Symmetric Virtual Constraints for Periodic Walking of Legged Robots

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***Abstract*— Dynamic and agile walking or running gaits for legged robots correspond to periodic orbits in the dynamic model. As the number of degrees of freedom of the models grows, obtaining periodic gaits for these systems through numerical optimization is not always straightforward. In contrast to common numerical search methods for periodic orbits, this paper introduces a class of holonomic constraints, called *symmetric virtual constraints*, which when enforced by controllers, ensure the existence of periodic orbits. The main advantage of symmetric virtual constraints is that they relax the need for online or offline parameter search for periodic orbits, and at the same time lead directly to feedback controllers that realize a periodic orbit.**

I. Introduction

It is known that, compared to static and ZMP walking [15], limit cycle walking can allow agile, fast and human-like walking [11], [16]. However, obtaining such periodic orbits for legged robots is not always straightforward.

Perhaps, the most common method of obtaining periodic orbits for a system is based on numerical search for fixed points of a Poincaré map [10], [7], [17], [5], [6], [4]. This numerical search, however, especially for higher order systems, is very cumbersome because the differential equations describing the system need to be integrated for each trial, and it is then checked wether the solution returns back to the starting point (in order to have a periodic orbit).

To reduce the computational costs of the numerical search, Grizzle et al. [7] have used the notion of virtual constraints and Hybrid Zero dynamics (HZD) to conduct the search on a lower dimensional system. Virtual constraints are relations between the generalized coordinates of the system that are enforced by controllers. By using a class of Bézier polynomials for virtual constraints, Grizzle et al. [8] have demonstrated the possibility of gait design together with optimization on energy, torque limit, etc.

In contrast to search methods, the symmetry method, presented in this paper, relaxes the need for any offline or online searches for periodic orbits. Based on this method, as described in Fig. 1, first, the natural symmetries of the legged robot are detected. Then virtual constraints are chosen such that the symmetry is preserved while the dimension of the system is reduced (i.e., the resulting HZD is symmetric) by enforcing the virtual constraints using controllers. Such virtual constraints are called Symmetric Virtual Constraints (SVC). It is then shown that the resulting HZD is a Symmetric Hybrid System (SHS) [14], and consequently, has an infinite number of symmetric periodic orbits, which can be identified easily (i.e., without any searches). The SVCs also allow gait design (with possible optimization on energy, torque limit, etc.). Moreover, it will be shown that with SVCs the resulting SHS automatically has a family of symmetric periodic gaits (rather than one single periodic orbit).

Even though the resulting SHS possesses symmetric periodic orbits, such periodic orbits at best are only neutrally stable. However, as described in Fig. 1, by introducing appropriate asymmetries to the system or by foot placement the neutrally stable periodic orbits become asymptotically stable. This paper, however, for the most part discusses steps 1 and 2 in Fig. 1.

The rest of the paper is organized as follows. In Section II, our notion of symmetry is defined intuitively through a few examples. In Section III, the notions of Symmetric Vector Fields and SHSs are presented, and it is shown that with a proper transition map the symmetric solutions of an SHS can become symmetric periodic orbits. Reduction of the dimension of the system by using an appropriate control law is explained in Section IV. SVCs are discussed in Section V. Section VI briefly discusses the stabilization mechanisms, and Section VII includes the concluding remarks.

It should be noted that due to limited space, some of the proofs are not provided, and this paper includes only 2D biped examples, while the notion of SVCs is applicable to 3D legged locomotion as well. In a related paper, which will serve as an extension of the current paper more detailed proofs together with 3D legged locomotion examples will be presented.

II. Notion of Symmetry

In the literature, the word "symmetry" has been used to refer to various concepts. To avoid possible confusion, this section includes a few examples to serve as a quick and intuitive introduction to the notion of symmetry for legged robots presented here.

The symmetry that we refer to in this paper is an invariance in the equations of motion under a discrete transformation of variables. For instance, the kinetic and potential energies of the 2D Double Inverted Pendulum (DIP) depicted in Fig.

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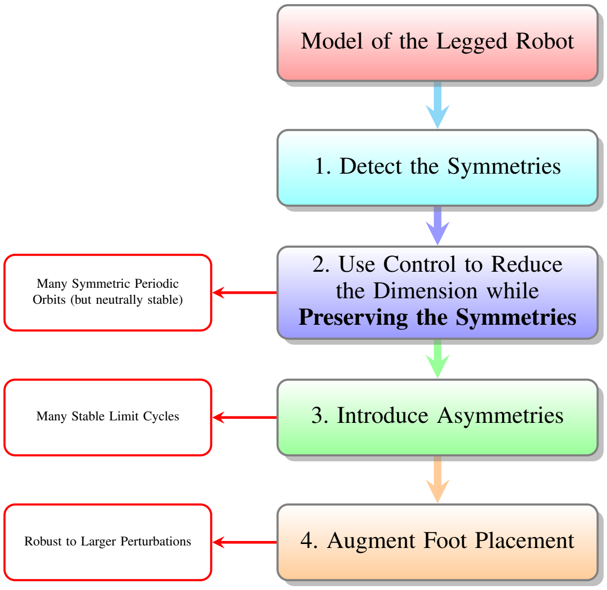

Fig. 1. High-level control algorithm for stable limit cycle walking. In the second step, we use symmetric virtual constraints which preserve the symmetry while reducing the dimension of the system.

2 are invariant under the map G which sends $(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ to $(-\theta_1, -\theta_2, \dot\theta_1, \dot\theta_2)$. As a result, in the equations of motion of the 2D DIP, which can be written as

$$\begin{aligned}\ddot\theta_1 &= f(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2),\\ \ddot\theta_2 &= g(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2),\end{aligned}$$

we have $f(-\theta_1, -\theta_2, \dot\theta_1, \dot\theta_2) = -f(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ and $g(-\theta_1, -\theta_2, \dot\theta_1, \dot\theta_2) = -g(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$.

Similarly, the Spring Loaded Inverted Pendulum (SLIP) and the 5-DOF biped as explained in Fig. 3 and Fig. 4 are symmetric under the specified transformations.

In fact, many legged robots have similar symmetries either exactly or approximately. Sources of possible asymmetries in a legged robot include the knees (which can rotate only in one direction), the feet and asymmetric mass distribution. However, such asymmetries are generally small compared to the overall symmetric structure of the system. Furthermore, as briefly described in Section VI and with more details in [14], such asymmetries could help with asymptotic stability of the symmetric periodic orbits.

