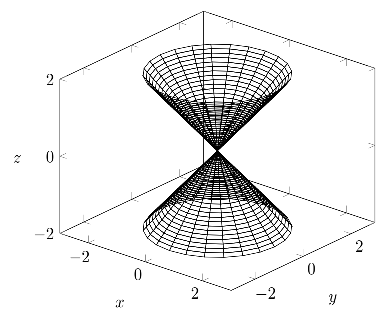
RESEARCH STATEMENT – RANKEYA DATTA

1. Introduction

Broadly speaking, I am interested in understanding the behavior of geometric objects in arbitrarily small neighborhoods of their points. The objects I study are solutions to multivariate polynomial equations, called *varieties*, with coefficients in fields like the real or the complex numbers, or even finite fields such as the prime fields $\mathbb{F}_p$. The investigation of local behavior of varieties borrows techniques from real, complex and non-Archimedean analysis, algebraic and differential topology, number theory and abstract algebra. Thus, although motivated by questions in algebraic geometry and commutative algebra, my research is at the crossroads of many areas of mathematics.

For an illustrative example of local behavior of varieties, consider the real number solutions of $x^2 + y^2 - z^2 = 0$ in three-dimensional space. The solution set looks like a double-sided infinite cone with vertex at the origin. The origin is qualitatively different from other points on the cone because at every other point there is a well-defined tangent plane. We emphasize this difference by calling the origin a *singularity* of the variety $x^2 + y^2 - z^2 = 0$.



One of the goals of algebraic geometry and commutative algebra is to find ways to rigorously characterize how 'bad' singularities of varieties can be. Hironaka, in his Fields Medal winning work, used one such characterization to show that any variety over the complex numbers, no matter how singular, can be transformed in a controlled manner into one without singularities [Hir64(a), Hir64(b)]. In contrast, a few decades before Hironaka's monumental achievement, Zariski championed a different approach to resolution of singularities by emphasizing the role of *valuations* of a field [Zar40, Zar42, Zar44]. Since valuations are central to my research, I would like to introduce them with the aid of an example which illustrates how they encode local information about geometric objects. Consider the collection $\mathcal{M}$ of functions *meromorphic* on an open disk of the complex plane centered at the origin. Then $\mathcal{M}$ comes equipped with a function

$$\mathrm{ord}_0 : \mathcal{M} - \{0\} \twoheadrightarrow \mathbb{Z},$$

the *order of vanishing* of a meromorphic function at the origin, such that for any $f, g \in \mathcal{M} - \{0\}$:

$$(1)\ \mathrm{ord}_0(fg) = \mathrm{ord}_0(f) + \mathrm{ord}_0(g) \qquad \text{and} \qquad (2)\ \mathrm{ord}_0(f+g) \geq \min\{\mathrm{ord}_0(f), \mathrm{ord}_0(g)\}.$$

The function ord_0 is an example of a valuation of the field $\mathcal{M}$. More generally, for an arbitrary field K and a *totally ordered* abelian group Γ, a *valuation* ν of K with *value group* Γ is any surjective function

$$\nu : K - \{0\} \twoheadrightarrow \Gamma$$

that satisfies conditions (1) and (2) above with ord_0 replaced by ν. Associated with ν is its *valuation ring* $R_\nu := \{x \in K - \{0\} : \nu(x) \geq 0\} \cup \{0\}$, which is a *local ring* with maximal ideal $\mathfrak{m}_\nu := \{x \in R_\nu : \nu(x) > 0\} \cup \{0\}$. For example, the valuation ring of ord_0 consists of meromorphic functions that are holomomorphic near the origin, while those that additionally vanish at the origin comprise the maximal ideal.

Valuation theory has a long history which goes back, at least, to the work of Hensel on p-adic numbers. Moreover, although Hironaka did not use valuations to resolve singularities, the only partial results on resolution of singularities over fields of prime characteristic rely heavily on valuation-theoretic techniques [Abh56(a), Abh66, CP08, CP09]. Valuations have played crucial roles in the development of number theory, model theory, birational algebraic geometry, tropical geometry, and various types of rigid geometries such as Tate's rigid analytic spaces [Tat71], Berkovich spaces [Ber90, Ber93] and Huber's adic spaces [Hub93, Hub94]. More recently, Berkovich and Huber's deep valuation theoretic techniques have been used as foundations for Kedlaya and Liu's relative p-adic Hodge theory [KL15] and Scholze's perfectoid spaces [Scho12]. Their works are already finding spectacular applications in solving long-standing conjectures in geometry and algebra [Scho12, And16, Bha16, HM17, MS17].

