A Simple Proof of the Generalized Cauchy's Theorem

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Abstract

The Cauchy's theorem for balance laws is proved in a general context using a simpler and more natural method in comparison to the one recently presented in [1]. By "generality" we mean that the ambient space is considered to be an orientable smooth manifold, and not only the Euclidean space.

Keywords Differentiable Manifolds . Cauchy's Theorem for Balance Laws . Continuum Mechanics

Mathematics Subject Classification (2010) 74A99 . 58Z05

1 Introduction

During the past decades there has been an interest to generalize different notions in continuum mechanics to smooth manifolds [e.g. 2, 3, 4]. This interest is mostly due to three advantages that smooth manifolds might have.

Firstly, manifolds are the suitable place where one can formulate many theories in physics, including continuum mechanics, because the notion of smooth manifolds makes it possible to formulate the theory without referring to any particular coordinate system. This allows us to state the theory in an arbitrary coordinate system without too much difficulty.

Secondly, as Marsden [2] notes, a basic message that we receive from Einstein is that any theory that purports to be fundamental ought to be generalizable so the underlying physical space is a manifold and not just Euclidean space. By implementing the geometry of manifolds, one can examine whether a specific law in continuum mechanics can be generalized to smooth manifolds or not, so if a law is generalizable it would be more fundamental. For example, it can be shown that if we interpret forces as vector fields the Conservation of Momentum law in its integral form cannot be stated on an arbitrary orientable manifold. This examination helps us to generalize the theory of continuum mechanics in a proper way.

Finally, there are practical examples in which modeling a body as an open subset of $\mathbb{R}^n$ is not possible. For instance, consider the motion of a shell in

$\mathbb{R}^3$. In this example the body cannot be considered as an open subset of $\mathbb{R}^3$, but it can be modeled as a 2-dimensional submanifold of $\mathbb{R}^3$. Another famous example involves liquid crystals (see [5]). In these materials the orientations of molecules affect the macroscopic behavior of the material. In order to include the orientation of molecules more degrees of freedom and hence, more dimensions are required. For example, liquid crystals with inextensible oriented rod molecules are modeled as specific submanifolds of $\mathbb{R}^3 \times S^2$.

The above discussion motivates generalizing different notions in continuum mechanics to smooth manifolds. This generalization includes both kinematics and balance laws [2]. As for the balance laws, a fundamental theorem, Generalized Cauchy's Theorem, is required to be proved on smooth manifolds. In this regard, different contributions have been made. For example, Marsden and Hughes [2], as they stated, proved the Cauchy's theorem in a three dimensional Riemannian manifold, although in their rough proof, the manifold is considered to be locally flat which is an additional assumption they made. Segev [1], proved the Cauchy's Theorem in the most general form. He considered an oriented manifold (not necessarily Riemannian) and replaced the scalar fields in the classical form of Cauchy's Theorem with suitable differential forms. The theorem, in this case, is called the Generalized Cauchy's Theorem, and the objective of the present paper is to prove this theorem by a simpler method in comparison to [1].

In our proof of the Generalized Cauchy's Theorem we first, prove the theorem in Euclidean space $\mathbb{R}^n$. Then using local coordinates which are local orientation preserving diffeomorphisms, we translate the statement of the Cauchy's theorem to $\mathbb{R}^n$. Finally, pulling back the result obtained in $\mathbb{R}^n$ by coordinate functions, and using a theorem concerning partition of unity on manifolds, we complete the proof of the Generalized Cauchy's Theorem.

2 Generalized Cauchy's Theorem

First, we state the ordinary form of Cauchy's Theorem in $\mathbb{R}^n$. Then, stating its generalized form, we explain the relationship between the classical and the generalized format of the theorem. The proofs are provided in the next section.

Definition 2.1 *A simple body $\mathcal{B}$ is an open subset of $\mathbb{R}^n$ with Lipschitz boundary. Furthermore, every open subset of $\mathcal{B}$ (usually denoted by $\mathcal{P}$) with Lipschitz boundary is called a material part of $\mathcal{B}$.*

Theorem 2.2 *Let $\mathcal{B} \subseteq \mathbb{R}^n$ be a simple body and $\alpha : \mathcal{B} \to \mathbb{R}$, $\beta : \mathcal{B} \to \mathbb{R}$ be bounded functions. Moreover, suppose that $\tau : \mathcal{UB} \to \mathbb{R}$, where $\mathcal{UB}$ is the unit tangent bundle of $\mathcal{B}$, is a continuous function. Also, for a material part $\mathcal{P} \subseteq \mathcal{B}$, define*

$$\tau_{\mathcal{P}}(X) := \tau(X, \boldsymbol{N}(X)) \qquad X \in \partial P,$$

where $\boldsymbol{N}$ is the smooth unit normal outward-pointing vector field on $\partial\mathcal{P}$. Now, *if the equation*

$$\int_{\mathcal{P}} \alpha dV = \int_{\mathcal{P}} \beta dV + \int_{\partial \mathcal{P}} \tau_{\mathcal{P}} dA,$$

is satisfied for every bounded material part $\mathcal{P} \subseteq \mathcal{B}$, then there exists a unique continuous vector field $\boldsymbol{\gamma}$ on $\mathcal{B}$ such that for every point $X \in B$

$$\tau(X, \boldsymbol{N}) = \langle \boldsymbol{\gamma}(X), \boldsymbol{N} \rangle.$$

Remark 2.3 *In the above theorem note that since the boundary is chosen to be Lipschitz the smooth unit normal vector field exists for almost every x on the boundary with respect to the Lebesgue measure induced on $\partial\mathcal{P}$. As a consequence, $\tau_{\mathcal{P}}$ is defined almost everywhere on the boundary.*

A common example of the application of the Cauchy's Theorem is in the case of First Law of Thermodynamics. When no work is involved α is regarded as the rate of change of internal energy per unit volume, β is considered to be the rate of energy per unit volume generated in the body and $\tau(\boldsymbol{N})$ is the rate of energy per unit area which leaves the boundary of the body (or the material part) in the $\boldsymbol{N}$ direction.

