Derivation of the Generalised Euler-Lagrange Equation

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1 Motivation

How would we minimise the quantity

\[ I = \int_a^b \left( \frac{d^2 f}{dx^2} \right)^2 \, dx \quad (1) \]

The typical form of the Euler-Lagrange equation

\[ \frac{d}{dx} \left\{ \frac{\partial \mathcal{L}}{\partial (\frac{df}{dx})} \right\} - \frac{\partial \mathcal{L}}{\partial f} = 0 \quad (2) \]

provides no information, so what is a necessary condition for \( f(x) \) to minimise \( I \)?

2 A More General Question

Given a functional of the form

\[ I \left[ f(x), \frac{df}{dx}, \ldots, \frac{d^n f}{dx^n}, x \right] = \int_a^b \mathcal{L} \left( f(x), \frac{df}{dx}, \ldots, \frac{d^n f}{dx^n}, x \right) \, dx \quad (3) \]

how do we minimise the quantity \( I \)?
3 Derivation

The classic derivation of the Euler-Lagrange equation is to break it apart into the optimal solution \( f^*(x) \), a variation \( u(x) \) and a constant \( \eta \) like so

\[
f(x) = f^*(x) + \eta u(x),
\]

(4)

In order to be consistent with the boundary value problem, we require that the variation and its derivatives (\( \{u, \ldots, \frac{d^n u}{dx^n}\} \)) all vanish at the end points (i.e. the support of \( u(x) \) is \([a, b]\)). Figure 1 shows some examples of \( f(x) \).

![Figure 1: Illustration of \( f(x) \) and \( f^*(x) \)](image)

Substituting (4) into (3) we obtain

\[
I[\cdots] = \int_a^b \mathcal{L} \left( f^*(x) + \eta u(x), \ldots, \frac{d^n f^*}{dx^n} + \eta \frac{d^n u}{dx^n}, x \right) dx.
\]

(5)

From calculus we expect a critical value (either a minimum or a maximum) at \( \eta = 0 \) per definition. Recall \( f^* \) is the optimal solution, so when \( \eta = 0 \), \( f = f^* \). Because we are looking for a critical value the obvious thing to do then is to take the derivative of \( I \) with respect to \( \eta \) and evaluate the expression when \( \eta = 0 \).

Changing the notation from Liebniz to Newton \( \left( \frac{df}{dx} = f'(x) \right) \), so I do not have to carry \( \frac{d}{dx} \)
all over) the derivative yields: (beware of passing the derivative through the integral)

\[
0 = \frac{\partial I}{\partial \eta} \bigg|_{\eta=0} = \frac{\partial}{\partial \eta} \int_a^b L \left( f^*(x) + \eta u(x), \ldots, \frac{d^n f^*}{dx^n} + \eta \frac{d^n u}{dx^n}, x \right) dx \bigg|_{\eta=0}
\]

\[
= \int_a^b \frac{\partial \partial I}{\partial \eta} \bigg|_{\eta=0} \L (\ldots) dx
\]

\[
= \int_a^b \frac{\partial \partial L}{\partial f} \frac{\partial f}{\partial \eta} + \cdots + \frac{\partial \partial L}{\partial f^{(n)}} \frac{\partial f^{(n)}}{\partial \eta} \bigg|_{\eta=0} dx
\]

\[
= \int_a^b \frac{\partial \partial L}{\partial f} u + \frac{\partial \partial L}{\partial f'} u' + \cdots + \frac{\partial \partial L}{\partial f^{(n)}} u^{(n)} \bigg|_{\eta=0} dx
\]

The trick is to recognise that we may use integration by parts to get each term to be multiplied by \( u(x) \).

