Stability of a Class of Coupled Hill's Equations and the Lorentz Oscillator Model

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Abstract

In this paper, we study the stability of a class of coupled Hill's equations. Different possible forms of solutions are discussed. Assuming certain odd-even symmetries, using Floquet theory, a simplified form of the monodromy matrix and a closed form formula for its eigenvalues, as a function of the first element of the monodromy matrix, is derived. The Lorentz oscillator model and its connection to the coupled Hill's equations is discussed and Lyapunov theory is used to prove its stability. Finally, using our formula for the eigenvalues, the stability diagrams of a system of coupled Mathieu equations, as an example of the coupled Hill's equations, are generated.

1 Introduction

In this paper, we study the stability of a class of coupled Hill's equations of the form

$$\begin{aligned} \ddot{x} + p(t)x &= -q(t)z, \\ \ddot{z} + p(t)z &= q(t)x\,, \end{aligned} \tag{1}$$

where $p(t)$ is an even continuous periodic function and $q(t)$ is an odd continuous periodic function so that $p(t+T) = p(t)$ and $q(t+T) = q(t)$ for a fixed $T > 0$. A particular example is the system of coupled Mathieu equations

$$\begin{aligned} \ddot{x} + [a + b\cos(2t)]\,x &= -c\sin(t)z, \\ \ddot{z} + [a + b\cos(2t)]\,z &= c\sin(t)x \end{aligned} \tag{2}$$

which, in a special case, is a transformation of the Lorentz Oscillator Model (LOM), a model of an electron bound to the nucleus by a harmonic potential that undergoes forced motion under electromagnetic fields subject to damping [4].

System (1) is in the form of a linear periodic differential equation, where Floquet theory is the standard tool of analysis [8]. In this paper, incorporating the odd-even symmetries in (1), we classify the solutions of this dynamical system based on the possible values of the Floquet multipliers and derive a formula for the Floquet multipliers of the system as a function of only the first element of the monodromy matrix. Finally, using our simplified formula for the Floquet multipliers, as an example, we generate the stability regions for system (2).

Further, the connection between the Lorentz Oscillator Model and system (2) is explained and stability of a general homogeneous LOM is discussed based on the Lyapunov theory.

There is a large literature on the 1-dimensional Hill's equation $\ddot{x} + p(t)x = 0$. The classic reference is [12]. This class of equations has many applications to dynamical systems, including the original question of lunar stability, inflationary dynamics in astrophysics, electron motion in crystals and accelerator physics for example. Estimates on the boundaries of stability regions of the 1-dimensional Hill's equation may be found in [6], for example. Further work may be found in e.g. Loud [11], where the case of $p(t)$ an even function is considered. Weinstein and Keller [16] studied the asymptotic behavior of the stability regions. In previous work, Hill's equation with random variation of some parameters (random forcing terms) was studied [1].

While much research has been done on the 1-dimensional Hill's equation, less work has been done on the coupled Hill's equations which is the subject of the current paper. Hsu [7] studied a restricted class of coupled Hill's equations which can be transformed to separate independent 1-dimensional Hill's equations. Mahmoud [13] analyzed a class of coupled Hill's equations using perturbation theory, where some parameters are assumed to be small. See also [**?**],

[?], [?]. Coupled Mathieu equations, which are special cases of coupled Hill's equations, have been studied in e.g. [3], [5] and [10]. Our work extends this literature to an interesting class of coupled Hill's equations motivated by the transformation of the Lorentz Oscillator Model. We remark that, in contrast to most systems analyzed in the literature, the system (2) does not have a canonical Hamiltonian structure (the coupling terms terms destroy the canonical structure) which makes the analysis of particular interest. While a noncanonical form can be found for certain values of the parameters the corresponding Hamiltonian is not postive in a useful sense. The system is also not of gyroscopic type (as in [2] for example).

The contents of the paper are as follows. In Section 2, we describe general coupled Hill's equations and the particular case of two such equations. In Section 3, we consider the particular symmetric case of interest in this paper and classify the various possible solutions. In Section 4, we analyze stability using Floquet theory and present a general result on the structure of the monodromy matrix. In particular, a formula for the eigenvalues of the monodromy matrix is derived as a function of only the first element of the matrix. In Section 5, we relate the coupled Hill's equations to the equations of the Lorentz Oscillator Model (LOM) and study the stability of the LOM. Section 6 presents numerical stability regions in parameter space for the system consisting of two coupled Mathieu equations.

2 General Coupled Hill's equations

We begin by defining a general system of coupled Hill's equations.

Definition 2.1. A general n-dimensional system of coupled Hill's equations (CHE) is a system of the form [7]

$$\ddot{\boldsymbol{x}} + B(t)\boldsymbol{x} \quad = \quad 0,$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $B(t)$ is an $n \times n$ periodic real matrix that is a continuous function of t.

In this paper, we study the the case when $n = 2$, that is, the system

$$\begin{aligned} \ddot{x} + p_x(t)x &= -q_x(t)z, \\ \ddot{z} + p_z(t)z &= q_z(t)x\,, \end{aligned} \tag{3}$$

where $p_x(t), q_x(t), p_z(t)$ and $q_z(t)$ are continuous periodic functions with a common period $T > 0$. Defining $\boldsymbol{v} = [x, \dot{x}, z, \dot{z}]^{tr}$, with ^{tr} denoting transpose, this system can be written in the form

$$\frac{d\boldsymbol{v}}{dt} = A(t)\boldsymbol{v}, \tag{4}$$

where

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -p_x(t) & 0 & -q_x(t) & 0 \\ 0 & 0 & 0 & 1 \\ q_z(t) & 0 & -p_z(t) & 0 \end{bmatrix}.$$

The matrix $A(t)$ is periodic with period T. If $\Phi(t)$ is a fundamental matrix of (4), then $\mathbb{M} = \Phi^{-1}(0)\Phi(T)$ is a monodromy matrix of (4). From standard Floquet theory [8], the determinant of the monodromy matrix is given by

$$\det(\mathbb{M}) = \exp\left[\int_0^T \operatorname{trace}(A(s))ds\right].$$

Since $\operatorname{trace}(A(t)) \equiv 0$, we have $\det(\mathbb{M}) = 1$.

For the rest of this paper we assume that $\mathbb{M} = \Phi(T)$, where $\Phi(t)$ is the fundamental matrix such that $\Phi(0)$ is the 4×4 identity matrix. We note that the Floquet analysis is independent of the choice of monodromy matrix.

Remark 2.2. Since $\det(\mathbb{M})$ is equal to the product of the Floquet multipliers (i.e., eigenvalues of the monodromy matrix), this result shows that the product of the Floquet multipliers is unity, that is,

$$\prod_{j=1}^{n} \rho_j = 1, \tag{5}$$

where ρ_j are the Floquet multipliers. Thus, the solutions of the general CHE are never asymptotically stable because if the system is stable, $|\rho_j| \leq 1$ for every Floquet multiplier ρ_j, and from equation (5), necessarily $|\rho_j| = 1$ for $j = 1, 2, \ldots, n$.

Note that stability of the solutions of system (4) means that all solutions remain bounded, or equivalently, the zero solution of the system is globally Lyapunov stable.

In the next section, where various forms of the solutions of a class of the CHE are derived, we use the result (5).

3 Classifying the Solutions of the Coupled Hill's Equations

In this section, we study a special class of the general CHE defined in equation (3). We assume that in system (3), $p_x = p_z = p$ and $q_x = q_z = q$ to obtain the forms

$$\begin{aligned} \ddot{x} + p(t)x &= -q(t)z, \\ \ddot{z} + p(t)z &= q(t)x\,, \end{aligned} \tag{6}$$

where $p(t)$ and $q(t)$ are continuous periodic functions. In Proposition 3.4 below, we classify solutions of this system. Later, these results will be used in the study of Floquet theory and stability of the CHE.

Lemma 3.1. Suppose that ρ is a Floquet multiplier of the CHE defined in equation (6), and assume that μ is a corresponding Floquet multiplier, that is, $\rho = e^{\mu T}$. Assuming that the CHE is non-trivial (i.e., $p(t)$ and $q(t)$ are not both zero functions), there exist T-periodic functions $p_1(t)$ and $p_2(t)$ such that

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} e^{\mu t} \quad \text{and} \quad \begin{bmatrix} -p_2(t) \\ p_1(t) \end{bmatrix} e^{\mu t}$$

are two independent solutions to the CHE.