As it is well-known, in the case of $n = 3$, i.e. $I\!R^3$, the above theorem is proved by considering tetrahedra whose volumes approach zero. When $n \neq 3$, tetrahedra cannot be used anymore, but instead, using n-simplexes, as it can be seen in the next section, we prove Theorem 2.2.
Before stating the Cauchy's Theorem on smooth manifolds, we need to present a more general definition of body.

Definition 2.4 *Let $\mathcal{M}$ be an oriented n-dimensional smooth manifold with corners. A subset $\mathcal{B}$ of $\mathcal{M}$ is said to be a body in the ambient space $\mathcal{M}$ if it is an orientable embedded submanifold with corners. By a material part of $\mathcal{B}$ we mean a subset $\mathcal{P} \subseteq \mathcal{B}$ which is an embedded submanifold with corners whose dimension is the same as the dimension of $\mathcal{B}$.*

In the above definition the dimension of $\mathcal{B}$ can be less than the dimension of the ambient space $\mathcal{M}$, but the dimension of $\mathcal{P}$ is always equal to the dimension of $\mathcal{B}$. Also note that Definition 2.1 is a special case of Definition 2.4.

Now, we are ready to state the generalized Cauchy's theorem. An analog of Theorem 2.2 is to be stated, however as Definition 2.4 suggests, the manifold is not necessarily Riemannian and hence, in general, there is no volume element as is in Theorem 2.2. This requires us to replace the scalar fields in Theorem 2.2 with suitable differential forms. We must replace the scalar fields α and β with n-forms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Hence, for example if $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are oriented vectors in the tangent space at x, $T_x\mathcal{B}$, then $\boldsymbol{\beta}(x)(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$ can be interpreted as the rate at which the property, denoted by β, is generated in the infinitesimal volume generated by $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_n$. Finally, we must replace $\tau : \mathcal{UB} \to I\!R$ with a suitable differential form field. In the case of a smooth manifold which is not Riemannian there is nothing known as unit tangent bundle $\mathcal{UB}$. However, noting that every unit vector at x actually specifies an oriented $(n-1)$-dimensional subspace of $T_x\mathcal{B}$, called a hyperplane, we can replace $\mathcal{UB}$ with the bundle of oriented hyperplanes on $\mathcal{B}$, denoted by $\overrightarrow{H}\mathcal{B}$. Therefore, instead of the continuous scalar field τ we assume the existence of a continuous function $\boldsymbol{\tau}$ which assigns an $(n-1)$-form in $T_x^{(n-1)}H$ to the oriented hyperplane H at x. That is if H is an oriented hyperplane at x, $\boldsymbol{\tau}(x,H)$ is an $(n-1)$-form on the vector space H.

Remark 2.5 If Ω is the orientation form on $\mathcal{B}$, $\overrightarrow{H}\mathcal{B}$ can be identified with a triple $(x,H,\boldsymbol{n})$ where $\boldsymbol{n}\in T_x\mathcal{B}\backslash H$ and the orientation form on H is $\boldsymbol{n}\lrcorner\Omega$ ($\boldsymbol{n}\lrcorner\Omega$ is the contraction of Ω with vector $\boldsymbol{n}$). So, if $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{n-1}$ are vectors in H, $(\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{n-1})$ is positively oriented in $(x,H,\boldsymbol{n})$ when $(\boldsymbol{n},\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{n-1})$ is positively oriented in $\mathcal{B}$. As a result,

$$(x,H,\boldsymbol{n})\sim(x,H,\boldsymbol{n}')\Leftrightarrow\exists c>0\quad s.t.\quad \boldsymbol{n}\lrcorner\Omega=c(\boldsymbol{n}'\lrcorner\Omega).$$

defines an equivalence relation on the set consisting of $(x,H,\boldsymbol{n})$'s. However, from now on for the sake of simplicity an equivalence class is shown by its representative say, $(x,H,\boldsymbol{n})$. It is obvious that every hyperplane H at x, adopts two different orientations such that for an arbitrary $\boldsymbol{n}\in T_x\mathcal{B}\backslash H$, one oriented hyperplane is $(x,H,\boldsymbol{n})$ and the other is $(x,H,-\boldsymbol{n})$.

In order to be able to state the generalized form of Theorem 2.2 we need to define the concept of continuity of a map whose domain is $\overrightarrow{H}\mathcal{B}$. To this end, we can simply define the continuity of the required map in the next theorem by implying coordinate systems as follows.

Lemma 2.6 *Let $\mathcal{B}$ be an oriented smooth manifold. If ψ is a coordinate system at $x\in M$ then to each oriented hyperplane $(x,H,\boldsymbol{n})$ corresponds a unit normal $\boldsymbol{N}$ at $X=\psi(x)$ such that*

$$(x,H,\boldsymbol{n})=(\psi^{-1}(X),\psi^*(\boldsymbol{N}^\perp),\psi^*(\boldsymbol{N})).$$

where ψ^* is the tangent map of ψ^{-1}.