Raibert [12] has noted the same notion of symmetry for planar legged robots, and has shown that for a planar hopper (even though many 3D experiments are presented as well), such symmetry can lead to steady-state running. The current paper and [14] generalize the notion of symmetry in [12] to a large class of legged robots (2D or 3D) and build a mathematical basis for what we call Symmetric Hybrid Systems. Moreover, a study of stability and a method for stabilization of symmetric periodic orbits are presented. Finally, this paper introduces the novel notion of Symmetric Virtual Constraints which allow gait design while exploiting the advantages of the notion of symmetry.

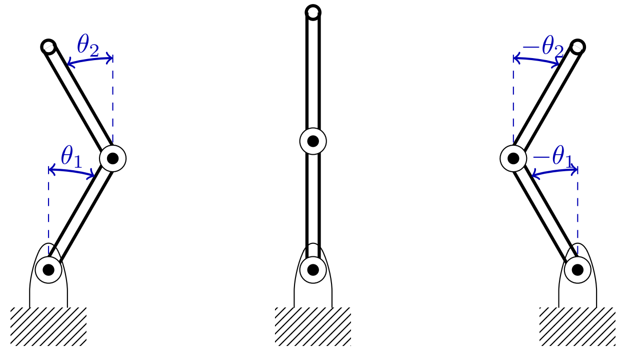

Fig. 2. The kinetic and potential energy of the Double Inverted Pendulum (DIP) is invariant under the map $(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2) \mapsto (-\theta_1, -\theta_2, \dot\theta_1, \dot\theta_2)$. The fixed points of this map, as shown in the middle figure, occurs at $\theta_1 = 0$ and $\theta_2 = 0$, with arbitrary $\dot\theta_1^*$ and $\dot\theta_2^*$.

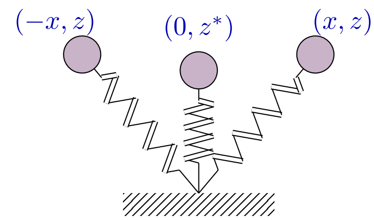

Fig. 3. The kinetic energy and potential energy of the SLIP are invariant under the map $(x, z, \dot x, \dot z) \mapsto (-x, z, \dot x, -\dot z)$. The fixed points of the map G, which correspond to the configuration in the middle figure the middle SLIP model in the figure above, are $(0, z^*, \dot x^*, 0)$ for arbitrary z^* and $\dot x^*$.

III. Symmetric Vector Fileds and Symmetric Hybrid Systems

The equations of motion of the examples presented in Section II all can be described with what we call Symmetric Vector Fields (SVF). In this section, we discuss SVFs, SHSs and their symmetric solutions. We show that the symmetric solutions of an SHS can become symmetric periodic orbits.

Definition 1. The smooth vector field X defined on a manifold $\mathcal{X}$ is said to be *symmetric* under the smooth map $G : \mathcal{X} \to \mathcal{X}$ (or in short G-symmetric) if

$$X \circ G(x) = -dG(x) \cdot X(x), \tag{1}$$

Moreover, G is said to be a *symmetry* map for X.

This type of symmetry of a vector field, which has been referred to as time reversal symmetry in [1], is closely related to the notion of equivariant vector fields [3].

Proposition 2. *Let X be a symmetric vector field (SVF) defined on a manifold $\mathcal{X}$, and let $G : \mathcal{X} \to \mathcal{X}$ be a symmetry map for X with a fixed point x^*, that is, $G(x^*) = x^*$. Then the solution of X (a.k.a. integral curve of X) passing through x^* is invariant under G. That is, if $x : \mathcal{I} \to X$, where $\mathcal{I} = (-a, a)$ is an open interval of $\mathbb{R}$ for some $a > 0$, and $\dot x(t) = X(x(t))$ with $x(0) = x^*$, and $\mathcal{C} = \{x(t) | t \in \mathcal{I}\}$, then $\mathcal{C}$ is invariant under G. Moreover, $G(x(t)) = x(-t)$ for all $t \in \mathcal{I}$.*

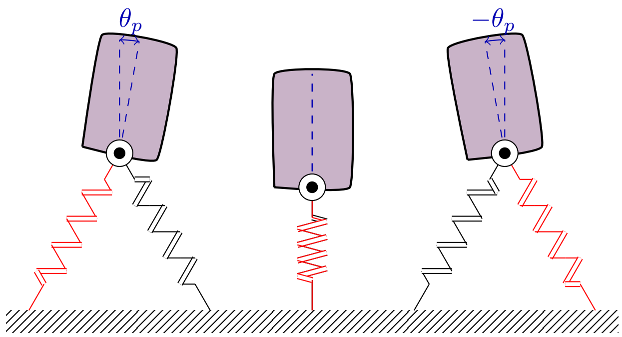

Fig. 4. Let $(x, z, \theta_p, x_{fh}, z_{fh})$ denote the generalized coordinates, where (x, z) is the position of the hip, θ_p is the pitch angle and (x_{fh}, z_{fh}) is the swing leg end position relative to hip. Assuming that the legs are identical and the mass distribution is uniform, the Lagrangian is invariant under the map G which maps $(x, z, \theta_p, x_{fh}, z_{fh}) \mapsto (-x, z, -\theta_p, -x_{fh}, z_{fh})$.

Proof. Define $\hat{x}(t) = G(x(-t))$ for $t \in \mathcal{I}$. We have $\hat{x}(0) = G(x(0)) = G(x^*) = x^*$. Therefore, $\hat{x}(t)$ and $x(t)$ satisfy the same initial conditions. Next we show that $\hat{x}(t)$ is an integral curve of X. By definition of $\hat{x}(t)$, $\dot{\hat{x}}(t) = -dG \cdot \dot{x}(-t)$. Thus, since $x(t)$ is a solution of X, $\dot{\hat{x}}(t) = -dG \cdot X(x(-t))$. From (1), $\dot{\hat{x}}(t) = X(G(x(-t)))$, and by definition of $\hat{x}(t)$, $\dot{\hat{x}}(t) = X(\hat{x}(t))$, which proves that $\hat{x}(t)$ is a solution of X. By uniqueness of the solution of the initial value problem, $\hat{x}(t) = x(t)$, that is, $G(x(-t)) = x(t)$ for all $t \in \mathcal{I}$; equivalently, $G(x(t)) = x(-t)$ for all $t \in \mathcal{I}$. $\square$

Example 3. Consider the following dynamical system defined on $\mathbb{R}^2$,

$$\begin{aligned} \dot{x}_1 &= x_2 \sin(x_2) + x_2 x_1^2, \\ \dot{x}_2 &= x_1^2 \sin(x_1) + 2x_1 x_2. \end{aligned}$$