Proof. From Floquet analysis [8], for the Floquet multiplier ρ, there exists a solution of the form

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} e^{\mu t}$$

for the CHE. Due to the symmetry that exists in (6), one can easily check that if $[x(t), z(t)]$ is a solution, so is $[-z(t), x(t)]$. As a result,

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -p_2(t) \\ p_1(t) \end{bmatrix} e^{\mu t}$$

is a second solution to the CHE. Since the CHE is assumed to be non-trivial,

$$\det\left(\begin{bmatrix} p_1(t) & -p_2(t) \\ p_2(t) & p_1(t) \end{bmatrix} \right) = p_1^2(t) + p_2^2(t) \neq 0.$$

As a result, the two solutions are independent. $\square$

Corollary 3.2. By the above lemma, the Floquet multipliers of the CHE defined in (6) have either multiplicity 2 or 4. Therefore, the Floquet multipliers can either be written as $\{\rho_1, \rho_2\}$, where each has multiplicity 2 or ρ_1 is the only Floquet multiplier of the CHE.

Note that in the case of the trivial CHE, that is, when $p(t) \equiv q(t) \equiv 0$, the above corollary holds, since in the case of the trivial CHE all the Floquet multipliers are equal to unity.

Proposition 3.3. Suppose that the Floquet multipliers of the CHE defined in equation (6) are $\lambda_1, \lambda_2, \lambda_3$ and λ_4. From Corollary 3.2, without loss of generality, assume that $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$. Let $\rho_1 = \lambda_1$ and $\rho_2 = \lambda_3$. We have $\rho_1\rho_2 = 1$.

Proof. From Remark 2.2, $(\rho_1\rho_2)^2 = 1$. We immediately see that $\rho_1\rho_2 = 1$ or $\rho_1\rho_2 = -1$. We show below that the second case cannot occur.

Each CHE can be identified by $(p(t), q(t))$ where $p(t)$ and $q(t)$ are continuous functions and $p(t+T) = p(t)$ and $q(t+T) = q(t)$. Let's denote the space of all such $(p(t), q(t))$ defined on $[0, T]$ by $\mathcal{Q}$. To each $(p, q) \in \mathcal{Q}$ corresponds a monodromy matrix $\mathbb{M}$. Therefore, we can define the function $\mathbb{M} : \mathcal{Q} \to \mathbb{R}^{4\times 4}$, where $\mathbb{M}(p, q)$ is the monodromy matrix of the CHE defined in equation (6), which can be written as a first order system

$$\frac{d\boldsymbol{v}}{dt} = A(t)\boldsymbol{v}, \tag{7}$$

where

$$A(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -p(t) & 0 & -q(t) & 0 \\ 0 & 0 & 0 & 1 \\ q(t) & 0 & -p(t) & 0 \end{bmatrix}, \tag{8}$$

and $\boldsymbol{v} = [x, \dot{x}, z, \dot{z}]^{tr}$. Since the space $\mathcal{Q}$ is closed under summation and scalar multiplication, it is a linear subspace of $C^0([0, T], \mathbb{R}^2)$, the space of continuous functions from $[0, T]$ to $\mathbb{R}^2$. We endow $\mathcal{Q}$ with sup norm, denoted by $\|\cdot\|_\infty$. Our goal is to show that the function $\mathbb{M}(\cdot, \cdot)$ is continuous at any point $(p, q) \in \mathcal{Q}$. To this end, let $(p_k(t), q_k(t))$ be a sequence in $\mathcal{Q}$ that approaches an element $(p(t), q(t)) \in \mathcal{Q}$, that is, $\|(p_k(t), q_k(t)) - (p(t), q(t))\|_\infty \to 0$. Denote the CHE corresponding to (p_k, q_k) by $\dot{\boldsymbol{v}} = A_k(t)\boldsymbol{v}$ as defined in equations (7) and (8), and denote the CHE corresponding to the point $(p, q) \in \mathcal{Q}$ by $\dot{\boldsymbol{v}} = A(t)\boldsymbol{v}$. Since $\|(p_k(t), q_k(t)) - (p(t), q(t))\|_\infty \to 0$, on $[0, T]$ we have

$$\left\| \int_0^t (A_k(s) - A(s)) ds \right\|_\infty \to 0.$$

As shown in [15], this result implies that with the same initial conditions, the solution $\boldsymbol{v}_k(t)$ of the system $\dot{\boldsymbol{v}} = A_k(t)\boldsymbol{v}$ uniformly converges to the solution $\boldsymbol{v}(t)$ of the system $\dot{\boldsymbol{v}} = A(t)\boldsymbol{v}$ as $k \to \infty$. This result proves the continuity of the function $\mathbb{M}(p, q)$ on $\mathcal{Q}$. Define

$$\begin{aligned} \mathcal{Q}_1 &= \{(p, q) \in \mathcal{Q} | \rho_1\rho_2 = 1\}, \\ \mathcal{Q}_2 &= \{(p, q) \in \mathcal{Q} | \rho_1\rho_2 = -1\}. \end{aligned}$$

Therefore, $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ and $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \emptyset$. If we show that $\mathcal{Q}_2$ is empty we are done. It is easy to check that $\mathcal{Q}_1$ is non-empty (for instance, one can numerically check that $(\cos(2\pi t/T), \sin(2\pi t/T)) \in \mathcal{Q}_1$). Let $(p_1, q_1) \in \mathcal{Q}_1$. Assume that $\mathcal{Q}_2$ is nonempty and let $(p_2, q_2) \in \mathcal{Q}_2$. We will show that the non-emptiness assumption of $\mathcal{Q}_2$ leads to a contradiction. Let $r(\tau)$ be a continuous curve in $\mathcal{Q}$ that starts from (p_1, q_1) and ends at (p_2, q_2), that is, $r(\tau)$ is defined on an interval $[\tau_1, \tau_2]$, and

$$\begin{aligned} r(\tau_1) &= (p_1, q_1) \in \mathcal{Q}_1, \\ r(\tau_2) &= (p_2, q_2) \in \mathcal{Q}_2. \end{aligned}$$

Let $m(\tau) = \mathbb{M}(r(\tau))$. Since $\mathbb{M}(p, q)$ and $r(\tau)$ are continuous, we conclude that $m(\tau)$ is a continuous curve in $\mathbb{R}^{4\times 4}$. Let $P_{m(\tau)}$ be the characteristic polynomial of $m(\tau)$. By endowing the complex numbers with a proper order, the function which maps the coefficients of $P_{m(\tau)}$ to its roots, (ρ_1, ρ_2), is a well-defined continuous function [14]. Therefore, the functions $\rho_1(\tau)$ and $\rho_2(\tau)$ are well-defined and continuous on $[\tau_1, \tau_2]$. Since $r(\tau_1) \in \mathcal{Q}_1$, $\rho_1(\tau_1)\rho_2(\tau_1) = 1$, and since $r(\tau_2) \in \mathcal{Q}_2$, $\rho_1(\tau_2)\rho_2(\tau_2) = -1$. However, this is a contradiction because 1 and -1 are the only values that the continuous function $\rho_1(\tau)\rho_2(\tau)$ can take. Since we know that $\mathcal{Q}_1$ is non-empty, we conclude that $\mathcal{Q}_2$ has to be empty. $\square$

Proposition 3.4. Let ρ_1, ρ_2 be the Floquet multipliers of the CHE defined in equation (6). Then one of the following cases holds.

1. ρ_1 is real and $|\rho_1| \neq 1$. In this case, $\rho_2 = 1/\rho_1$ and the system is unstable. Moreover, if $\rho_1 > 0$, $\rho_1 = \exp(\mu T)$ for a real number μ and there exist T-periodic functions $p_1(t), p_2(t), p_3(t)$ and $p_4(t)$ such that

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} e^{\mu t}, \begin{bmatrix} -p_2(t) \\ p_1(t) \end{bmatrix} e^{\mu t}, \begin{bmatrix} p_3(t) \\ p_4(t) \end{bmatrix} e^{-\mu t} \quad \text{and} \quad \begin{bmatrix} -p_4(t) \\ p_3(t) \end{bmatrix} e^{-\mu t}$$

are four independent solutions of the system. When $\rho_1 < 0$, $\rho_1 = \exp(\mu T + i\pi)$ for some real number μ. In this case, the four independent solutions are in the form

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} e^{\mu t}, \begin{bmatrix} -q_2(t) \\ q_1(t) \end{bmatrix} e^{\mu t}, \begin{bmatrix} q_3(t) \\ q_4(t) \end{bmatrix} e^{-\mu t} \quad \text{and} \quad \begin{bmatrix} -q_4(t) \\ q_3(t) \end{bmatrix} e^{-\mu t}$$

for $2T$-periodic functions $q_1(t), q_2(t), q_3(t)$ and $q_4(t)$.

2. $\rho_1 = \rho_2 = 1$ and the system can be stable or unstable. In the stable case, there exist T-periodic functions $p_1(t), p_2(t), p_3(t)$ and $p_4(t)$ such that

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}, \begin{bmatrix} -p_2(t) \\ p_1(t) \end{bmatrix}, \begin{bmatrix} p_3(t) \\ p_4(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -p_4(t) \\ p_3(t) \end{bmatrix}$$

are four independent solutions of the system.