Proof Denoting tangent map of ψ by ψ_*, there are exactly two unit normals to $\psi_*(H)$. Choose the one, say $\boldsymbol{N}$, for which $(\psi^*\boldsymbol{N})\lrcorner\Omega=c(\boldsymbol{n}\lrcorner\Omega)$ for a positive number c, where Ω is the orientation form of $\mathcal{B}$. Referring to Remark 2.5, the above equation is clear. □

Lemma 2.7 *Let $\mathcal{B}$ be an oriented smooth manifold. Suppose $\boldsymbol{\tau}$ is a function on the bundle of oriented hyperplanes $\overrightarrow{H}\mathcal{B}$ such that for each $(x,H,\boldsymbol{n})$ in $\overrightarrow{H}\mathcal{B}$, $\boldsymbol{\tau}(x,H,\boldsymbol{n})$ is an $(n-1)$-form on hyperplane H. If ψ is a coordinate system at $x\in M$ then there exists a scalar function τ_ψ such that* [1]

$$\tau_\psi(X,\boldsymbol{N})(\boldsymbol{N}\lrcorner d\boldsymbol{V})|_{\boldsymbol{N}^\perp}=(\psi^{-1})^*\boldsymbol{\tau}(x,H,\boldsymbol{n})$$

where $\boldsymbol{N}$ is the unit normal at $X=\psi(x)$ for which

$$(x,H,\boldsymbol{n})=(\psi^{-1}(X),\psi^*(\boldsymbol{N}^\perp),\psi^*(\boldsymbol{N})).$$

[1] Here, $(\psi^{-1})^*$ is considered to be pullback under ψ^{-1}.

Proof By lemma 2.6, noting that $(\boldsymbol{N} \lrcorner d\boldsymbol{V})$ is an $(n-1)$-form the proof is obvious. $\square$

In the above lemma τ_ψ is a scaler function defined on the unit tangent bundle of an open subset of Euclidean space. This permits us to define the continuity of the map $\boldsymbol{\tau}$ as follows.

Definition 2.8 *Map $\boldsymbol{\tau}$ in the above lemma is said to be continuous if τ_ψ is continuous for every coordinate system ψ.*

Now, having found suitable substitutions for the notions in Theorem 2.2, we are prepared to state the Generalized Cauchy's Theorem.

Theorem 2.9 *Let $\mathcal{M}$ be an oriented smooth manifold with corners and $\mathcal{B}$ be an n-dimensional body in $\mathcal{M}$. Suppose that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are bounded n-forms on $\mathcal{B}$ and $\boldsymbol{\tau}$ is a continuous function on the bundle of oriented hyperplanes $\overrightarrow{H}\mathcal{B}$ such that for each $(x, H, \boldsymbol{n})$ in $\overrightarrow{H}\mathcal{B}$, $\boldsymbol{\tau}(x, H, \boldsymbol{n})$ is an $(n-1)$-form on hyperplane H. Also, for a material part $\mathcal{P} \subseteq \mathcal{B}$, define*

$$\boldsymbol{\tau}_{\mathcal{P}}(x) := \boldsymbol{\tau}(x, T_x\partial\mathcal{P}, \boldsymbol{n}(x)) \qquad x \in \partial P,$$

where $\boldsymbol{n}$ is a smooth outward-pointing vector field on $\partial\mathcal{P}$. Now, if the equation

$$\int_{\mathcal{P}} \boldsymbol{\alpha} = \int_{\mathcal{P}} \boldsymbol{\beta} + \int_{\partial\mathcal{P}} \boldsymbol{\tau}_{\mathcal{P}},$$

is satisfied for every compact material part $\mathcal{P} \subseteq \mathcal{B}$, then there exists a unique continuous $(n-1)$-dimensional form field $\boldsymbol{\sigma}$ on $\mathcal{B}$ such that for every $x \in \mathcal{B}$, if H is a hyperplane at x and $\boldsymbol{n} \in T_x\mathcal{B}\backslash H$ we have

$$\boldsymbol{\sigma}(x)|_H = \boldsymbol{\tau}(x, H, \boldsymbol{n}).$$

The above theorem says that if the stated integral equation holds for every compact material part $\mathcal{P}$, then there exists a global form $\boldsymbol{\sigma}$ whose restriction to each hyperplane equals $\boldsymbol{\tau}$. This is similar to the existence of the global vector field $\boldsymbol{\gamma}$ in Theorem 2.2, whose projection in any direction equals τ calculated in that direction.

The proof of Theorem 2.9 is postponed until section four, after stating the proof of the Cauchy's Theorem on $\mathbb{R}^n$ in section three. Later, when Theorem 2.9 is proved it would be clear that the Cauchy's Theorem on $\mathbb{R}^n$ is a special form of Theorem 2.9.
Also note that a similar description as in remark 2.3 is stated in Theorem 4.1.

3 Proof of Cauchy's Theorem in $\mathbb{R}^n$

As mentioned before to prove the Generalized Cauchy's Theorem, we first prove the Cauchy's Theorem in $\mathbb{R}^n$.