This system can be written as $[\dot{x}_1; \dot{x}_2] = X(x_1, x_2)$, where $X(x_1, x_2) = [x_2 \sin(x_2) + x_2 x_1^2; x_1^2 \sin(x_1) + 2x_1 x_2]$. Define $G : \mathbb{R}^2 \to \mathbb{R}^2$ by $G(x_1, x_2) = (-x_1, x_2)$. Since $X \circ G(x_1, x_2) = X(-x_1, x_2) = (x_2 \sin(x_2) + x_2 x_1^2; -x_1^2 \sin(x_1) - 2x_1 x_2)$, and

$$\begin{aligned} -dG \cdot X(x_1, x_2) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} x_2 \sin(x_2) + x_2 x_1^2 \\ x_1^2 \sin(x_1) + 2x_1 x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_2 \sin(x_2) + x_2 x_1^2 \\ -x_1^2 \sin(x_1) - 2x_1 x_2 \end{bmatrix}, \end{aligned}$$

we conclude that $X \circ G(x) = -dG \cdot X(x)$; hence, X is G-symmetric.

As the next proposition shows, for a Lagrangian system we can check the symmetry of the corresponding vector field by just looking at the Lagrangian.

Proposition 4. *Let L be the Lagrangian defined on the configuration space $\mathcal{Q}$, and let $F : \mathcal{Q} \to \mathcal{Q}$ be a smooth map. Define $G : \mathcal{TQ} \to \mathcal{TQ}$ by*

$$G(q, \dot{q}) = (F(q), -dF(q) \cdot \dot{q}).$$

If L is invariant under G, that is,

$$L \circ G(q, \dot{q}) = L(q, \dot{q}),$$

and $x^* = (q^*, \dot{q}^*)$ is a fixed point of G, then for the solution $x(t) = (q(t), \dot{q}(t))$ defined on $\mathcal{I} = (-a, a)$ for $a > 0$, with $x(0) = x^*$, we have $G(x(t)) = x(-t)$. Equivalently,

$$F(q(t)) = q(-t). \tag{2}$$

Finally, if X is the vector field defining the state space representation of the Lagrangian system, then X is symmetric under G.

Proof. Suppose that $x(t) = (q(t), \dot{q}(t))$ is the solution of the Lagrangian system for which $x(0) = x^*$, where x^* is a fixed point of G. Define $\hat{x}(t) = G(x(-t))$. At $t = 0$, $\hat{x}(0) = G(x(0)) = G(x^*) = x^*$. Therefore, $\hat{x}(t)$ and $x(t)$ satisfy the same initial conditions. To prove that $\hat{x}(t) = x(t)$ for all $t \in \mathcal{I}$, we show that $\hat{x}(t)$ satisfies the Euler-Lagrange equations of motion. To this end, from the Hamilton's principle [2], it suffices to show that $\delta \int_{-t_i}^{t_i} L(\hat{x}(t))dt = 0$. However, by definition of $\hat{x}(t)$, $\delta \int_{-t}^{t} L(\hat{x}(t))dt = \delta \int_{-t}^{t} L(G(x(-t)))dt$. Invariance of L under G yields $L(G(x(-t)) = L(x(-t))$. Thus for any $t_i \in (-a, a)$ we have $\delta \int_{-t_i}^{t_i} L(\hat{x}(t))dt = \delta \int_{-t_i}^{t_i} L(x(-t))dt = \delta \int_{-t_i}^{t_i} L(x(t))dt = 0$, where the second equality is obtained by simple substitution of $t \to -t$, and the last equality follows from the fact that $x(t)$ is a solution to the Lagrangian system. Therefore, $\hat{x}(t)$ satisfies the Euler-Lagrange equations as $x(t)$ does, and since $\hat{x}(t)$ and $x(t)$ both satisfy the same initial conditions, by uniqueness of the solution of the initial value problem, we have $\hat{x}(t) = x(t)$; thus, $G(x(-t)) = x(t)$, as desired. $\square$

It should be noted that in the above proposition G is not a coordinate transformation because G is defined as $(F, -dF)$ not (F, dF).

Example 5. (2D Double Inverted Pendulum (2D DIP) Consider the double inverted pendulum depicted in Figure 2. In the coordinates (θ_1, θ_2), the kinetic and potential energies are

$$\begin{aligned} K &= \frac{1}{2} m_1 (l_1^2 \dot{\theta}_1^2) + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2), \\ V &= m_1 l_1 \cos(\theta_1) + m_2 (l_1 \cos(\theta_1) + l_2 \cos(\theta_2)). \end{aligned}$$

Define $F(\theta_1, \theta_2) = (-\theta_1, -\theta_2)$. Thus, as defined in Proposition 4, $G(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (-\theta_1, -\theta_2, \dot{\theta}_1, \dot{\theta}_2)$. Clearly, the Lagrangian $L = K - V$ is invariant under G. Since, $x^* = (0, 0, \dot{\theta}_1^*, \dot{\theta}_2^*)$ for $\dot{\theta}_1^*, \dot{\theta}_2^* \in \mathbb{R}$ are fixed points of G, the solutions $x(t) = (\theta_1(t), \theta_2(t), \dot{\theta}_1(t), \dot{\theta}_2(t))$ for which $x(0) = x^*$ satisfy the equation $F(\theta_1(t), \theta_2(t)) = (-\theta_1(t), -\theta_2(t))$; equivalently, $\theta_1(-t) = -\theta_1(t)$, $\theta_2(-t) = -\theta_2(t)$.

Next we discuss the notion of symmetry for hybrid systems. Throughout this paper we adopt the notion of hybrid systems as in [16]. The following proposition shows that the symmetric solutions of a symmetric system can become periodic orbits if an appropriate transition map is added to the system.

Proposition 6. *Consider the following hybrid system defined on a manifold $\mathcal{X}$.*

$$\Sigma = \begin{cases} \dot{x} &= X(x) \quad x \notin \mathcal{S}, \\ x^+ &= \Delta(x^-) \quad x \in \mathcal{S}, \end{cases} \tag{3}$$

where $\mathcal{S}$ is a hypersurface called the switching surface, and Δ is the impact map (a.k.a. transition map). Suppose that X is symmetric under a map $G : \mathcal{X} \to \mathcal{X}$, and let $x^* \notin \mathcal{S}$ be a fixed point of G. If $x(t)$ is a solution to the hybrid system Σ such that $x(0) = x^*$, then $G(x(t)) = x(-t)$. Moreover, if the set $A := \{t > 0 | x(t) \in \mathcal{S}\}$ is non-empty with a minimum t_i, and for $x_0 = x(t_i)$

$$\Delta(x_0) = G(x_0), \tag{4}$$

then $x(t)$ is a periodic solution of Σ with period $T = 2t_i$.