In the unstable case, there exist T-periodic functions $p_1(t), p_2(t), p_3(t)$ and $p_4(t)$ such that

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}, \begin{bmatrix} -p_2(t) \\ p_1(t) \end{bmatrix}, \begin{bmatrix} tp_1(t) + p_3(t) \\ tp_2(t) + p_4(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -tp_2(t) - p_4(t) \\ tp_1(t) + p_3(t) \end{bmatrix}$$

are four independent solutions of the system.

3. $\rho_1 = \rho_2 = -1$ and the system can be stable or unstable. In the stable case, there exist $2T$-periodic functions $q_1(t), q_2(t), q_3(t)$ and $q_4(t)$ such that

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}, \begin{bmatrix} -q_2(t) \\ q_1(t) \end{bmatrix}, \begin{bmatrix} q_3(t) \\ q_4(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -q_4(t) \\ q_3(t) \end{bmatrix}$$

are four independent solutions of the system.

In the unstable case, there exist $2T$-periodic functions $q_1(t), q_2(t), q_3(t)$ and $q_4(t)$ such that

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}, \begin{bmatrix} -q_2(t) \\ q_1(t) \end{bmatrix}, \begin{bmatrix} tq_1(t) + q_3(t) \\ tq_2(t) + q_4(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -tq_2(t) - q_4(t) \\ tq_1(t) + q_3(t) \end{bmatrix}$$

are four independent solutions of the system.

4. ρ_1 is not real and $\rho_2 = \bar{\rho}_1$, that is, ρ_1 and ρ_2 are complex conjugates. In this case, $\rho_1 = \exp(i\sigma T)$ and $\rho_2 = \exp(-i\sigma T)$ for some $\sigma > 0$, and the system is stable. Moreover, there exist real T-periodic functions $p_1(t), r_1(t), p_2(t)$ and $r_2(t)$ such that the real and imaginary parts of

$$\begin{bmatrix} p_1(t) + ir_1(t) \\ p_2(t) + ir_2(t) \end{bmatrix} e^{i\sigma t} \quad \text{and} \quad \begin{bmatrix} -p_2(t) - ir_2(t) \\ p_1(t) + ir_1(t) \end{bmatrix} e^{i\sigma t}$$

define four independent solutions of the system.

Proof. By Proposition 3.3, $\rho_1\rho_2 = 1$. We have the following cases.

Case 1. Since $|\rho_1| \neq 1$, at least one of the Floquet multipliers lies outside the unit circle. Hence, the system is unstable. By standard Floquet theory, since ρ_1 is real, if $\rho_1 > 0$, there exists a real number μ such that $\rho_1 = \exp(\mu T)$ and there exist T-periodic functions $p_1(t)$ and $p_2(t)$ such that

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} e^{\mu t}$$

is a solution. By Lemma 3.1 a second solution

$$\begin{bmatrix} -p_2(t) \\ p_1(t) \end{bmatrix} e^{\mu t}$$

exists. Since $\rho_2 = 1/\rho_1$, $\rho_2 = \exp(-\mu T)$. Similarly, there exist T-periodic functions $p_3(t)$ and $p_4(t)$ such that

$$\begin{bmatrix} p_3(t) \\ p_4(t) \end{bmatrix} e^{-\mu t} \quad \text{and} \quad \begin{bmatrix} -p_4(t) \\ p_3(t) \end{bmatrix} e^{-\mu t}$$

are solutions. In the case when $\rho_1 < 0$, we have $\rho_1 = \exp((\mu + i\pi/T)T)$ for some real number μ. The results follow similarly to the case $\rho_1 > 0$, except that since $\exp(i\pi t/T) = \cos(\pi t/T) + i\sin(\pi t/T)$, we obtain $2T$-periodic functions q_1, q_2, q_3 and q_4 instead of T-periodic functions p_1, p_2, p_3 and p_4.

Case 2. This case is similar to the case above with $\mu = 0$. However, from standard Floquet theory, in the unstable case, a t multiple of the first two solutions has to be added to the third and fourth solutions.

Case 3. Since $\rho_1 = -1$, we can set $\mu = i\pi/T$. In the stable case, there exist T-periodic functions $p_1(t)$ and $p_2(t)$ such that

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} e^{\frac{i\pi}{T}t} \tag{9}$$

is a solution. Writing $\exp(i\pi t/T) = \cos(\pi t/T) + i\sin(\pi t/T)$, since $\cos(\pi t/T)$ and $\sin(\pi t/T)$ are $2T$-periodic real functions, from equation (9), we conclude that there exist $2T$-periodic functions $q_1(t)$ and $q_2(t)$ such that

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

is a solution. By Lemma 3.1, a second solution of the form

$$\begin{bmatrix} -q_2(t) \\ q_1(t) \end{bmatrix}$$

exists. Similarly, there exist $2T$-periodic functions $q_3(t)$ and $q_4(t)$ such that

$$\begin{bmatrix} q_3(t) \\ q_4(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -q_4(t) \\ q_3(t) \end{bmatrix}$$

are solutions. In the unstable case, t multiples of the first two solutions are added to the third and fourth solutions.

Case 4. Since $\rho_2 = \bar{\rho}_1$ and $\rho_1\rho_2 = 1$, we get $|\rho_1| = 1$. Therefore, $\rho_1 = \exp(i\sigma T)$ for some real number σ. The rest follows from standard Floquet theory and Lemma 3.1. □

4 Stability of the Symmetric Coupled Hill's Equations: Floquet Theory

In this section, we impose further conditions on the CHE studied in the previous section and study the stability of the resulting system by applying the Floquet theory. Consider the CHE

$$\begin{aligned} \ddot{x} + p(t)x &= -q(t)z, \\ \ddot{z} + p(t)z &= q(t)x\,, \end{aligned} \tag{10}$$

where $p(t)$ is now assumed to be an even continuous periodic function and $q(t)$ an odd continuous periodic function of common period $T > 0$. Because of the odd-even symmetries, we refer to this system as *symmetric* CHE or simply SCHE. We show that out of 16 elements of the monodromy matrix of the SCHE at most 4 of them are independent. Moreover, we derive a formula for the Floquet multipliers as a function of only the first element of the monodromy matrix. In Section 6, using this result, the stability regions of an SCHE are found numerically.

Lemma 4.1. Let $[x(t), y(t)]$ be a solution of the T-periodic SCHE with the following initial data:

$$x(0) = 1, \dot{x}(0) = 0, z(0) = 0, \dot{z}(0) = 0.$$

Then

1. $x(t)$ is an even function and $z(t)$ is an odd function.

2. $\dot{z}(T) = 0$.

Proof. Since $[x(t), z(t)]$ is a solution,

$$\begin{array}{rcl} \ddot{x}(-t) + p(-t)x(-t) & = & -q(-t)z(-t), \\ \ddot{z}(-t) + p(-t)z(-t) & = & q(-t)x(-t)\,. \end{array} \tag{11}$$

Since $p(t)$ is odd and $q(t)$ is even, this reduces to

$$\begin{array}{rcl} \ddot{x}(-t) + p(t)x(-t) & = & q(t)z(-t), \\ \ddot{z}(-t) + p(t)z(-t) & = & -q(t)x(-t)\,. \end{array}$$

If $\hat{x}(t) = x(-t)$ and $\hat{z}(t) = -z(-t)$, then $\ddot{\hat{x}}(t) = \ddot{x}(-t)$ and $\ddot{\hat{z}}(t) = -\ddot{z}(-t)$. Substituting these equations into the system above, we obtain

$$\begin{array}{rcl} \ddot{\hat{x}}(t) + p(t)\hat{x}(t) & = & -q(t)\hat{z}(t), \\ \ddot{\hat{z}}(t) + p(t)\hat{z}(t) & = & q(t)\hat{x}(t). \end{array}$$

Consequently, $[\hat{x}(t), \hat{z}(t)]$ is a solution of the SCHE as well. On the other hand, by definition of $\hat{x}$ and $\hat{z}$ and from the initial conditions, the following holds:

$$\begin{array}{rcccl} \hat{x}(0) & = & x(0) & = & 1, \\ \dot{\hat{x}}(0) & = & -\dot{x}(0) & = & 0, \\ \hat{z}(0) & = & -z(0) & = & 0, \\ \dot{\hat{z}}(0) & = & \dot{z}(0) & = & 0. \end{array}$$

Therefore, $[\hat{x}(t), \hat{z}(t)]$ satisfies the same initial conditions as $[x(t), y(t)]$. By the uniqueness property of the initial value problem, we conclude that $\hat{x}(t) = x(t)$ and $\hat{z}(t) = z(t)$. Thus, by definition of $\hat{x}(t)$ and $\hat{z}(t)$, we get $x(-t) = x(t)$ and $z(-t) = -z(t)$, that is, $x(t)$ is even and $z(t)$ is odd.