Proof of Theorem 2.2 In case of $n = 3$ the theorem is proved by considering a tetrahedron as a material part whose volume approaches zero. Here, in our proof, in case of an arbitrary natural number n, an n-simplex is used instead of a tetrahedron. For the sake of simplicity we prove the equation $\tau(X, \boldsymbol{N}) = \langle \boldsymbol{\gamma}(X), \boldsymbol{N} \rangle$ at $X = 0$ for a body $\mathcal{B}$ containing the point $X = 0$.
Suppose that for nonzero real numbers $a_1, a_2, \ldots, a_n$ and the real variable $t \in I\!R$, $P_0, P_1^t, \ldots, P_n^t$ are the vertices of the n-simplex $S^t = (P_0, P_1^t, \ldots, P_n^t)$ where

$$\begin{array}{rlll} P_0 = & (0, & 0, & \ldots \quad ,0) \\ P_1^t = & t(a_1, & 0, & \ldots \quad ,0) \\ \vdots & & & \\ P_n^t = & t(0, & 0, & \ldots \quad ,a_n) \end{array}$$

Clearly, there exists an $\epsilon > 0$ such that for every $t \in (o, \epsilon)$, $S^t \subseteq \mathcal{B}$. Since the stated integral equation holds for every bounded material part $\mathcal{P} \subseteq \mathcal{B}$ we have

$$\int_{S^t} \alpha dV = \int_{S^t} \beta dV + \int_{\partial S^t} \tau_{S^t} dA \qquad 0 < t < \epsilon,$$

or,

$$\int_{S^t} (\alpha - \beta) dV = \int_{\partial S^t} \tau_{S^t} dA \qquad 0 < t < \epsilon. \tag{1}$$

To make use of this equation properly, we need to compute ∂S^t as follows

$$\partial S^t = \bigcup_{i=0}^{n} S_i^t \quad s.t. \quad S_i^t = (P_0, P_1^t, \ldots, P_{i-1}^t, P_{i+1}^t, \ldots, P_n^t).$$

Now, if the unit normal vector field on S_i^t is denoted by $\boldsymbol{N}_i$, then for $i \neq o$

$$\boldsymbol{N}_i(X) = -\boldsymbol{e_i} \qquad \forall X \in S_i^t,$$

where $\boldsymbol{e_i} = (0, 0, \ldots, \underbrace{sign(a_i)}_{i^{th}}, \ldots, 0)$.

For $i = 0$,

$$S_0^t = \left\{ (X_1, X_2, \ldots, X_n) \quad s.t. \quad \frac{1}{a_1 t} X_1 + \frac{1}{a_2 t} X_2 + \ldots + \frac{1}{a_n t} X_n = 1 \right\},$$

thus, defining $\boldsymbol{b} = (a_1^{-1}, a_2^{-2}, \ldots, a_n^{-1})$, we have

$$\boldsymbol{N}_0(X) = \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|} \qquad \forall X \in S_0^t.$$

Define

$$K = sup \left\{ |\alpha(X) - \beta(X)| \quad s.t. \quad X \in S^t \right\}.$$

From equation (1)

$$\left|\int_{\partial S^t} \tau_{S^t} dA\right| \leq KV^t,$$

where V^t is the volume of S^t. From the above inequality

$$\left|\int_{S_0^t} \tau_{S^t} dA + \sum_{i=1}^{n} \int_{\partial S_i^t} \tau_{S^t} dA\right| \leq KV^t. \tag{2}$$

Using the mean value theorem for integral on $I\!R^n$, there exist points $Q_i^t \in S_i^t$ such that if A_i^t is the area of S_i^t,

$$\int_{S_i^t} \tau_{S^t} dA = \tau_{S^t}(Q_i^t)A_i^t = \tau(Q_i^t, \boldsymbol{N}_i(Q_i^t))A_i^t.$$

Substituting this equation into (2) and dividing by A_0^t

$$\left|\tau(Q_0^t, \boldsymbol{N}_0(Q_0^t)) + \sum_{i=1}^{n} \tau(Q_i^t, \boldsymbol{N}_i(Q_i^t))\frac{A_i^t}{A_0^t}\right| \leq K\frac{V^t}{A_0^t}.$$

Now let t approach zero, due to the continuity of τ

$$\lim_{t\to 0} \tau(Q_i^t, \boldsymbol{N}_i(Q_i^t)) = \tau(0, \boldsymbol{N}_i(0)).$$

Furthermore,

$$V^t = \frac{a_1 a_2 \dots a_n}{n!}t^n \Rightarrow \qquad \lim_{t\to 0}\frac{V^t}{A_0^t} = 0.$$

Also, considering the unit normal vector to the area A_i^t and A_0^t,

$$\lim_{t\to 0}\frac{A_i^t}{A_0^t} = \frac{b_i}{\|\boldsymbol{b}\|} \qquad i \neq 0,$$

where b_i is the i^{th} component of the vector $\boldsymbol{b}$ displayed in the basis $\{\boldsymbol{e}_i\}$. Substituting the above equation into the last inequality we have

$$\tau(0, \boldsymbol{N}_0(0)) + \sum_{i=1}^{n} \frac{b_i}{\|\boldsymbol{b}\|}\tau(0, -\boldsymbol{e}_i) = 0. \tag{3}$$

Now, since a_i's are arbitrary nonzero real number, b_i is also an arbitrary nonzero real number. As a result, in (3) we can let $\boldsymbol{N}_0(0)$ approach $\boldsymbol{e}_i$ and consequently from (3) and continuity of τ

$$\tau(0, \boldsymbol{e}_i) = -\tau(0, -\boldsymbol{e}_i).$$

Substituting this into (3) and considering the the equation we had for $\boldsymbol{N}_0$,

$$\tau(0, \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|}) = \sum_{i=1}^{n} \frac{b_i}{\|\boldsymbol{b}\|}\tau(0, \boldsymbol{e}_i). \tag{4}$$

If the vector field $\boldsymbol{\gamma}$ at 0 is defined as

$$\boldsymbol{\gamma}(0) = \sum_{i=1}^{n} \tau(0, \boldsymbol{e}_i)\boldsymbol{e}_i,$$

equation (4) implies

$$\tau(0, \boldsymbol{N}) = \langle \boldsymbol{\gamma}(0), \boldsymbol{N} \rangle.$$