In my work so far, I have focused on understanding how notions of singularities in prime characteristic commutative algebra behave in the setting of valuation rings, in order to then be able to use valuation-theoretic techniques to answer questions about varieties and Noetherian rings. The starting point of prime characteristic singularity theory is the astonishing fact that a simple ring homomorphism is able to detect how far a Noetherian ring is from being *regular* or *non-singular*. In order to describe this homomorphism, let $p > 0$ be a prime number and R

be a ring of characteristic p, that is, $p \cdot 1_R = 0$ in R. Then for any $r, s \in R$, the binomial theorem implies that $(r+s)^p = r^p + s^p$ since multiples of p equal zero in R. Thus the function

$$F : R \to R$$

that maps $r \mapsto r^p$ is a ring homomorphism, called the *(absolute) Frobenius map* of R. Kunz made the crucial discovery that the Frobenius map is able to detect completely the non-singularity of a Noetherian ring.

Theorem 1.0.1. [Kun69, Theorem 2.1] *Let R be a Noetherian ring of prime characteristic. Then R is regular if and only if the Frobenius map $F : R \to R$ is a flat ring homomorphism.*

The various notions of 'F-singularities' that have been defined and studied since Kunz's result such as F-purity, Frobenius splitting, weak F-regularity, F-rationality, strong F-regularity, etc., arise from systematically weakening flatness of the Frobenius map, thereby weakening regularity or non-singularity of a ring.

Even though valuation rings are rarely Noetherian, the Frobenius map is always flat for such rings in prime characteristic. This observation inspired my work with Karen Smith on using Frobenius methods to study valuation rings [DS16, DS17(a), Dat17(c)]. The techniques we develop are novel because, before our work, F-singularities were extensively studied mainly for Noetherian rings. Flatness of Frobenius for valuation rings also led Bhargav Bhatt to ask if it could be exploited to study local cohomology in the valuative setting, which culminated in surprising characteristic independent vanishing results [Dat17(b)]. A deeper goal of my research is to ascertain to what extent the various notions of F-singularities imply that rings possessing such singularities have to be Noetherian, and valuation rings provide the most natural setting to examine how F-singularities behave for non-Noetherian objects. This goal is in the spirit of having prime characteristic analogues of finite generation results originally considered by Zariski [Zar62] and Mumford [Mum62] and established recently in characteristic 0 using Mori theory [BCHM10] and analytic techniques [Siu06].

Structure of this statement: My work, summarized in the next section, is divided into three main topics:

- Uniform approximation of Abhyankar valuation ideals in prime characteristic (Section 1.2).
- p^{-e}-linear maps, finiteness of Frobenius and test ideals (Section 1.3).
- F-singularities and valuation rings (Section 1.4).

With each topic, I also discuss ongoing and future projects under the title 'Future research'. There is a preliminary section in which the notion of an *Abhyankar valuation* is introduced (Section 1.1).

2. Overview of my work

2.1. **Abhyankar valuations.** A significant portion of my research has involved understanding the behavior of higher rank analogues of *divisorial valuations* associated with prime divisors on normal varieties in prime characteristic. These higher rank analogues, called *Abhyankar valuations*, are introduced in this section.

Suppose K is a finitely generated field extension of a field k, that is, K is a *function field* over k. A valuation ν of K/k is a valuation of K such that $\nu(k - \{0\}) = \{0\}$. Then its valuation ring R_ν and residue field κ_ν are both k-algebras. If Γ_ν is the value group of ν, one defines the *rational rank of* ν to be $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$ and the *transcendence degree of* ν to be the transcendence degree of the field extension κ_ν/k. The following fundamental inequality links the rational rank and transcendence degree of a valuation [Bou89, VI, §10.3, Cor. 1]:

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/k \le \operatorname{tr.deg} K/k. \tag{2.1.0.1}$$

Definition 2.1.1. An **Abhyankar valuation** of K/k is a valuation for which equality holds in (2.1.0.1).