To prove $\dot{z}(T) = 0$, we examine all possible solutions discussed in Proposition 3.4.

Case 1. ρ_1 is a positive real number and $|\rho_1| \neq 1$. In this case, by Proposition 3.4, the general solution is

$$\begin{array}{rcl} x(t) & = & (Ap_1(t) - Bp_2(t))e^{\mu t} + (Cp_3(t) - Dp_4(t))e^{-\mu t}, \\ z(t) & = & (Ap_2(t) + Bp_1(t))e^{\mu t} + (Cp_4(t) + Dp_3(t))e^{-\mu t}\,. \end{array}$$

Using the fact that $z(t)$ is odd, we can show that

$$z(t) \quad = \quad A(p_2(t)e^{\mu t} - p_2(-t)e^{-\mu t}) + B(p_1(t)e^{\mu t} - p_1(-t)e^{-\mu t})\,.$$

Since $\dot{z}(0) = 0$, we have

$$A(\dot{p}_2(0) + \mu p_2(0)) + B(\dot{p}_1(0) + \mu p_1(0)) \quad = \quad 0\,. \tag{12}$$

On the other hand, since $p_1(t)$ and $p_2(t)$ are T-periodic,

$$\begin{aligned} \dot{z}(T) \quad &= \quad [A(\dot{p}_2(0) + \mu p_2(0)) + B(\dot{p}_1(0) + \mu p_1(0))]e^{\mu T} \\ &+ \quad [A(\dot{p}_2(0) + \mu p_2(0)) + B(\dot{p}_1(0) + \mu p_1(0))]e^{-\mu T}\,. \end{aligned}$$

Therefore, by equation (12), $\dot{z}(T) = 0$.

When $\rho_1 < 0$, same calculations, with μ replaced by $\mu + i\pi/T$, proves that $\dot{z}(T) = 0$.

Case 2. $\rho_1 = \rho_2 = 1$. In the stable case, the solutions are T-periodic. Hence, $\dot{z}(T) = \dot{z}(0) = 0$. In the unstable case,

$$z(t) \quad = \quad Ap_2(t) + Bp_1(t) + Ctp_4(t) + Dtp_3(t)\,.$$

Since z is odd, and p_j are all periodic functions, we conclude that $p_1(t)$ and $p_2(t)$ are odd, and $p_3(t)$ and $p_4(t)$ are even functions. Since $\dot{z}(0) = 0$, we have

$$A\dot{p}_2(0) + B\dot{p}_1(0) + Cp_4(0) + Dp_3(0) \quad = \quad 0\,. \tag{13}$$

Using the fact that p_1, p_2, p_3 and p_4 are T-periodic,

$$\dot{z}(T) = A\dot{p}_2(0) + B\dot{p}_1(0) + Cp_4(0) + Dp_3(0) + T(C\dot{p}_4(0) + D\dot{p}_3(0))\,.$$

By equation (13) and the fact that p_3 and p_4 are even functions, from the equation above, $\dot{z}(T) = 0$.

Case 3. $\rho_1 = \rho_2 = -1$. In the stable case,

$$z(t) \quad = \quad Aq_2(t) + Bq_1(t) + Cq_4(t) + Dq_3(t),$$

where q_1 and q_2 are $2T$-periodic functions. Therefore, $z(t)$ is an odd $2T$-periodic function. Since $z(t)$ is continuous, it has a convergent Fourier series expansion:

$$z(t) = b_1 \sin(\frac{\pi}{T}t) + b_3 \sin(\frac{3\pi}{T}t) + b_5 \sin(\frac{5\pi}{T}t) + \cdots$$

Because $\dot{z}(0) = 0$, we get $b_1 + 3b_3 + 5b_5 + \cdots = 0$. From this equality, it is easy to see that $\dot{z}(T) = 0$. In the unstable case,

$$z(t) \quad = \quad Aq_2(t) + Bq_1(t) + Cq_4(t) + Dq_3(t) + Ctq_2(t) + Dtq_1(t)\,.$$

Since $z(t)$ is an odd function, and q_j are periodic functions, we conclude that $Aq_2(t)+Bq_1(t)+Cq_4(t)+Dq_3(t)$ is odd and $Cq_2(t)+Dq_1(t)$ is an even function. For simplicity we write $z(t) = z_1(t) + tz_2(t)$ with $z_1(t) = Aq_2(t) + Bq_1(t) + Cq_4(t) + Dq_3(t)$ and $z_2(t) = Cq_2(t) + Dq_1(t)$, where $z_1(t)$ is an odd $2T$-periodic and $z_2(t)$ is an even $2T$-periodic function. Since $\dot{z}(0) = 0$,

$$\dot{z}_1(0) + z_2(0) = 0\,.$$

On the other hand,

$$\dot{z}(T) = \dot{z}_1(T) + z_2(T) + T\dot{z}_2(T)\,. \tag{14}$$

Since $\dot{z}_1(t) + z_2(t)$ is an even $2T$-periodic function and $\dot{z}_1(0) + z_2(0) = 0$, from its Fourier series expansion, it is easy to show that $\dot{z}_1(T) + z_2(T) = 0$. On the other hand, because $\dot{z}_2(t)$ is an odd $2T$-periodic function, from its Fourier series expansion, one can show that $\dot{z}_2(T) = 0$. Hence, as desired, from equation (14), $\dot{z}(T) = 0$.

Case 4. ρ_1 is not real. For convenience we denote $p_1(t)$ by p_1, $p_2(t)$ by p_2, $\cos(\sigma t)$ by c and $\sin(\sigma t)$ by s. In this case, by Proposition 3.4, the general solution is

$$z(t) = (Ap_2 + Br_2 + Cp_1 + Dr_1)c + (-Ar_2 + Bp_2 - Cr_1 + Dp_1)s\,.$$

Since ρ_1 is not real, $\sigma \neq 2n\pi/T$ for any integer n. Using the fact that $z(t)$ is odd, p_j and r_j are T-periodic, and $\sigma \neq 2n\pi/T$, we can show that $R(t) := Ap_2+Br_2+Cp_1 + Dr_1$ is an odd T-periodic function and $S(t) := -Ar_2 + Bp_2 - Cr_1 + Dp_1$ is an even T-periodic function. With these definitions of $R(t)$ and $S(t)$, we have

$$z(t) = R(t)\cos(\sigma t) + S(t)\sin(\sigma t).$$

From the fact that $\dot{z}(0) = 0$, $R(t)$ is odd and $S(t)$ is even, we have

$$\dot{R}(0) + \sigma S(0) = 0. \tag{15}$$

On the other hand, because $R(t)$ and $S(t)$ are T-periodic,

$$\dot{z}(T) = (\dot{R}(0) + \sigma S(0))\cos(\sigma T) + (-\sigma R(0) + \dot{S}(0))\sin(\sigma T).$$

Using equation (15) and the fact that $R(t)$ is odd and $S(t)$ is even, we conclude that $\dot{z}(T) = 0$. $\square$

Lemma 4.2. Let $[x(t), y(t)]$ be a solution of the T-periodic SCHE with the following initial data

$$x(0) = 0, \dot{x}(0) = 1, z(0) = 0, \dot{z}(0) = 0\,.$$

Then

1. $x(t)$ is an odd function and $z(t)$ is an even function.

2. $z(T) = 0$.

Proof. The proof is similar to that of Lemma 4.1. □

Lemma 4.3. The monodromy matrix of a T-periodic SCHE has the form

$$\mathbb{M} = \begin{bmatrix} \mathbb{A} & -\mathbb{B} \\ \mathbb{B} & \mathbb{A} \end{bmatrix},$$

where $\mathbb{A}$ and $\mathbb{B}$ are 2×2 matrices.