As mentioned before we can construct the above proof for every $X \in \mathcal{B}$ just as the case $X = 0$, consequently if for $X \in \mathcal{B}$

$$\boldsymbol{\gamma}(X) := \sum_{i=1}^{n} \tau(X, \boldsymbol{e}_i)\boldsymbol{e}_i,$$

we have

$$\tau(X, \boldsymbol{N}) = \langle \boldsymbol{\gamma}(X), \boldsymbol{N} \rangle \quad \forall X \in \mathcal{B}.$$

The continuity and uniqueness of the vector field $\boldsymbol{\gamma}$ follows readily from the above equation. □

An interesting point in the above proof is that the equation $\tau(0, \boldsymbol{e}_i) = -\tau(0, -\boldsymbol{e}_i)$ is shown during the main proof, so the following corollary doesn't need a separate proof.

Corollary 3.1 *With the assumptions of Theorem 2.2*

$$\tau(X, \boldsymbol{N}) = -\tau(X, -\boldsymbol{N}) \qquad \forall X \in \mathcal{B}.$$

where $\boldsymbol{N}$ is any unit vector at X.

4 Proof of The Generalized Cauchy's Theorem

Before starting off with the proof of Theorem 2.9, let's recall a few famous theorems from Geometry of Manifolds (e.g. see [6]). In the following theorems, as usual, the smooth manifold $\mathcal{B}$ is assumed to be Hausdorff and second countable.

Theorem 4.1 *If $\mathcal{B}$ is any smooth manifold with corners, there is a smooth outward-pointing vector field for almost every $x \in \partial\mathcal{B}$. In other words, if (ψ, U) is a boundary chart the image under ψ of points at which this vector field does not exist has measure zero with respect to Lebesgue measure induced on $\psi(\partial\mathcal{B} \cap U)$.*

Theorem 4.2 *Let $\mathcal{B}$ be an oriented smooth manifold with boundary. Then $\partial\mathcal{B}$ is orientable, and the orientation determined by any outward-pointing vector field along $\partial\mathcal{B}$ is independent of the choice of vector field.*

Note that if $\mathcal{B}$ is a smooth manifold with corners its boundary is be a union of smooth manifolds each of which is orientable using an outward-pointing vector field.

Theorem 4.3 *If $\mathcal{B}$ is a smooth manifold with corners and $\mathcal{U} = \{U_\alpha\}$ is any open cover of $\mathcal{B}$, there exists a smooth partition of unity subordinate to $\mathcal{U}$.*

Lemma 4.4 *Let $\psi : N \to P$ be a diffeomorphism, and N_1 be an embedded submanifold of N, if $\widehat{\psi} : N_1 \to P_1$, where $P_1 = \psi(N_1)$, is the restriction of ψ to N_1, then for every tangent vector $\boldsymbol{v} \in T_{n_1}N_1$*

$$(\widehat{\psi})_*(\boldsymbol{v}) = \psi_*(\boldsymbol{v}),$$

in particular

$$\psi_*(\boldsymbol{v}) \in T_{p_1}P_1,$$

where $p_1 = \psi(n_1)$.

Proof First, note that since ψ is a diffeomorphism, $\widehat{\psi}$ is differentiable the left hand side of the above equation is well-defined. Every tangent vector on N_1, such as $\boldsymbol{v}$, can be thought of as a tangent to a curve, say θ, in N_1. However,

$$(\widehat{\psi})_*(\boldsymbol{v}) = (\widehat{\psi})_*\dot{\theta} = \dot{\overline{\widehat{\psi} o \theta}} = \dot{\overline{\psi o \theta}} = (\psi)_*(\boldsymbol{v}).$$

Note that since N_1 and P_1 are embedded submanifolds, $T_{n_1}N_1$ and $T_{p_1}P_1$ can be considered as subspaces of $T_{n_1}N$ and $T_{p_1}P$, respectively. $\square$

Lemma 4.5 *If $\boldsymbol{\gamma}$ is a vector field on a set $U \subseteq I\!\!R^n$ and $\boldsymbol{N}$ is a unit vector at X*

$$(\boldsymbol{\gamma}(X) \lrcorner\, d\boldsymbol{V}(X))|_{\boldsymbol{N}^\perp} = \langle \boldsymbol{\gamma}(X), \boldsymbol{N} \rangle (\boldsymbol{N} \lrcorner\, d\boldsymbol{V}(X))|_{\boldsymbol{N}^\perp}.$$

Proof If $(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-1})$ is a basis for $\boldsymbol{N}^\perp$, then $(\boldsymbol{N}, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-1})$ is a basis for the tangent space at X, and therefore there exist scalars $a_1, a_2, \ldots, a_n$ such that

$$\boldsymbol{\gamma}(X) = \langle \boldsymbol{\gamma}(X), \boldsymbol{N} \rangle \; \boldsymbol{N} + \sum_{i=1}^{n-1} a_i \boldsymbol{v}_i.$$

Substituting this into $(\boldsymbol{\gamma}(X) \lrcorner\, d\boldsymbol{V}(X))|_{\boldsymbol{N}^\perp}$ the desired equation follows. $\square$

Using 2.2, 2.7 and 4.1 to 4.5 we are ready to prove the Generalized Cauchy's Theorem.