Performing integration by parts:

\[
\int_a^b \frac{\partial \partial L}{\partial f'} u' \bigg|_a^b dx = \frac{\partial \partial L}{\partial f'} u \bigg|_a^b dx - \int_a^b \frac{d}{dx} \left\{ \frac{\partial \partial L}{\partial f'} \right\} u dx
\]

\[
0 \text{ because } u(x) \text{ vanishes}
\]

\[
g = \frac{\partial \partial L}{\partial f'} \quad \frac{df}{dx} = \frac{du}{dx}
\]

\[
d_g \frac{dx}{dx} = d \frac{dx}{dx} \left\{ \frac{\partial \partial L}{\partial f'} \right\} f = u
\]

On the third term (we begin to see the necessity of the variation and its derivatives vanishing at the endpoints):

\[
\int_a^b \frac{\partial \partial L}{\partial f^{(n)}} u^{(n)} (x) \bigg|_a^b dx = \frac{\partial \partial L}{\partial f^{(n)}} u' \bigg|_a^b dx - \int_a^b \frac{d}{dx} \left\{ \frac{\partial \partial L}{\partial f^{(n)}} \right\} u' dx
\]

\[
0 \text{ because } u'(x) \text{ vanishes}
\]

\[
g = \frac{\partial \partial L}{\partial f^{(n)}} \quad \frac{d^2 f}{dx^2} = \frac{d^2 u}{dx^2}
\]

\[
d_g \frac{dx}{dx} = d \frac{dx}{dx} \left\{ \frac{\partial \partial L}{\partial f^{(n)}} \right\} f = u
\]

\[
= - \left( \frac{d}{dx} \left\{ \frac{\partial \partial L}{\partial f^{(n)}} \right\} u(x) \bigg|_a^b dx - \int_a^b \frac{d^2}{dx^2} \left\{ \frac{\partial \partial L}{\partial f^{(n)}} \right\} u(x) dx \right)
\]

\[
0 \text{ because } u(x) \text{ vanishes}
\]

\[
g = \frac{d}{dx} \left\{ \frac{\partial \partial L}{\partial f^{(n)}} \right\} \quad \frac{d^2 f}{dx^2} = \frac{du}{dx}
\]

\[
d_g \frac{dx}{dx} = \frac{d^2}{dx^2} \left\{ \frac{\partial \partial L}{\partial f^{(n)}} \right\} f = u
\]

\[
= \int_a^b \frac{d^2}{dx^2} \left\{ \frac{\partial \partial L}{\partial f^{(n)}} \right\} u(x) dx
\]
We can continue to perform integration by parts on the remaining terms to get:

$$0 = \int_a^b \left[ \frac{\partial L}{\partial f} - \frac{d}{dx} \left\{ \frac{\partial L}{\partial f'} \right\} + \frac{d^2}{dx^2} \left\{ \frac{\partial L}{\partial f''} \right\} - \ldots + (-1)^n \frac{d^n}{dx^n} \left\{ \frac{\partial L}{\partial f^{(n)}} \right\} \right] u(x) \, dx.$$ 

Or in a more compact form:

$$0 = \int_a^b \left[ \sum_{k=0}^n (-1)^k \frac{d^k}{dx^k} \left\{ \frac{\partial L}{\partial f^{(k)}} \right\} \right] u(x) \, dx.$$ 

Because $u(x)$ may take on almost any bounded function that vanishes at its endpoints it is necessary that

$$0 = \sum_{k=0}^n (-1)^k \frac{d^k}{dx^k} \left\{ \frac{\partial L}{\partial f^{(k)}} \right\} \quad (6)$$

and this is the generalised Euler-Lagrange equation.

### 4 Back to the Example

What is the curve, $f(x)$, such that (2) is minimised?

Substituting $\mathcal{L} = (f'')^2$ into (6) we are left with

$$\frac{d^2}{dx^2} \left\{ \frac{\partial L}{\partial f''} \right\} = \frac{d^2}{dx^2} \{2f''\} = 2f^{(4)}(x) = 0$$

$$\Rightarrow f^{(4)}(x) = 0 \quad (7)$$

We see that (7) is a differential equation to solve

$$f^{(4)}(x) = 0 \Rightarrow f^{(3)}(x) = a_3$$
$$\Rightarrow f^{(2)}(x) = a_3 x + a_2$$
$$\Rightarrow f'(x) = \frac{a_3}{2} x^2 + a_2 x + a_1$$
$$\Rightarrow f(x) = \frac{a_3}{6} x^3 + \frac{a_2}{2} x^2 + a_1 x + a_0$$

or that the curve that minimises the square of the second derivative is a cubic polynomial.