Equation (4) simply says that the impact map sends the solution back to the point $x(-t_i)$ in which case $x(t)$ is periodic, and is said to be a *symmetric periodic* orbit of Σ.

Example 7. (2D Linear Inverted Pendulum (LIP) Biped) As shown in [14], the 2D LIP biped taking constant swing foot end to hip strides of length x_0 is a hybrid system with the following equations.

$$\begin{aligned} \ddot{x} &= \omega^2 x, \\ \mathcal{S} &= \{(x, \dot{x}) | x = x_0 > 0\}, \\ \Delta(x^-, \dot{x}^-) &= (-x_0, \dot{x}^-). \end{aligned}$$

In the state space representation of this system with $x_1 = x$ and $x_2 = \dot{x}$, the vector field $X(x_1, x_2) = (x_2, \omega^2 x_1)$ which is symmetric under the map $G(x_1, x_2) = (-x_1, x_2)$ has fixed points of the form $(0, \dot{x}^*)$. Let $(x(t), \dot{x}(t))$ denote a solution with $x(0) = 0$ and $\dot{x}(0) = \dot{x}^* > 0$. Noting that $\dot{x}(0) > 0$, and $\ddot{x} > 0$ for $x > 0$, $(x(t), \dot{x}(t))$ crosses the switching surface $\mathcal{S}$ at $x = x_0$ at which point the velocity is $\dot{x}^-$. However, $G(x_0, \dot{x}^-) = (-x_0, \dot{x}^-) = \Delta(x_0, \dot{x}^-)$. Hence, by Proposition 6, $x(t)$ is a symmetric periodic solution of the system.

Example 8. (2D Spring-Loaded Inverted Pendulum (2D SLIP)) Next we look at the 2D SLIP model in Fig. 3. Let z denote the length of the spring, and let θ denote the angle of the leg with respect to the center-line. Suppose that $V(\theta, z)$ denotes the potential and $K(\theta, z)$ denotes the kinetic energy of the system. If k is the spring constant, and l_0 is the no-load length of the spring,

$$\begin{aligned} V &= mgz\cos(\theta) + \tfrac{1}{2}k(z - l_0)^2, \\ K &= \tfrac{1}{2}m(\dot{z}^2 + z^2\dot{\theta}^2). \end{aligned}$$

The Euler-Lagrange equations of motion result in

$$\ddot{\theta} = -2\frac{\dot{z}}{z}\dot{\theta} - \frac{g}{z}\sin(\theta), \tag{5}$$

$$\ddot{z} = z\dot{\theta}^2 + \omega^2(l_0 - z) - g\cos(\theta). \tag{6}$$

To derive the equation of the transition map, we note that the flight phase starts when the spring length reaches its no-load length (i.e., $z = l_0$); therefore, the switching surface is defined as $\mathcal{S} = \{(\theta, z, \dot{\theta}, \dot{z}) | z = l_0\}$.

The flight phase consists of a projectile motion at the end of which, when $z = l_0$, the next stance phase starts. We assume that at the start of each step the leg is at an angle $-\theta_0$. Therefore,

$$\theta^+ = -\theta_0, \quad z^+ = l_0. \tag{7}$$

Hence, the transition occurs when the height of the mass is $l_0\cos(\theta_0)$. Writing the equations of motion of a projectile yields

$$\begin{aligned} \dot{x}^+ &= \dot{x}^-, \\ \dot{y}^+ &= -((\dot{y}^-)^2 - 2g(y^- - y_0))^{1/2}, \end{aligned} \tag{8}$$

where, $x = z\sin(\theta)$, $y = z\cos(\theta)$ and $y_0 = l_0\cos(\theta_0)$. Equation (8) implicitly defines the transition map of the SLIP. Equations (5) to (8) define the equations of motion of the SLIP, excluding the flight phase.

Looking at the SLIP kinetic and potential energies, it is clear that the Lagrangian $L = K - V$ is invariant under the map $F(\theta, z) = (-\theta, z)$ (note that according to Proposition 4, $G(\theta, z, \dot{\theta}, \dot{z}) = (-\theta, z, \dot{\theta}, -\dot{z})$). As a consequence, the corresponding vector field is symmetric under G. The fixed points of G are of the form $\chi^* = (0, z^*, \dot{\theta}^*, 0)$. Let $\phi(t, \chi^*) = (\theta(t), z(t), \dot{\theta}(t), \dot{z}(t))$ be the solution for which $\phi(0, \chi^*) = \chi^*$. Based on Proposition 6, $\phi(t, \chi^*)$ is invariant under G, in the sense that $G(\phi(t, \chi^*)) = \phi(-t, \chi^*)$; equivalently, $\theta(t)$ is an odd function, and $z(t)$ is an even function of t:

$$\theta(-t) = -\theta(t), \; z(-t) = z(t).$$

With numerical simulations it can be shown that there are infinitely many symmetric solutions $\phi(t, \chi^*)$ that cross the switching surface for different values of χ^*. Let $\chi(t)$ denote one of those solutions and assume that $\chi(t)$ crosses $\mathcal{S}$ at $\chi^- = (\theta_0, l_0, \dot{\theta}^-, \dot{z}^-)$. From the impact map defined by equations (7) and (8), we can show that $\Delta(\theta_0, l_0, \dot{\theta}^-, \dot{z}^-) = (-\theta_0, l_0, \dot{\theta}^-, -\dot{z}^-)$ which is equal to $G(\theta_0, l_0, \dot{\theta}^-, \dot{z}^-)$. Therefore, by Proposition 6, $\chi(t)$ is a symmetric periodic solution of the SLIP model.

IV. Symmetric Zero Dynamics and Symmetric Hybrid Zero Dynamics

Even though an SHS can have an infinite number of symmetric periodic solutions, generally, for these solutions to be stable we need to use control. However, the control laws, if not chosen carefully, can easily destroy the natural symmetry of the system. In this section, we show that with an appropriate choice of control laws, the resulting zero dynamics or hybrid zero dynamics is still symmetric and hence will have the properties of the SVFs or SHSs, while having lower dimensions compared to the original system.