Proof of Theorem 2.9 From 4.1 and 4.2 it is clear that $\boldsymbol{\tau}_{\mathcal{P}}$ is well-defined almost everywhere (in the sense explained in 4.1). Since $\mathcal{B}$ is a smooth manifold, it is second countable locally Euclidean and consequently, $\mathcal{B}$ admits a countable atlas of coordinate charts $\{(U_m, \psi_m)\}$ such that ψ_m is orientation preserving on U_m. We show that for every m there exist an $(n-1)$-form $\boldsymbol{\sigma}_m$ on U_m with the property stated in Theorem 2.9.

Pick a compact submanifold with corners $\mathcal{P} \subset U_m$ such the interior points of $\psi_m(\mathcal{P})$ form an open set with Lipschitz boundary. Writing the integral equation in Theorem 2.9 for $\mathcal{P}$

$$\int_{\mathcal{P}} \boldsymbol{\alpha} = \int_{\mathcal{P}} \boldsymbol{\beta} + \int_{\partial \mathcal{P}} \boldsymbol{\tau}_{\mathcal{P}}.$$

Now, since ψ_m is an orientation preserving diffeomorphism from U_m to $\psi_m(U_m)$ by diffeomorphism invariance property of integrals on manifolds

$$\int_{\psi_m(\mathcal{P})} (\psi_m^{-1})^* \boldsymbol{\alpha} = \int_{\psi_m(\mathcal{P})} (\psi_m^{-1})^* \boldsymbol{\beta} + \int_{\partial\psi_m(\mathcal{P})} (\widehat{\psi}_m^{-1})^* \boldsymbol{\tau}_{\mathcal{P}}, \tag{5}$$

where $\widehat{\psi}_m$ is the restriction of ψ_m to $\partial\mathcal{P}$, and so itself is an orientation preserving diffeomorphism on $\partial\mathcal{P}$. Define

$$\boldsymbol{\alpha}_m := (\psi_m^{-1})^* \boldsymbol{\alpha}, \quad \boldsymbol{\beta}_m := (\psi_m^{-1})^* \boldsymbol{\beta},$$

and,

$$\boldsymbol{\tau}_{\psi_m(\mathcal{P})} := (\widehat{\psi}_m^{-1})^* \boldsymbol{\tau}_{\mathcal{P}},$$

$$\boldsymbol{\tau}_m(X, \boldsymbol{N}) := (\psi_m^{-1})^* \boldsymbol{\tau}(\psi_m^{-1}(X), \psi_m^* \boldsymbol{N}^{\perp}, \psi_m^*(\boldsymbol{N})), \tag{6}$$

where $\boldsymbol{N}$ is any unit vector at $X \in U_m$ and $\boldsymbol{N}^{\perp}$ is the $(n-1)$-dimensional subspace normal to $\boldsymbol{N}$. Note that due to the fact that ψ_m is diffeomorphism, the boundedness and continuity properties of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\tau}$ are transferred to $\boldsymbol{\alpha}_m$, $\boldsymbol{\beta}_m$ and $\boldsymbol{\tau}_m$ respectively. In order to apply Theorem 2.2 to (5) we require to show that for the unit normal outward-pointing vector field $\boldsymbol{N}$ on $\partial\psi_m(\mathcal{P})$

$$\boldsymbol{\tau}_{\psi_m(\mathcal{P})}(X) = \boldsymbol{\tau}_m(X, \boldsymbol{N}(X)).$$

If $X \in \partial\mathcal{P}$ by definition of $\boldsymbol{\tau}_{\psi_m(\mathcal{P})}$ and $\boldsymbol{\tau}_{\mathcal{P}}$(see Theorem 2.9) for $x = \psi_m^{-1}(X)$ we have

$$\boldsymbol{\tau}_{\psi_m(\mathcal{P})}(X) = (\widehat{\psi}_m^{-1})^*(\boldsymbol{\tau}_{\mathcal{P}}(x)) = (\widehat{\psi}_m^{-1})^*(\boldsymbol{\tau}(x, T_x\partial\mathcal{P}, \boldsymbol{n}(x)), \tag{7}$$

where $\boldsymbol{n}$ is an outward-pointing vector field on $\partial\mathcal{P}$. Note that because $\boldsymbol{N}(X)$ is normal to $\partial\psi_m(\mathcal{P})$, we have

$$T_X\partial\psi_m(\mathcal{P}) = \boldsymbol{N}(X)^{\perp},$$

and so,

$$T_x\partial\mathcal{P} = \psi_m^*(\boldsymbol{N}(X)^{\perp}).$$

Since $\boldsymbol{N}$ is the smooth outward-pointing vector field on $\partial\psi_m(\mathcal{P})$, $\psi_m^*(\boldsymbol{N})$ is a smooth outward-pointing vector field on $\partial\mathcal{P}$. Therefore, since $\partial\mathcal{P}$ is an embedded submanifold, using lemma 4.4 , from (7)

$$\boldsymbol{\tau}_{\psi_m(\mathcal{P})}(X) = (\psi_m^{-1})^*(\boldsymbol{\tau}(x, \psi_m^*(\boldsymbol{N}(X)^{\perp}), \psi_m^*(\boldsymbol{N})(x)).$$

Finally, using (6)

$$\boldsymbol{\tau}_{\psi_m(\mathcal{P})}(X) = \boldsymbol{\tau}_m(X, \boldsymbol{N}(X)).$$

Now, substituting the above results into (5) we have

$$\int_{\psi_m(\mathcal{P})} \boldsymbol{\alpha}_m = \int_{\psi_m(\mathcal{P})} \boldsymbol{\beta}_m + \int_{\partial\psi_m(\mathcal{P})} \boldsymbol{\tau}_{\psi_m(\mathcal{P})}.$$

where,

$$\boldsymbol{\tau}_{\psi_m(\mathcal{P})}(X) = \boldsymbol{\tau}_m(X, \boldsymbol{N}(X)).$$