Proof. Let $\boldsymbol{\xi}(t) = [x(t), \dot{x}(t), z(t), \dot{z}(t)]^{tr}$. Let $\boldsymbol{e}_j$ be the jth column of the $4{\times}4$ identity matrix. Then,

$$\mathbb{M} = [\boldsymbol{\xi}_1(T), \boldsymbol{\xi}_2(T), \boldsymbol{\xi}_3(T), \boldsymbol{\xi}_4(T)],$$

where $\boldsymbol{\xi}_j(0) = \boldsymbol{e}_j$. If $[x(t), z(t)]$ is a solution to the T-periodic SCHE, so is $[-z(t), x(t)]$. Therefore, if $\boldsymbol{\xi}_1(t) = [x_1(t), \dot{x}_1(t), z_1(t), \dot{z}_1(t)]^{tr}$ is the solution to the system with initial condition $\boldsymbol{\xi}_1(0) = \boldsymbol{e}_1$, $\hat{\boldsymbol{\xi}}_1(t) = [-z_1(t), -\dot{z}_1(t), x_1(t), \dot{x}_1(t)]^{tr}$ is a solution to the system with initial condition $\hat{\boldsymbol{\xi}}_1(0) = \boldsymbol{e}_3$. Similarly, if $\boldsymbol{\xi}_2(t) = [x_2(t), \dot{x}_2(t), z_2(t), \dot{z}_2(t)]^{tr}$ is the solution to the system with initial condition $\boldsymbol{\xi}_2(0) = \boldsymbol{e}_2$, then $\hat{\boldsymbol{\xi}}_2(t) = [-z_2(t), -\dot{z}_2(t), x_2(t), \dot{x}_2(t)]^{tr}$ is a solution to the system with initial condition $\hat{\boldsymbol{\xi}}_2(0) = \boldsymbol{e}_4$. Consequently,

$$\begin{array}{cccccccccccc} \mathbb{M}_{13} & = & -\mathbb{M}_{31}, & \mathbb{M}_{14} & = & -\mathbb{M}_{32}, & \mathbb{M}_{23} & = & -\mathbb{M}_{41}, & \mathbb{M}_{24} & = & -\mathbb{M}_{42}, \\ \mathbb{M}_{33} & = & \mathbb{M}_{11}, & \mathbb{M}_{34} & = & \mathbb{M}_{12}, & \mathbb{M}_{43} & = & \mathbb{M}_{21}, & \mathbb{M}_{44} & = & \mathbb{M}_{22}. \end{array}$$

This proves that the monodromy matrix has the form

$$\mathbb{M} = \begin{bmatrix} \mathbb{A} & -\mathbb{B} \\ \mathbb{B} & \mathbb{A} \end{bmatrix},$$

where $\mathbb{A}$ and $\mathbb{B}$ are 2×2 matrices. □

Theorem 4.4. The monodromy matrix of a T-periodic SCHE has the form

$$\mathbb{M} = \begin{bmatrix} a_{11} & a_{12} & -a_{31} & 0 \\ a_{21} & a_{11} & 0 & a_{31} \\ a_{31} & 0 & a_{11} & a_{12} \\ 0 & -a_{31} & a_{21} & a_{11} \end{bmatrix}. \tag{16}$$

The eigenvalues of $\mathbb{M}$ (i.e., the Floquet multipliers) have the form

$$\lambda = a_{11} \pm \sqrt{a_{11}^2 - 1}\,. \tag{17}$$

Proof. By Lemma 4.1, 4.2 and 4.3, the monodromy matrix $\mathbb{M}$ has the form

$$\mathbb{M} = \begin{bmatrix} a_{11} & a_{12} & -a_{31} & 0 \\ a_{21} & a_{22} & 0 & -a_{42} \\ a_{31} & 0 & a_{11} & a_{12} \\ 0 & a_{42} & a_{21} & a_{22} \end{bmatrix}.$$

Suppose $\boldsymbol{\xi}_j(0) = \boldsymbol{e}_j$ for $j = 1, 2, 3, 4$. Define

$$\mathbb{M}' = [\boldsymbol{\xi}_1(-T), \boldsymbol{\xi}_2(-T), \boldsymbol{\xi}_3(-T), \boldsymbol{\xi}_4(-T)].$$

By odd-even symmetries that exist in the SCHE, one can show that

$$\mathbb{M}' = \begin{bmatrix} a_{11} & -a_{12} & a_{31} & 0 \\ -a_{21} & a_{22} & 0 & a_{42} \\ -a_{31} & 0 & a_{11} & -a_{12} \\ 0 & -a_{42} & -a_{21} & a_{22} \end{bmatrix}.$$

From standard Floquet theory [8], we have $\mathbb{M}\mathbb{M}' = \mathbb{I}_{4\times4}$, where $\mathbb{I}_{4\times4}$ is the 4×4 identity matrix. This equation gives rise to the following equalities:

$$\begin{aligned} a_{12}(a_{22} - a_{11}) &= 0, \\ a_{21}(a_{22} - a_{11}) &= 0, \\ a_{21}(a_{42} + a_{31}) &= 0, \\ a_{12}(a_{42} + a_{31}) &= 0, \\ a_{11}^2 + a_{31}^2 - a_{12}a_{21} &= 1, \\ a_{42}^2 + a_{22}^2 - a_{12}a_{21} &= 1. \end{aligned}$$

This system has two sets of solutions:

1. $a_{11} = a_{22}, a_{42} = -a_{31}, a_{11}^2 + a_{31}^2 = 1 + a_{12}a_{21}$.

2. $a_{12} = 0, a_{21} = 0, a_{11}^2 + a_{31}^2 = a_{42}^2 + a_{22}^2 = 1$.

In the first case, the monodromy matrix has the form

$$\mathbb{M}_1 = \begin{bmatrix} a_{11} & a_{12} & -a_{31} & 0 \\ a_{21} & a_{11} & 0 & a_{31} \\ a_{31} & 0 & a_{11} & a_{12} \\ 0 & -a_{31} & a_{21} & a_{11} \end{bmatrix}, \tag{18}$$

and its characteristic equation is

$$\det(\mathbb{M}_1 - \rho\mathbb{I}) = (\rho^2 - 2a_{11}\rho + 1)^2.$$

From this equation, the eigenvalues are found to be

$$\rho_1 = a_{11} + \sqrt{a_{11}^2 - 1} \quad \text{and} \quad \rho_2 = a_{11} - \sqrt{a_{11}^2 - 1},$$

where each has a multiplicity 2, which is consistent with Corollary 3.2.

In the second case, the monodromy matrix is

$$\mathbb{M}_2 = \begin{bmatrix} a_{11} & 0 & -a_{31} & 0 \\ 0 & a_{22} & 0 & -a_{42} \\ a_{31} & 0 & a_{11} & 0 \\ 0 & a_{42} & 0 & a_{22} \end{bmatrix},$$

and its characteristic equation is

$$\det(\mathbb{M}_2 - \rho\mathbb{I}) = (\rho^2 - 2a_{11}\rho + 1)(\rho^2 - 2a_{22}\rho + 1).$$

From this equation, the eigenvalue are found to be

$$\rho_1 = a_{11} \pm \sqrt{a_{11}^2 - 1} \quad \text{and} \quad \rho_2 = a_{22} \pm \sqrt{a_{22}^2 - 1}.$$

However, by Corollary 3.2 either $a_{11} = a_{22}$ or $a_{11} = 1$ and $a_{22} = -1$ or $a_{11} = -1$ and $a_{22} = 1$. The case $a_{11} = a_{22}$ is the same as the first scenario above. Therefore, the only other possible case that could occur is when $\rho_1\rho_2 = -1$. However, this is a contradiction to Proposition 3.3. Consequently, the only possible form of the monodromy matrix is the one given in equation (18) for which the eigenvalues have the form

$$\rho_1 = a_{11} + \sqrt{a_{11}^2 - 1} \quad \text{and} \quad \rho_2 = a_{11} - \sqrt{a_{11}^2 - 1}.$$

□

In Section 6, as an example of the SCHE we study the coupled Mathieu equations which were defined in equation (2).

5 Lorentz Oscillator Model and Coupled Hills Equations

In this section, we describe the Lorentz Oscillator Model (LOM) and its connection to the coupled Mathieu equations defined in (2). In particular, we study the stability of the LOM and will show that, under a suitable transformation it is equivalent, in a special case, to a class of coupled Mathieu equations. In Section 6, where the stability diagrams of the coupled Mathieu equations are found, we will use the results of this section.

5.1 Lorentz Oscillator Model

In the Lorentz Oscillator Model, an electron is bound to the nucleus by a harmonic potential and undergoes forced motion subject to damping [4]. Under external electromagnetic fields one can show that the equations of motion of the electron are [4]:

$$\begin{array}{rcl} \ddot{\chi} + \gamma_\chi \dot{\chi} + \omega_\chi^2 \chi & = & \frac{qE_0}{m}\cos(\omega t) - \frac{qB_0}{m}\cos(\omega t)\dot{\zeta}\,, \\ \ddot{\zeta} + \gamma_\zeta \dot{\zeta} + \omega_\zeta^2 \zeta & = & \frac{qB_0}{m}\cos(\omega t)\dot{\chi}\,, \end{array} \tag{19}$$

where (χ, ζ) is the relative position of the electron with respect to the nucleus, $\gamma_\chi, \gamma_\zeta$ are the phenomenological damping coefficients, and $\omega_\chi, \omega_\zeta$ are the natural frequencies in the χ and ζ directions. The charge and mass of the electron are q and m, respectively, while E_0 and B_0 are the electric and magnetic fields amplitudes.