Proposition 9. *Consider the following n-dimensional control system on $\mathcal{X} \times \mathcal{U}$*

$$\dot{x} = X(x, u),$$

such that $X(x,u)=f(x)+g(x)u$, and $u\in\mathcal{U}\subset\mathbb{R}^m$ is a control input with $m<n$. Suppose that $\mathcal{Z}$ is an $(n-m)$-dimensional zero dynamics submanifold of $\mathcal{X}$ enforced by $u(x)$ (thus $X(z,u(z))$ is tangent to $\mathcal{Z}$ for all $z\in\mathcal{Z}$)[1]. If there exists a map $G:\mathcal{X}\to\mathcal{X}$ and an isomorphism $H:\mathcal{U}\to\mathcal{U}$ such that

1) *$X(x,0)$ is symmetric under G,*
2) *$(g\circ G(x))H(u)=-dG\cdot g(x)u$ for all $x\in\mathcal{X}$ and $u\in\mathcal{U}$,*
3) *$\mathcal{Z}$ is invariant under G,*

then letting $X_{\mathcal{Z}}$ and $G_{\mathcal{Z}}$ denote restrictions of $X(x,u(x))$ and G to $\mathcal{Z}$, we have

$$X_{\mathcal{Z}}\circ G_{\mathcal{Z}}(z)=-dG_{\mathcal{Z}}\cdot X_{\mathcal{Z}}(z). \tag{9}$$

Moreover, if $x^*\in\mathcal{Z}$ is a fixed point of G, then the solution $x(t):\mathcal{I}\to X$ for which $x(0)=x^*$ lies on $\mathcal{Z}$, and $G(x(t))=x(-t)$ for all $t\in\mathcal{I}$. Finally, if $u(x)$ is the control law on the zero dynamics, and $v(x)=g(x)u(x)$, then $v\circ G(z)=-dG(z)\cdot v(z)$ on $\mathcal{Z}$.

As we shall see later, this proposition is very helpful in choosing virtual constraints for periodic walking of legged robots.

Example 10. Consider the following control system defined on $\mathbb{R}^2$,

$$\begin{aligned}\dot{x}_1 &= x_2\sin(x_2)+x_2x_1^2,\\ \dot{x}_2 &= x_1^2\sin(x_1)+2x_1x_2+u(x_1,x_2).\end{aligned}$$

This system can be written as $\dot{x}=X(x,u)$ such that $X(x,0)$ is the vector field in Example 3 which was shown to be symmetric under the map $G:(x_1,x_2)\mapsto(-x_1,x_2)$. We can write the above system in the form $\dot{x}=f(x)+g(x)u$, with $f(x)=X(x,0)$ and $g(x)=[0;1]$. It can be checked that $g\circ G=dG\cdot g$. We define the zero dynamics submanifold to be $\mathcal{Z}=\{(x_1,x_2)|x_2=h(x_1)\}$ such that h is an even function of x_1. This choice of h renders $\mathcal{Z}$ invariant under G. The zero dynamics then will be

$$\dot{x}_1=h(x_1)\sin(h(x_1))+h(x_1)x_1^2,$$

which satisfies (9). Moreover, on $\mathcal{Z}$

$$\begin{aligned}u(x_1) &= \frac{\partial h(x_1)}{\partial x_1}(h(x_1)\sin(h(x_1))+h(x_1)x_1^2)\\ &\quad -x_1^2\sin(x_1)-2x_1h(x_1),\end{aligned}$$

which satisfies $v\circ G(z)=-dG(z)\cdot v(z)$, where $v=[0;1]u$, or equivalently, $u(-z)=-u(z)$ for $z\in\mathcal{Z}$. Since $x^*=(0,h(0))$ is the fixed point of G on $\mathcal{Z}$, by Proposition 9, for solutions $x(t)=(x_1(t),x_2(t))$ with $x(0)=x^*$ we have $x(t)\in\mathcal{Z}$, and $G(x(t))=x(-t)$; that is, $x_2(t)=h(x_1(t))$, and $x_1(-t)=-x_1(t)$.

Example 11. (2D DIP Zero Dynamics) Consider the 2D DIP in Example 5. If u is an actuator that controls the angle between the two links, the equations of motion are

$$\begin{aligned}(m_1+m_2)l_1^2\ddot{\theta}_1+m_2l_1l_2\ddot{\theta}_2-(m_1+m_2)l_1\sin(\theta_1)&=-u,\\ m_2l_2^2\ddot{\theta}_2+m_2l_1l_2\ddot{\theta}_1-m_2l_2\sin(\theta_2)&=u.\end{aligned}$$

Recall that in Example 5 we showed that the 2D DIP Lagrangian is invariant under the map $F(\theta_1,\theta_2)=(-\theta_1,-\theta_2)$. Define the zero dynamics manifold to be

$$\mathcal{Z}=\{(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2)|\theta_2=h(\theta_1),\dot{\theta}_2=\frac{\partial h}{\partial\theta_1}\dot{\theta}_1\},$$

where h is an odd function of θ_1. Note that $\mathcal{Z}$ is invariant under $G=(F,-dF)$ which maps $(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2)$ to $(-\theta_1,-\theta_2,\dot{\theta}_1,\dot{\theta}_2)$. The restriction of G to $\mathcal{Z}$ is $G_{\mathcal{Z}}(\theta_1,\dot{\theta}_1)=(-\theta_1,\dot{\theta}_1)$ whose fixed points are of the form $(0,\dot{\theta}_1^*)$. Therefore, by Proposition 9, there are infinitely many solutions $\theta_1(t)$ that lie on $\mathcal{Z}$ and are invariant under G, that is, $\theta_1(-t)=-\theta_1(t)$. Moreover, the torque $u(\theta_1,\dot{\theta}_1)$ on $\mathcal{Z}$ is an odd function of θ_1, that is, $u(-\theta_1,\dot{\theta}_1)=-u(\theta_1,\dot{\theta}_1)$.

Next we look at the zero dynamics of a hybrid system with control input.