Note that by compactness of $\mathcal{P}$ the set $\psi_m(\mathcal{P})$ is bounded. Therefore, since ψ_m is a diffeomorphism, from the above integral equation we deduce that for every bounded open subset $\mathcal{Q}$ of $\psi_m(U_m)$ with Lipschitz boundary

$$\int_{\mathcal{Q}} \boldsymbol{\alpha}_m = \int_{\mathcal{Q}} \boldsymbol{\beta}_m + \int_{\partial\mathcal{Q}} \boldsymbol{\tau}_{\mathcal{Q}}, \tag{8}$$

where

$$\boldsymbol{\tau}_{\mathcal{Q}}(X) = \boldsymbol{\tau}_m(X, \boldsymbol{N}(X)).$$

There exist unique scalar functions α_m and β_m, τ_m on $\psi_m(U_m)$ such that

$$\boldsymbol{\alpha}_m = \alpha_m \ d\boldsymbol{V}, \ \boldsymbol{\beta}_m = \beta_m \ d\boldsymbol{V}$$

$$\boldsymbol{\tau}_m(X, \boldsymbol{N}) = \tau_m(X, \boldsymbol{N}) \ d\boldsymbol{A}(X, \boldsymbol{N})$$

where $d\boldsymbol{V}$ is the volume form on $I\!R^n$, $\boldsymbol{N}$ is a unit vector at X and $d\boldsymbol{A}(X, \boldsymbol{N})$ is the area form $\boldsymbol{N} \lrcorner d\boldsymbol{V}(X)$ restricted to $\boldsymbol{N}^{\perp}$. Furthermore, for every $\mathcal{Q} \subseteq \psi_m(U_m)$ there exists a unique function $\tau_{\mathcal{Q}}$ on $\partial\mathcal{Q}$ such that

$$\boldsymbol{\tau}_{\mathcal{Q}} = \tau_{\mathcal{Q}} \ d\boldsymbol{A}_{\mathcal{Q}},$$

where $d\boldsymbol{A}_{\mathcal{Q}}$ is the area form on $\partial\mathcal{Q}$ which is oriented by an outward-pointing vector field. As a consequence of these definitions,

$$\tau_{\mathcal{Q}}(X) = \tau_m(X, \boldsymbol{N}(X)).$$

Substituting these equations into (8)

$$\int_{\mathcal{Q}} \alpha_m \ d\boldsymbol{V} = \int_{\mathcal{Q}} \beta_m \ d\boldsymbol{V} + \int_{\partial\mathcal{Q}} \tau_{\mathcal{Q}} \ d\boldsymbol{A}_{\mathcal{Q}},$$

Since the above equation is an integral equation in $I\!R^n$ we can interpret the integral of forms as multiple integrals, that is for every $\mathcal{Q} \subseteq \psi_m(U_m)$,

$$\int_{\mathcal{Q}} \alpha_m \ dV = \int_{\mathcal{Q}} \beta_m \ dV + \int_{\partial\mathcal{Q}} \tau_{\mathcal{Q}} \ dA,$$

such that,

$$\tau_{\mathcal{Q}}(X) = \tau_m(X, \boldsymbol{N}(X)).$$

By assumption and definition 2.8, τ_m is continuous. Therefore, applying Theorem 2.2 there exists a unique continuous vector field $\boldsymbol{\gamma}_m$ on $\psi_m(U_m)$ such that for every point $X \in \psi_m(U_m)$ and every unit vector $\boldsymbol{N}$ at X

$$\tau_m(X, \boldsymbol{N}) = \langle \boldsymbol{\gamma_m}(X), \boldsymbol{N} \rangle.$$

Define the $(n-1)$-form field $\boldsymbol{\sigma}_m$ on U_m as follows

$$\boldsymbol{\sigma}_m = (\psi_m)^*(\boldsymbol{\gamma}_m \lrcorner \ d\boldsymbol{V}). \tag{9}$$

We claim that for every $x \in U_m$, if H is a hyperplane at x and $\boldsymbol{n} \in T_x\mathcal{B}\backslash H$ we have

$$\boldsymbol{\sigma}_m(x)|_H = \boldsymbol{\tau}(x, H, \boldsymbol{n}). \tag{10}$$

To show this, by lemma 2.6 it suffices to show that for every unit normal $\boldsymbol{N}$ at $X \in \psi_m(U_m)$

$$\boldsymbol{\sigma}_m(x)|_{\psi_m^*(\boldsymbol{N}^\perp)} = \boldsymbol{\tau}(x, \psi_m^*\boldsymbol{N}^\perp, \psi_m^*\boldsymbol{N}).$$

From (9) the previous equation reduces to

$$(\psi_m)^*(\boldsymbol{\gamma}_m(X) \lrcorner\, d\boldsymbol{V}(X))|_{\boldsymbol{N}^\perp} = \boldsymbol{\tau}(x, \psi_m^*\boldsymbol{N}^\perp, \psi_m^*\boldsymbol{N}).$$

Thus, by (6) and definition of τ_m it remains to show

$$(\boldsymbol{\gamma}_m(X) \lrcorner\, d\boldsymbol{V}(X))|_{\boldsymbol{N}^\perp} = \tau_m(X, \boldsymbol{N})(\boldsymbol{N} \lrcorner\, d\boldsymbol{V}(X))|_{\boldsymbol{N}^\perp}.$$

However, since $\tau_m(X, \boldsymbol{N}) = \langle \boldsymbol{\gamma_m}(X), \boldsymbol{N}\rangle$, the above equation is an immediate result of lemma 4.5 and hence (10) is satisfied.