Our goal is to analyze the stability of the homogenous LOM and establish its connection with CHE. To this end, we first write a dimensionless form of (19). We divide all of the terms by ω^2, where ω is the frequency of the driving terms on the right-hand-side of the differential equations, and scale the parameters as follows:

$$t \to \omega t\,, \qquad \gamma \to \frac{\gamma}{\omega}\,, \qquad \text{and} \qquad \Omega \to \frac{\Omega}{\omega}\,.$$

Similarly, we define the coupling amplitudes

$$\epsilon = \frac{qE_0}{m\omega^2} \qquad \text{and} \qquad \beta = \frac{qB_0}{m\omega}\,.$$

The dimensionless LOM then becomes

$$\begin{aligned} \ddot{\chi} + \gamma_\chi \dot{\chi} + \Omega_\chi^2 \chi &= \epsilon\cos(t) - \beta\cos(t)\dot{\zeta}\,, \\ \ddot{\zeta} + \gamma_\zeta \dot{\zeta} + \Omega_\zeta^2 \zeta &= \beta\cos(t)\dot{\chi}\,. \end{aligned}$$

The homogenous form of the dimensionless LOM, which we will study, becomes

$$\begin{aligned} \ddot{\chi} + \gamma_\chi \dot{\chi} + \Omega^2 \chi &= -\beta\cos(t)\dot{\zeta}\,, \\ \ddot{\zeta} + \gamma_\zeta \dot{\zeta} + \Omega^2 \zeta &= \beta\cos(t)\dot{\chi}\,. \end{aligned}$$

5.2 Rotation of the symmetric homogenous LOM to CHE

The LOM is said to be *symmetric* if

$$\gamma_\chi = \gamma_\zeta = \gamma, \quad \Omega_\chi^2 = \Omega_\zeta^2 = \Omega^2.$$

Therefore, the equations of the symmetric homogenous LOM are

$$\begin{aligned} \ddot{\chi} + \gamma \dot{\chi} + \Omega^2 \chi &= -\beta\cos(t)\dot{\zeta}\,, \\ \ddot{\zeta} + \gamma \dot{\zeta} + \Omega^2 \zeta &= \beta\cos(t)\dot{\chi}\,. \end{aligned} \tag{20}$$

We can use a transformation to write this system as a CHE. Define

$$\chi + i\zeta = (x + iz)\exp\left[-\frac{1}{2}\gamma t + \frac{i}{2}\beta\sin(t)\right]\,. \tag{21}$$

Let

$$a = \Omega^2 - \frac{\gamma^2}{4} + \frac{\beta^2}{8} \qquad \text{and} \qquad b = \frac{\beta^2}{8}\,. \tag{22}$$

Under this transformation, one can show that the differential equations governing (x, z) are

$$\begin{array}{rcl} \ddot{x} + [a + b\cos(2t)]\, x &=& \frac{1}{2}\beta\,(\gamma\cos(t) - \sin(t))\, z, \\ \ddot{z} + [a + b\cos(2t)]\, z &=& -\frac{1}{2}\beta\,(\gamma\cos(t) - \sin(t))\, x\,. \end{array} \tag{23}$$

Clearly, this is a CHE which we call the equivalent CHE of the symmetric LOM defined in (20).

Although these equations seem simple and straightforward, the relationship between these functions, x and z, and the original coordinates, χ and ζ, is complicated:

$$\chi = \exp\left[-\frac{1}{2}\gamma t\right]\left\{x\cos\left[\frac{1}{2}\beta\sin(t)\right] - z\sin\left[\frac{1}{2}\beta\sin(t)\right]\right\}, \tag{24}$$

$$\zeta = \exp\left[-\frac{1}{2}\gamma t\right]\left\{z\cos\left[\frac{1}{2}\beta\sin(t)\right] + x\sin\left[\frac{1}{2}\beta\sin(t)\right]\right\}. \tag{25}$$

If we assume that the damping, γ, is zero, system (23) becomes an SCHE as defined in equation (10).

Proposition 5.1. The undamped LOM

$$\begin{array}{rcl} \ddot{\chi} + \Omega^2\chi &=& -\beta\cos(t)\dot{\zeta}, \\ \ddot{\zeta} + \Omega^2\zeta &=& \beta\cos(t)\dot{\chi} \end{array}$$

is stable if and only if its equivalent CHE is stable.

Proof. From equations (24) and (25), if we set $\gamma = 0$, a solution $(\chi(t), \zeta(t))$ of the undamped LOM can be found from a solution $(x(t), z(t))$ of the equivalent CHE by a rotation of angle $\frac{1}{2}\beta\sin(t)$. Hence, the undamped LOM is stable if and only if its equivalent CHE is stable. □

5.3 Stability of the Homogenous LOM

Below, we show that the homogenous LOM is stable for any parameter set $(\gamma_\chi, \Omega_\chi^2, \beta, \gamma_\zeta, \Omega_\zeta^2)$ for which $\gamma_\chi > 0$ and $\gamma_\zeta > 0$. We can show this by the Lyapunov method (see e.g. [9]). The general homogenous LOM is in the form

$$\begin{array}{rcl} \ddot{\chi} + \gamma_\chi\dot{\chi} + \Omega_\chi^2\chi &=& -\beta\cos(t)\dot{\zeta}, \\ \ddot{\zeta} + \gamma_\zeta\dot{\zeta} + \Omega_\zeta^2\zeta &=& \beta\cos(t)\dot{\chi}. \end{array} \tag{26}$$

Let

$$V = \frac{1}{2}(\dot{\chi}^2 + \dot{\zeta}^2 + \Omega_\chi^2\chi^2 + \Omega_\zeta^2\zeta^2).$$

Function V is a Lyapunov function for the time-varying system (26). We have

$$\begin{aligned}
\dot{V} &= \dot{\chi}\ddot{\chi} + \dot{\zeta}\ddot{\zeta} + \Omega_\chi^2 \chi\dot{\chi} + \Omega_\zeta^2 \zeta\dot{\zeta} \\
&= \dot{\chi}(-\gamma_\chi \dot{\chi} - \Omega_\chi^2 \chi + \beta\cos(t)\dot{\zeta}) + \\
&\quad \dot{\zeta}(-\gamma_\zeta \dot{\zeta} - \Omega_\zeta^2 \zeta - \beta\cos(t)\dot{\chi}) + \\
&\quad \Omega_\chi^2 \chi\dot{\chi} + \Omega_\zeta^2 \zeta\dot{\zeta} \\
&= -\gamma_\chi \dot{\chi}^2 - \gamma_\zeta \dot{\zeta}^2 \\
&\leq 0.
\end{aligned}$$

Since V is a positive definite function which does not depend on t, and $\dot{V} \leq 0$, system (26) is stable [9].

Note that experimental realizations of the LOM system sometimes exhibit instability [4], which must result from the forcing terms that are not included in this present analysis.

6 Stability Regions for the Coupled Mathieu Equations

In this section, we study the stability regions for the coupled Mathieu equations both in the case where it is a transformation of the LOM and for general choices of the parameters. In the former case, an analytic result is presented.

Recall the equations of the coupled Mathieu equations are given by:

$$\begin{aligned}
\ddot{x} + [a + b\cos(2t)]\, x &= -c\sin(t)z, \\
\ddot{z} + [a + b\cos(2t)]\, z &= c\sin(t)x.
\end{aligned} \tag{27}$$

Proposition 6.1. The system of coupled Mathieu equations is

1. stable if $a > b > 0$ and $c = \sqrt{2b}$.
2. unstable if $0 < a < b$ and $c = \sqrt{2b}$.

Proof. 1. Define $\beta = \sqrt{8b}$. Since $c = \sqrt{2b}$, we have $c = \beta/2$. Therefore, with these parameters, the coupled Mathieu equations become

$$\begin{aligned}
\ddot{x} + [a + b\cos(2t)]\, x &= -\tfrac{\beta}{2}\sin(t)z, \\
\ddot{z} + [a + b\cos(2t)]\, z &= \tfrac{\beta}{2}\sin(t)x\,.
\end{aligned} \tag{28}$$

Let $\Omega = \sqrt{a - b}$. One can easily check that with these definitions, equation (22) holds with $\gamma = 0$. Hence, under transformation (21), the governing differential equations of (χ, ζ) are the undamped LOM that are equivalent to the coupled

Mathieu equations defined above. Thus, the proof follows from Proposition 5.1 and the fact that the homogeneous LOM is always stable.