Proposition 12. *Consider the following n-dimensional hybrid system*

$$\Sigma=\begin{cases}\dot{x} &= X(x,u) \quad x\notin\mathcal{S},\\ x^+ &= \Delta(x^-) \quad x\in\mathcal{S},\end{cases} \tag{10}$$

such that $X(x,u)=f(x)+g(x)u$ is defined on $\mathcal{X}\times\mathcal{U}$, and u is an m-dimensional control input in $\mathcal{U}\subset\mathbb{R}^m$, and $m<n$. Assume that $\mathcal{Z}$ is a hybrid zero dynamics (HZD[2]) submanifold of Σ enforced by $u(x)$, that is, $\mathcal{Z}$ is a zero dynamics submanifold of $\mathcal{X}$, and $\Delta(\mathcal{Z}\cap\mathcal{S})\subset\mathcal{Z}$. Suppose that there exists a smooth map $G:\mathcal{X}\to\mathcal{X}$ and an isomorphism $H:\mathcal{U}\to\mathcal{U}$ such that

1) *$X(x,0)$ is a symmetric vector field under G,*
2) *$(g\circ G(x))H(u)=-dG\cdot g(x)u$ for all $x\in\mathcal{X}$ and $u\in\mathcal{U}$,*
3) *$\mathcal{Z}$ is invariant under G.*

If $X_{\mathcal{Z}}$ denotes the vector field $X(x,u(x))$ restricted to $\mathcal{Z}$, and $G_{\mathcal{Z}}$ denotes the map G restricted to $\mathcal{Z}$, then $X_{\mathcal{Z}}$ is symmetric under $G_{\mathcal{Z}}$, that is,

$$X_{\mathcal{Z}}\circ G_{\mathcal{Z}}(z)=-dG_{\mathcal{Z}}\cdot X_{\mathcal{Z}}(z), \tag{11}$$

In addition, $v(x)=g(x)u(x)$ satisfies $v\circ G(z)=-dG(z)\cdot v(z)$ on $\mathcal{Z}$. Moreover, if $x^*\in\mathcal{Z}$ is a fixed point of G, then the solution $x(t):\mathcal{I}\to X$ with $x(0)=x^*$ lies on $\mathcal{Z}$, and $G(x(t))=x(-t)$. Finally, if the set $A:=\{t>0|x(t)\in\mathcal{S}\}$ is non-empty with a minimum t_i, and for $x_0=x(t_i)$ we have

$$\Delta(x_0)=G(x_0),$$

then $x(t)$ is a periodic solution of Σ lying on $\mathcal{Z}$ with period $T=2t_i$.

Example 13. (3-DOF Biped HZD) Consider the 2D biped in Fig. 5, which is a simple 2D model of the bipedal robot

[1]For a detailed discussion of zero dynamics see [9].

[2]For a detailed discussion of HZD see [16].

MARLO [?]. Assuming that the legs are massless, this biped has 3 degrees of freedom (DOF). Suppose that the torso has a mass of m and a moment of inertia I about the center of mass (COM), and let l be the distance from the hip joint to the COM. Let (x, z) denote the hip position and let θ_p denote the pitch angle of the torso. The actuators include a motor at the hip which applies a torque u_θ to control the angle between the thigh and torso and an actuator which controls the knee angle. Without loss of generality (for non-zero knee angles), we can replace the torque at knee by a force f_l along the line connecting the support point to the hip; f_l controls the length of the leg.

The kinetic energy and potential energies of the biped are

$$\begin{aligned} K &= \frac{1}{2}(I+ml^2)\dot{\theta}_p^2 + \frac{1}{2}m(\dot{x}^2+\dot{z}^2+2l\dot{x}\dot{\theta}_p\cos(\theta_p) - \\ &\quad 2l\dot{z}\dot{\theta}_p\sin(\theta_p)), \\ V &= mg(z+l\cos(\theta_p)). \end{aligned}$$

To simplify the equations of motion, without loss of generality, we replace x/l by x, z/l by z, f_l/ml by f_l, and u_θ/ml^2 by u_θ. With these assignments, the equations of motion are:

$$\begin{aligned} &\ddot{x}+\ddot{\theta}_p\cos(\theta_p)-\dot{\theta}_p\sin(\theta_p) &&= F_1, \\ &\ddot{z}-\ddot{\theta}_p\sin(\theta_p)-\dot{\theta}_p^2\cos(\theta_p)+\tfrac{g}{l} &&= F_2, \\ &(\tfrac{I}{ml^2}+1)\ddot{\theta}_p+\cos(\theta_p)\ddot{x}-\sin(\theta_p)\ddot{z}- \\ &\dot{x}\dot{\theta}_p\sin(\theta_p)-\dot{z}\dot{\theta}_p\cos(\theta_p)-\tfrac{g}{l}\sin(\theta_p) &&= -u_\theta, \end{aligned} \tag{12}$$

where

$$F_1 = \frac{f_l x}{\sqrt{x^2+z^2}} + \frac{u_\theta z}{x^2+z^2},\ F_2 = \frac{f_l z}{\sqrt{x^2+z^2}} - \frac{u_\theta x}{x^2+z^2}.$$

Suppose that the biped is taking constant swing leg end to hip strides, that is, if $q = (x, z, \theta_p)$ and $\dot{q} = (\dot{x}, \dot{z}, \dot{\theta}_p)$, then the switching surface is $\mathcal{S} = \{(q, \dot{q}) | x = x_0\}$, and $x^+ = x_0$ for some $x_0 > 0$. With this assumption, the impact map is $\Delta = (\Delta_q, \Delta_{\dot{q}})$, where

$$\Delta_q(x^-, z^-, \theta_p^-) = (-x_0, z^-, \theta_p^-),\ \Delta_{\dot{q}}(q^-, \dot{q}^-) = \dot{q}^+,$$

and $\dot{q}^+$ can be found by using conservation of angular momentum about the swing leg end and knee joint before and after impact [10]. Our goal is to choose the virtual constraints that define the zero dynamics such that the zero dynamics is hybrid invariant and has periodic solutions. From the equations of kinetic and potential energies, the Lagrangian is invariant under the map

$$F(x, z, \theta_p) = (-x, z, -\theta_p). \tag{13}$$

Therefore, by Proposition 4, the vector field X corresponding to the equations of motion with $u_\theta = 0$ and $f_l = 0$ in (12) is symmetric under $G = (F, -dF)$.

Define the submanifold $\mathcal{Z}$ as follows:

$$\mathcal{Z} = \{(q, \dot{q}) | z = h_1(x), \theta_p = h_2(x), \dot{z} = \frac{\partial h_1}{\partial x}\dot{x}, \dot{\theta}_p = \frac{\partial h_2}{\partial x}\dot{x}\}.$$

If $h_1(x) = z_0$ and $h_2(x) = 0$, then $\mathcal{Z}$ is invariant under $G = (F, -dF)$. In this case, the fixed points of G lying on $\mathcal{Z}$ are $(q, \dot{q}^*)$, where

$$q^* = (0, z_0, 0), \dot{q}^* = (\dot{x}^*, 0, 0), \tag{14}$$

and the zero dynamics equation is simply that of the 2D LIP:

$$\ddot{x} = \frac{g/l}{1+z_0}x,$$

with $\dot{x}^+ = \dot{x}^-$ and $x^+ = -x_0$, which consequently has an infinite number of symmetric periodic orbits [14] as predicted by Proposition 12 as well.