To complete the proof we construct $\boldsymbol{\sigma}$ on $\mathcal{B}$ using Theorem 4.3. Let $\mathcal{U} = \{U_m\}$, by Theorem 4.3 there is a collection of smooth functions $\{\phi_m : \mathcal{B} \to I\!R\}$ subordinate to $\mathcal{U}$. Define $\boldsymbol{\sigma}$ as follows

$$\boldsymbol{\sigma} = \sum_m \phi_m \boldsymbol{\sigma}_m.$$

Now, by (10) for a hyperplane H at $x \in \mathcal{B}$

$$\boldsymbol{\sigma}(x)|_H = \sum_m \phi_m(x)\boldsymbol{\sigma}_m(x)|_H = \sum_m \phi_m(x)\boldsymbol{\tau}(x, H, \boldsymbol{n}).$$

Finally, since for every $x \in \mathcal{B}$, $\sum_m \phi_m(x) = 1$, the last equation implies

$$\boldsymbol{\sigma}(x)|_H = \boldsymbol{\tau}(x, H, \boldsymbol{n}) \qquad \forall x \in \mathcal{B}, \quad \boldsymbol{n} \in T_x\mathcal{B}\backslash H.$$

The uniqueness and continuity of $\boldsymbol{\sigma}$ follows from the above equation. $\square$

Similar to Theorem 2.2, here, there is an important result that follows readily from Theorem 2.9.

Corollary 4.6 *With the assumptions of Theorem 2.9,*

$$\boldsymbol{\tau}(x, H, \boldsymbol{n}) = \boldsymbol{\tau}(x, H, -\boldsymbol{n}) \qquad \forall x \in \mathcal{B}, \quad \boldsymbol{n} \in T_x\mathcal{B}\backslash H.$$

In other words, the value of $\boldsymbol{\tau}$ at a hyperplane H is independent of the orientation of the hyperplane H.

This result might seem to be in contradiction to Corollary 3.1 and our physical intuition. However, in fact, it is completely consistent with Corollary 3.1 and our physical intuition! What does $\boldsymbol{\tau}(x, H, \boldsymbol{n})$ show? It is an $(n-1)$-form on the oriented hyperplane H such that if $(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-1})$ is an oriented basis for H, $\boldsymbol{\tau}(x, H, \boldsymbol{n})(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-1})$ is the amount of the physical property, for example heat flux, that leaves the infinitesimal area generated by $(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-1})$ in the oriented hyperplane $(x, H, \boldsymbol{n})$. Now let us reverse the orientation of H. We expect that the amount that leaves $(x, H, -\boldsymbol{n})$ be minus of the amount that leaves $(x, H, \boldsymbol{n})$.(For example the heat flux in a body in $I\!\!R^3$ that flows in direction $\boldsymbol{n}$ is negative to the heat flux that flows in direction $-\boldsymbol{n}$). To compute this amount we need to choose an oriented basis for $(x, H, -\boldsymbol{n})$, but since $(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-1})$ is oriented in $(x, H, \boldsymbol{n})$, $(-\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-1})$ is oriented in $(x, H, -\boldsymbol{n})$. So, $\boldsymbol{\tau}(x, H, -\boldsymbol{n})(-\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-1})$ is the amount of physical property that leaves $(x, H, -\boldsymbol{n})$. However, since $\boldsymbol{\tau}(x, H, -\boldsymbol{n}) = \boldsymbol{\tau}(x, H, \boldsymbol{n})$ is an $(n-1)$-form on H

$$\boldsymbol{\tau}(x, H, -\boldsymbol{n})(-\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-1}) = -\boldsymbol{\tau}(x, H, \boldsymbol{n})(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-1}),$$

and this is exactly the expected result. Furthermore, the consistency of Corollaries 3.1 and 4.6 can be shown as follows. Suppose that in Theorem 2.9 Ω is the orientation form on $\mathcal{B}$. Let $(x, H, \boldsymbol{n})$ be an arbitrary oriented hyperplane. Clearly, $\boldsymbol{n} \lrcorner \Omega(x)$ is an $(n-1)$-form at x, so for the oriented hyperplane $(x, H, \boldsymbol{n})$, there exists a scalar $\tau(x, H, \boldsymbol{n})$ such that,

$$\boldsymbol{\tau}(x, H, \boldsymbol{n}) = \tau(x, H, \boldsymbol{n})\ (\boldsymbol{n} \lrcorner \Omega(x))|_H.$$

This equation defines a scalar function τ on oriented hyperplanes. $\tau(x, H, \boldsymbol{n})$ can be thought of as a representative of the amount of the physical property that leaves the oriented hyperplane $(x, H, \boldsymbol{n})$. Now, if the orientation of the hyperplane is reversed

$$\boldsymbol{\tau}(x, H, \boldsymbol{n}) = \boldsymbol{\tau}(x, H, -\boldsymbol{n}).$$

So,

$$\tau(x, H, \boldsymbol{n})\ (\boldsymbol{n} \lrcorner \Omega(x))|_H = \tau(x, H, -\boldsymbol{n})\ (-\boldsymbol{n} \lrcorner \Omega(x))|_H.$$

this equation is satisfied if and only if,

$$\tau(x, H, \boldsymbol{n}) = -\tau(x, H, -\boldsymbol{n}),$$

and this is what we had in Corollary 3.1.

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