2. In this case, define $\Omega = \sqrt{b-a}$, and $\beta = \sqrt{8b}$. If we transform equation (28) back to LOM using the inverse of transformation (21), we get

$$\begin{array}{rcl} \ddot{x} - \Omega^2 x &=& -\beta \cos(t)\dot{z}, \\ \ddot{z} - \Omega^2 z &=& \beta \cos(t)\dot{x}. \end{array} \tag{29}$$

It is easy to check that the curves

$$\frac{1}{2}(\dot{x}^2 + \dot{z}^2 - \Omega^2 x^2 - \Omega^2 z^2) = C$$

are integral curves of system (29) for arbitrary constant C. These curves are hyperbolic type curves which show that system (29) is unstable. Since $\gamma = 0$, similar to the proof of Proposition 5.1 we can show that the stability of system (29) is equivalent to that of (28). Hence, in this case, because system (29) is unstable, the system of coupled Mathieu equations is unstable as well. $\square$

Fig. 1 to 3 show the stability regions in the parameter space (a, b, c) of equation (27). To generate these stability regions, for each given set of parameters, (a, b, c), the monodromy matrix $\mathbb{M}$ is calculated by integrating the differential equations. Then, the largest eigenvalue of $\mathbb{M}$ is calculated from equation (17). We note that this method compared to the standard algorithms for calculation of eigenvalues is faster and more accurate.

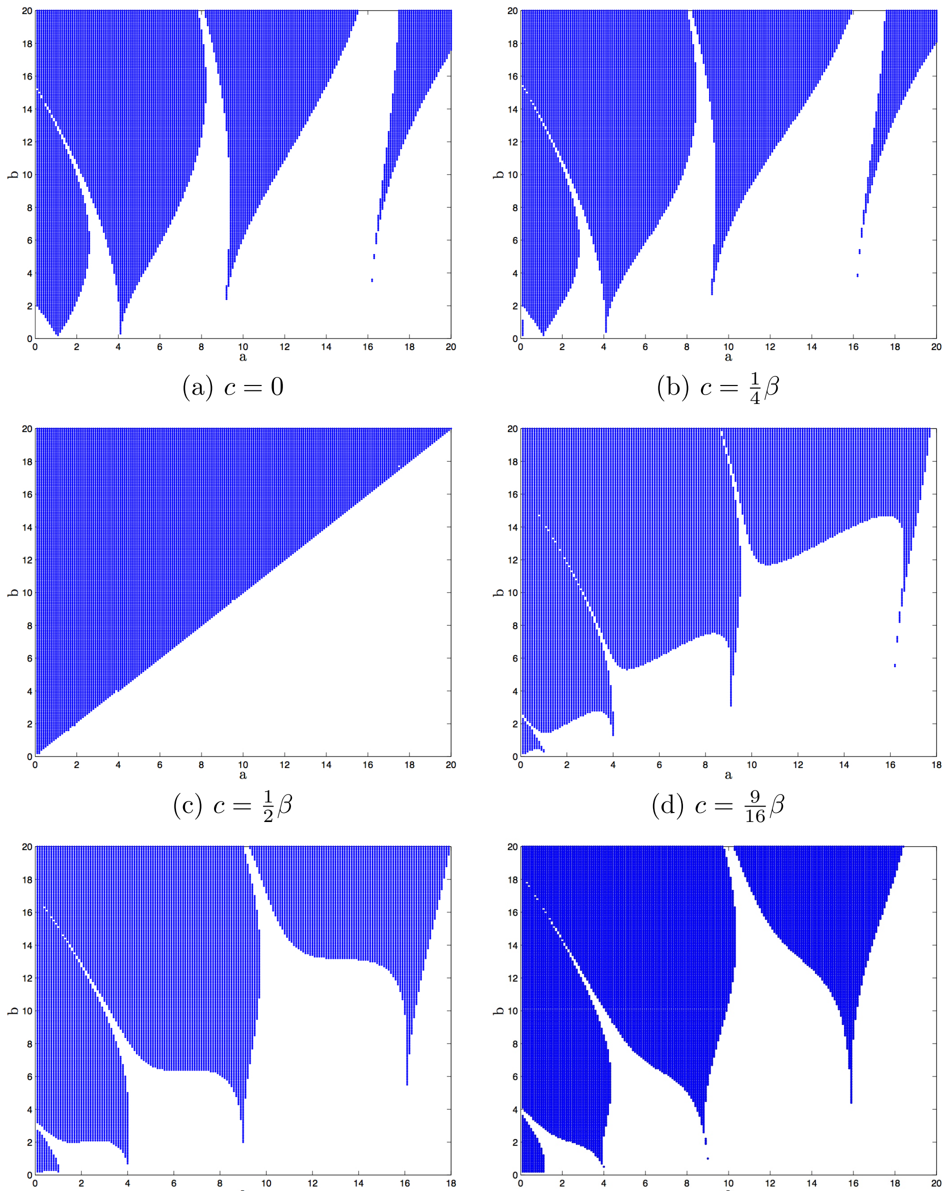
Fig. 1 shows the stability regions when $c = k\sqrt{2b}$ for different values of k. The numerical simulation in part (c) illustrates Proposition 6.1. Fig. 2 plots, in addition, level sets of the Floquet multiplier, where the special case of the LOM again appears in part (b).

Fig. 3 fixes c and varies a and b. We note that the pictures generalize the Arnold tongue picture found in the usual Mathieu equation.

7 Conclusion

In this paper, we studied a class of Coupled Hill's equations in the form

$$\begin{array}{rcl} \ddot{x} + p(t)x &=& -q(t)z, \\ \ddot{z} + p(t)z &=& q(t)x\,, \end{array}$$

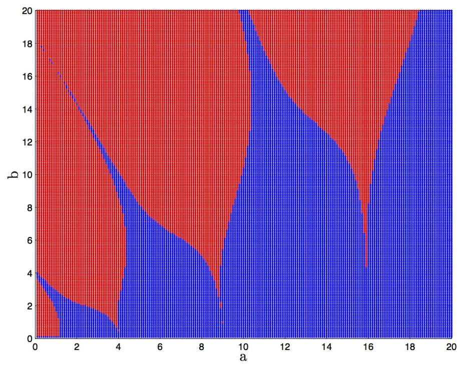


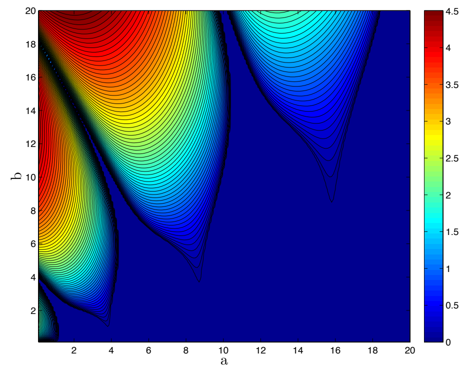
(a) $c = 0$ (b) $c = \frac{1}{4}\beta$

(c) $c = \frac{1}{2}\beta$ (d) $c = \frac{9}{16}\beta$

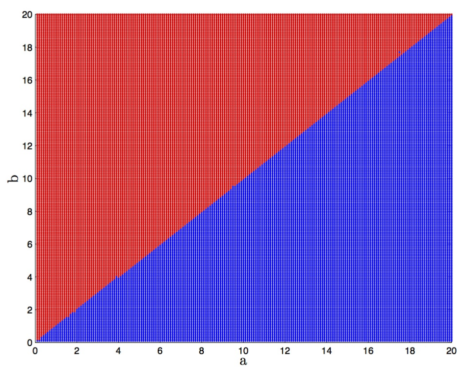
(e) $c = \frac{3}{4}\beta$ (f) $c = \beta$

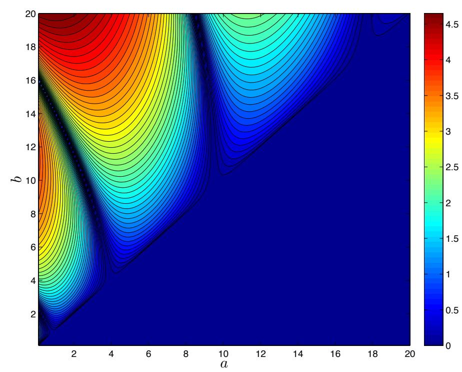
Figure 1: Stability regions for the Hill's equations for different values of c, varying a and b between 0 and 20, where $\beta = \sqrt{8b}$. Shaded area shows unstable region. Part (c) is clearly consistent with Proposition 6.1.