Another set of holonomic constraints that can render $\mathcal{Z}$ invariant under G is defined by $h_1(x) = z_0 - a\cos((\pi/x_0)x)$ and $h_2(x) = 0$, for which the HZD is

$$\ddot{x} = \frac{(g/l)x + ax(\frac{\pi}{x_0})^2\cos((\frac{\pi}{x_0})x)\dot{x}^2}{1+z_0-a\cos((\frac{\pi}{x_0})x) - ax\frac{\pi}{x_0}\sin((\frac{\pi}{x_0})x)}, \tag{15}$$

with $\dot{x}^+ = \dot{x}^-$ and $x^+ = -x_0$. The right hand side of (15) is an odd function of x which is consistent with (11), where $G_{\mathcal{Z}}(x, \dot{x}) := (-x, \dot{x})$ is the restriction of $G = (F, -dF)$ to $\mathcal{Z}$ with F defined in (13). By Proposition 12, since $(x^+, \dot{x}^+) = G(x_0, x^-)$, this HZD has an infinite number of symmetric periodic orbits which are invariant under G. Fig. 6 shows one of such periodic orbits.

The two sets of virtual constraints defined above preserve the

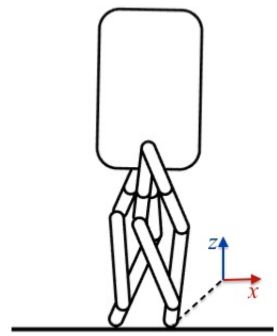

Fig. 5. A model of the bipedal robot MARLO [?] in 2D.

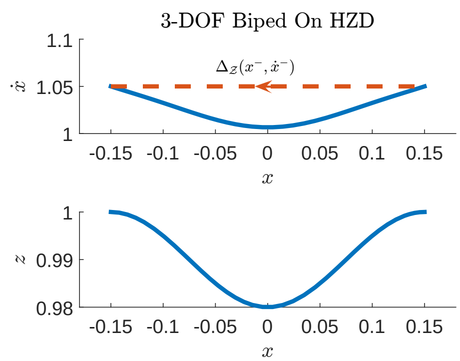

Fig. 6. Multiple symmetric periodic solutions of the 3-DOF biped on the HZD defined by $h_1(x) = z_0 - a\cos((\pi/x_0)x)$ and $h_2(x) = 0$. Note that $\dot{x}$ and z are both even functions of x.

symmetry of the system on its HZD. More general virtual constraints which can preserve the symmetry of the system will be discussed in the next section.

V. Symmetric Virtual Constraints for Periodic Walking

In Proposition 12, we saw that in a G-symmetric SHS, if $\mathcal{Z}$ is hybrid invariant and is invariant under G, then the resulting HZD is G-symmetric SHS and can have an infinite number of periodic orbits. In this section, we particularly show how $\mathcal{Z}$ can be defined by virtual constraints to be both hybrid invariant and invariant under G. Such virtual constraints are called *Symmetric Virtual Constraints* (SVC). With SVCs the HZD of the bipedal robot is an SHS, and has an infinite number of symmetric periodic orbits. Thus, SVCs allow periodic gait design without the need for searching for periodic orbits.

A. The 3-DOF Biped

In Example 13, we defined the holonomic constraints that rendered the HZD an SHS. However, the holonomic constraints defined in Example 13 were only examples of SVCs. Below we present sufficient conditions in order for the virtual constraints to be symmetric.

Define the submanifold $\mathcal{Z}$ as follows

$$\mathcal{Z} = \{(q,\dot{q})|z = h_1(x), \theta_p = h_2(x), \dot{z} = \frac{\partial h_1}{\partial x}\dot{x}, \dot{\theta}_p = \frac{\partial h_2}{\partial x}\dot{x}\}.$$

If there exists $x_0 > 0$ such that

1) h_1 is an even function of x in the interval $[-x_0, x_0]$, and $(\partial h_1/\partial x)|_{x_0} = 0$,
2) h_2 is an odd function of x in the interval $[-x_0, x_0]$, and $h_2(x_0) = 0$,

then $\mathcal{Z}$ is hybrid invariant (i.e., is a zero dynamics submanifold which is invariant under the impact map) and is invariant under the symmetry map $G = (F, -dF)$, where F is defined in (13). The fixed points of G lying on $\mathcal{Z}$ are $(q, \dot{q}^*)$, where

$$q^* = (0, h_1(0), 0), \dot{q}^* = (\dot{x}^*, 0, \frac{\partial h_2}{\partial x}|_{x=0}\dot{x}^*). \tag{16}$$

Consequently, by proposition 9, any solutions $\phi(t) = (q(t), \dot{q}(t))$ for which $\phi(0) = (q^*, \dot{q}^*)$, lies on $\mathcal{Z}$ and $\phi(-t) = G(\phi(t))$, or equivalently, if $q(t) = (x(t), z(t), \theta_p(t))$, then

$$x(-t) = -x(t), z(-t) = z(t), \theta_p(-t) = -\theta_p(t). \tag{17}$$

Since h_1 is even and h_2 is odd, $\mathcal{Z}$ is invariant under G. To prove hybrid invariance of $\mathcal{Z}$ we need to show that $\Delta(\mathcal{S} \cap \mathcal{Z}) \subset \mathcal{Z}$. First, we derive the impact map on the zero dynamics. Since $(\partial h_1/\partial x)|_{x_0} = 0$ and $h_2(x_0) = 0$, right before the impact the whole torso is moving parallel to the ground. Writing the conservation of angular momentum about the swing leg end and swing leg knee before and after impact (see [10] for derivation of impact map for planar bipedal robots using conservation of angular momentum) yields

$$\dot{x}^+ = \dot{x}^-, \dot{z}^+ = \dot{z}^-, \dot{\theta}_p^+ = \dot{\theta}_p^-. \tag{18}$$

Moreover, from geometry and after swapping the legs,

$$x^+ = -x_0, \theta_p^+ = \theta_p^-, z^+ = z^-. \tag{19}$$

To check $\Delta(\mathcal{S} \cap \mathcal{Z}) \subset \mathcal{Z}$, we need to verify that

$$\begin{array}{rclrcl} z^+ &=& h_1(x^+), & \theta_p^+ &=& h_2(x^+), \\ \dot{z}^+ &=& \frac{\partial h_1}{\partial x}(x^+)\dot{x}^+, & \dot{\theta}_p^+ &=& \frac{\partial h_2}{\partial x}(x^+)\dot{x}^+. \end{array}$$

However, all these equalities are satisfied by using the conditions on h_1 and h_2 (note that $\dot{z}^- = 0$, and $\theta_p^- = 0$), and equations (18) and (19). Therefore, the zero dynamics is hybrid invariant. Based on Proposition 12, and the impact map equations on the HZD, if the symmetric solutions $\phi(t)$, as defined in (17), cross the switching surface, they are periodic solutions. From (16), these symmetric periodic solutions, which are infinitely many, can be indexed by $\dot{x}^*$.

