\partial}{\partial y}(B_3) & \frac{\partial}{\partial z}(B_3) \end{bmatrix} - \begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x}(A_1) & \frac{\partial}{\partial y}(A_1) & \frac{\partial}{\partial z}(A_1) \\ \frac{\partial}{\partial x}(A_2) & \frac{\partial}{\partial y}(A_2) & \frac{\partial}{\partial z}(A_2) \\ \frac{\partial}{\partial x}(A_3) & \frac{\partial}{\partial y}(A_3) & \frac{\partial}{\partial z}(A_3) \end{bmatrix} \\
= & \vec{A}^{\times} J(\vec{B}) - \vec{B}^{\times} J(\vec{A})
\end{aligned}$$