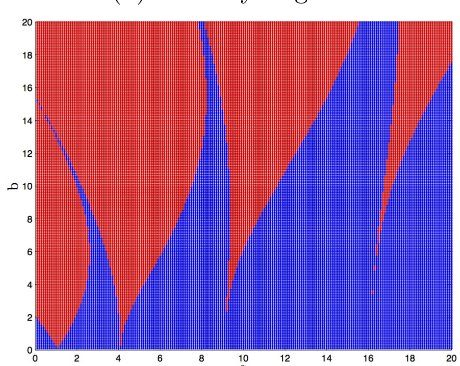


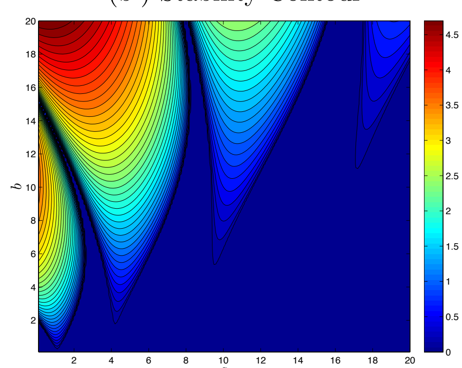
(a) Stability Regions (a′) Stability Contour





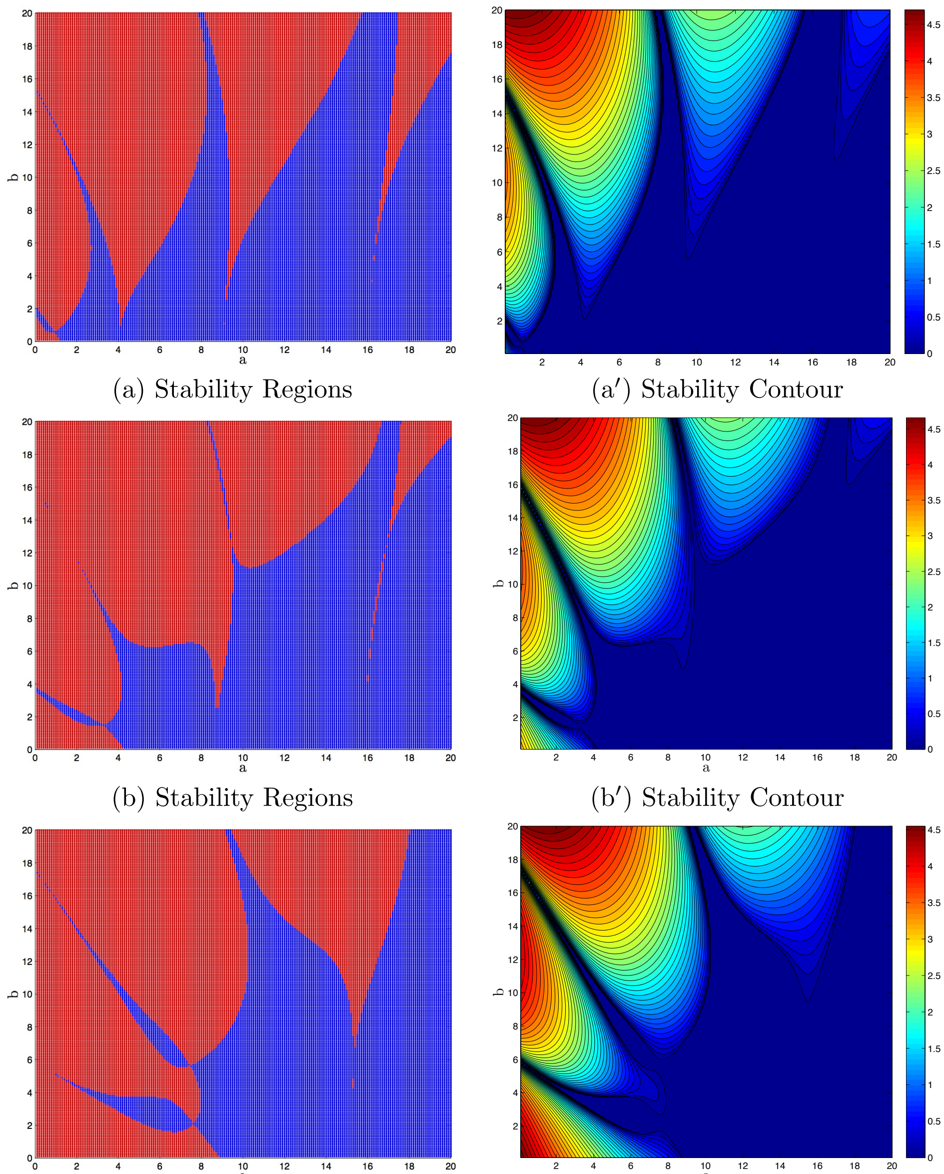
(b) Stability Regions (b′) Stability Contour





(c) Stability Regions (c′) Stability Contour

Figure 2: (a, a′) Stability diagrams for $c = \beta$. (b, b′) Stability diagrams for $c = \beta/2$. (c, c′) Stability diagrams for $c = 0$. In the stability regions, blue denotes the stable region and red denotes the unstable region. In the stability contours, the color bar shows the $\log_{10}$ of the magnitude of the largest eigenvalue of the monodromy matrix.



(a) Stability Regions

(a′) Stability Contour

(b) Stability Regions

(b′) Stability Contour

(c) Stability Regions

(c′) Stability Contour

Figure 3: (a, a′) Stability diagrams for $c = 1$, where a and b vary from 0 to 20. (b, b′) Stability diagrams for $c = 5$, where a and b vary from 0 to 20. (c, c′) Stability diagrams for $c = 10$, where a and b vary from 0 to 20.

where $p(t+T) = p(t)$ and $q(t+T) = q(t)$ for some $T > 0$. Floquet theory was used to identify different possible forms of the solutions of this system. Assuming that $p(t)$ and $q(t)$ are even and odd functions, respectively, exploiting the existing symmetries, we proved that out of 16 elements of the monodromy matrix only 4 of them are independent. A general simplified form of the monodromy matrix, as in equation (16), was derived and it was shown that the eigenvalues of the monodromy matrix are only functions (see equation (17)) of the first element of the matrix. The formula that was found for eigenvalues is also numerically useful, as it gives a closed form for the eigenvalues of the monodromy matrix which will be more accurate and computationally less cumbersome compared to the standard algorithms that are used to find eigenvalues. Next, the Lorentz Oscillator model and its connection to the system of coupled Mathieu equations (which is an example of the coupled Hill's equations) was introduced. It was shown that this system in its homogenous case is always stable. Finally, the stability diagrams of the coupled Mathieu equations were presented and it was shown that the results are consistent with the stability of the corresponding Lorentz oscillator model.

In future work, we intend to study this problem in presence of stochastic disturbances as well as the related quantum problem.

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References

[1] Fred C. Adams and Anthony M. Bloch. Hill's equation with random forcing terms. *SIAM Journal on Applied Mathematics*, 68(4):947–980, 2008.

[2] J. Baillieul. Energy methods for stability of bilinear systems with oscillatory inputs. *Internat. J. Robust Nonlinear Control*, 5(4):285–301, 1995. Control of nonlinear mechanical systems.

[3] Lamberto Cesari. *Asymptotic behavior and stability problems in ordinary differential equations*. Springer Berlin, 1963.

[4] WM. Fisher and SC. Rand. Light-induced dynamics in the Lorentz oscillator model with magnetic forces. *Physical Review A*, 82(1):013802, 2010.

[5] J. Hansen. Stability diagrams for coupled Mathieu equations. *Ingenieur-Archiv*, 55(6):463–473, 1985.

[6] Harry Hochstadt. Estimates on the stability intervals for Hill's equation. *Proceedings of the American Mathematical Society*, 14(6):930–932, 1963.

[7] Chieh Su. Hsu. On a restricted class of coupled Hills equations and some applications. *Journal of Applied Mechanics*, 28(4):551–556, 1961.

[8] Vladimir Andreevich Iakubovich and Viacheslav Mikhailovich Starzhinski. *Linear differential equations with periodic coefficients*, volume 1. Krieger Pub Co, 1975.

[9] Hassan K. Khalil. *Nonlinear systems*, volume 3. Prentice hall New Jersey, 1996.

[10] H. Landa, M. Drewsen, B. Reznik, and A. Retzker. Classical and quantum modes of coupled Mathieu equations. *Journal of Physics A: Mathematical and Theoretical*, 45(45):455305, 2012.

[11] WS. Loud. Stability regions for Hill's equation. *Journal of Differential Equations*, 19(2):226–241, 1975.

[12] Wilhelm Magnus and Stanley Winkler. *Hill's equation*. Courier Corporation, 2013.

[13] Gamal M. Mahmoud. Stability regions for coupled Hill's equations. *Physica A: Statistical Mechanics and its Applications*, 242(1):239–249, 1997.

[14] Raul Naulin and CarLos Pabst. The roots of a polynomial depend continuously on its coefficients. *Rev. Colombiana Mat*, 28:35–37, 1994.

[15] Zdzisław Opial. Continuous parameter dependence in linear systems of differential equations. *Journal of Differential Equations*, 3(4):571–579, 1967.

[16] Michael I. Weinstein and Joseph B. Keller. Asymptotic behavior of stability regions for Hill's equation. *SIAM Journal on Applied Mathematics*, 47(5):941–958, 1987.