B. The 5-DOF Biped

Consider the planar biped with point feet as depicted in Fig. 4. Assuming that the legs have mass, this biped has 5 DOFs and 4 actuators (two in each leg to control the leg length and the angle between the leg and torso), hence, 1 degree of underactuation. To describe the biped's configuration, we use the generalized coordinates $q = (x, z, \theta_p, x_{fh}, z_{fh})$, where (x, z) is the position of the hip, θ_p is the torso pitch angle as shown in Fig 4, and (x_{fh}, z_{fh}) is the position of the swing leg foot relative to the hip; that is if (x_f, z_f) is the coordinate of the swing leg end in the inertial frame attached to the support point (i.e., stance leg end point), then $(x_{fh}, z_{fh}) = (x_f - x, z_f - z)$.

Proposition 14. *If the equations of motion of the 5-DOF bipedal robot in Fig. 4 is written in the form $\dot{x} = X(x,u) = f(x) + g(x)u$, and $G = (F, -dF)$, where $F(x, z, \theta_p, x_{fh}, z_{fh}) = (-x, z, -\theta_p, -x_{fh}, z_{fh})$, then*

1) $X(x, 0)$ is symmetric under G,
2) $(g \circ G(x))H(u) = -(dG \cdot g(x))u$,

for an isomorphism $H : \mathcal{U} \to \mathcal{U}$.

As explained in the following proposition, based on the symmetry map in Proposition 14, SVCs are chosen such that the HZD of the 5-DOF biped is an SHS. It is noted that naturally, it is assumed that the transition occurs when the swing leg hits the ground, that is, the switching surface is assumed to be $\mathcal{S} = \{(q,\dot{q})|z_f(q) = 0\}$.

Proposition 15. *In the 5-DOF biped, define the zero dynamics submanifold $\mathcal{Z}$ by the virtual constraints $z = h_1(x), \theta_p = h_2(x), x_{fh} = h_3(x)$, $z_{fh} = h_4(x)$ and their derivatives. If*

$$\begin{array}{l} h_1(-x) = h(x), \frac{\partial h_1}{\partial x}|_{x=x_0} = 0, \\ h_2(-x), = -h_2(x), h_2(x_0) = 0, \\ h_3(-x) = -h_3(x), h_3(x_0) = x_0, \frac{\partial h_3}{\partial x}|_{x=x_0} = -1, \\ h_4(-x) = h(x), h_4(x_0) = -h_1(x_0), \frac{\partial h_4}{\partial x}|_{x=x_0} = 0, \\ h_4(x) + h_1(x) > 0, \text{if } x \in (-x_0, x_0), \end{array}$$

for some $x_0 > 0$, then if the zero dynamics is well defined, it is hybrid invariant (i.e., is an HZD), the impact map restricted to $\mathcal{S} \cap \mathcal{Z}$ and its switching surface are

$$(x^+, \dot{x}^+) = (-x_0, \dot{x}^-),\ \mathcal{S} \cap \mathcal{Z} = \{(x, \dot{x}) | x = x_0\},$$

and the HZD is an SHS under the map $G_{\mathcal{Z}}(x, \dot{x}) = (-x, \dot{x})$, where G is defined in Proposition 14. Consequently, the continuous phase of equations on the HZD can be written as $\ddot{x} = f(x, \dot{x})$, where $f(-x, \dot{x}) = -f(x, \dot{x})$.

Proposition 15 will be extended to 3D bipeds in a related paper.

Example 16. Based on Proposition 15 the following holonomic constraints together with their derivative are SVCs for the 5-DOF biped:

$$\begin{aligned} z &= z_0 - a_1 \cos(\frac{\pi x}{x_0}), \\ \theta_p &= b_1 \sin(\frac{\pi x}{x_0}), \\ x_{fh} &= x + \frac{2x_0}{\pi} \sin(\frac{\pi x}{x_0}), \\ z_{fh} + z &= a_2(x_0^4 - 2x_0^2 x^2 + x^4) \end{aligned}$$

It should be noted that the conditions on virtual constraints in Proposition 15 can all be satisfied by just using polynomials, and in particular, by Bézier polynomials[3].

Corollary 17. *There exists $x_0 > 0$ such that with the SVCs satisfying the conditions in Proposition 15, the HZD of the 5-DOF biped has an infinite number of symmetric periodic solutions of the form $(\phi(t), \dot{\phi}(t))$ defined on intervals of the form $[-t_i, t_i]$ with $t_i > 0$ such that $\phi(t_i) = x_0$ and $\phi(-t) = -\phi(t)$.*

VI. Stabilization of Symmetric Periodic Orbits

In the previous section, we showed that with the SVCs the HZD becomes an SHS, and consequently, the 5-DOF has an infinite number of symmetric periodic orbits. However, as shown in [13] for the LIP and in [14] for SHSs, these periodic orbits are only neutrally stable, that is, the eigenvalue of the Poincaré map of a symmetric periodic orbit of the 5-DOF biped restricted to the HZD is 1. In [14] it is demonstrated that introducing asymmetries to the SHS can modify the symmetric periodic orbits to become asymptotically stable. In the related paper, it will be shown that by slightly modifying the SVCs the symmetric periodic orbits of the system can be modified to become asymptotically stable limit cycles. It will also be shown that using only foot placement, the symmetric periodic orbits of the system (without any modifications) can become stable limit cycles.

VII. Conclusion

In this paper, the notion of symmetric virtual constraints (SVC) is introduced. It is shown that with SVCs the resulting HZD of a symmetric legged robot possesses an infinite number of symmetric periodic orbits which can be easily identified. The main advantage of the SVCs is that without requiring any searches for periodic orbits, they allow gait design and provide a family of periodic orbits rather than one single periodic orbit. A few planar biped examples are included. In a related paper, more detailed proofs, 3D legged locomotion examples, and a discussion of the stability mechanisms will be presented.

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[3]For a discussion of Bézier polynomials and their application in virtual constraints see [16].

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