A classical result of Zariski shows that valuations of rational rank 1 satisfy equality in (2.1.0.1) precisely when they are divisorial [SZ60, VI, §14, Thm. 31]. Thus, Abhyankar valuations are higher rational rank analogues of divisorial valuations. Non-divisorial Abhyankar valuations have found many applications in geometry, such as in [Spi90, ELS03, FJ04, FJ05, JM12, RS14, Tei14, Pay14, Blu16].

2.2. **Uniform approximation of Abhyankar valuation ideals in prime characteristic.** Let X be a variety over a field k with function field K. Suppose ν is a real-valued valuation of K/k centered on X. Then for any $m \in \mathbb{R}_{\ge 0}$, we have the *valuation ideal* $\mathfrak{a}_m$ consisting of local sections f such that $\nu(f) \ge m$.

In a follow-up to their paper establishing surprising uniform bounds for symbolic power ideals of smooth varieties over fields of characteristic 0 [ELS01], Ein, Lazarsfeld and Smith used asymptotic multiplier ideals in another ingenious way to compare the ideals $\mathfrak{a}_m$ to their powers $\mathfrak{a}_m^\ell$. More precisely, they showed:

Theorem 2.2.1. [ELS03, Theorem A] *Suppose K is a function field over a field k of characteristic 0. If ν is a real-valued Abhyankar valuation of K/k centered on a regular local ring R, essentially of finite type over k with fraction field K, then there exists $e > 0$ such that for all $m \in \mathbb{R}_{\geq 0}$ and $\ell \in \mathbb{N}$,*

$$\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{\ell m} \subseteq \mathfrak{a}_{m-e}^\ell.$$

Heuristically, one can think of Theorem 2.2.1 as saying that the associated Rees-algebra $\bigoplus_{m\in\mathbb{R}_{\geq 0}} \mathfrak{a}_m$ of a real-valued Abhyankar valuation centered on a regular variety, although rarely finitely generated, is close to being so.

The proof of Theorem 2.2.1 requires existence of log resolutions and deep vanishing theorems, neither of which are available in prime characteristic. Nevertheless, I was able to show the following result:

Theorem 2.2.2. [Dat17(a), Theorem A] *Let X be a regular variety over a perfect field k of prime characteristic with function field K. For any non-trivial, real-valued Abhyankar valuation ν of K/k centered on X, there exists a real number $e > 0$ such that for all $m \in \mathbb{R}_{\geq 0}$ and $\ell \in \mathbb{N}$,*

$$\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{\ell m} \subseteq \mathfrak{a}_{m-e}^\ell.$$

My proof of Theorem 2.2.2 employed two key ingredients. First, in order to bypass resolution of singularities, I used a characteristic independent local uniformization result of Knaf and Kuhlmann shown below. Their result implies that one can always find a regular local center of any real-valued Abhyankar valuation whose valuation ideals are monomial in a regular system of parameters [Dat17(a), Prop. 2.3.3].

Theorem 2.2.3. [KK05, Theorem 1] *Let K be a finitely generated field extension of a field k of any characteristic, and let ν be an Abhyankar valuation of K/k such that the residue field κ_ν is separable over k. Then for any finite set $Z \subseteq R_\nu$, R_ν is centered on a regular local ring $(A, \mathfrak{m}_A, \kappa_A)$ essentially of finite type over k with fraction field K satisfying the following properties:*

1. *The Krull dimension of A equals $d := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$.*
2. *$Z \subseteq A$, and there exists a regular system of parameters $\{x_1, \ldots, x_d\}$ of A such that every $z \in Z$ admits a factorization*
$$z = ux_1^{a_1} \ldots x_d^{a_d},$$
for some $u \in A^\times$, and $a_i \in \mathbb{N} \cup \{0\}$.

Second, I utilized an asymptotic version of *test ideals of pairs* which are a prime characteristic analogue of asymptotic multiplier ideals of pairs. Test ideals arose in Hochster and Huneke's work on *tight closure* [HH90, HH94], following which Smith [Smi00] and Hara [Har01] forged a striking connection between multiplier and test ideals. The Japanese school of commutative algebraists then extended the notion of test ideals to pairs and showed that test ideals of pairs satisfy many properties of multiplier ideals of pairs even in the absence of vanishing theorems [HY03, HT04, Har05, Tak06]. In my work, instead of utilizing tight closure techniques, I exploited a dual reformulation of test ideals due to Schwede [Sch10, Sch11], which is more amenable to geometric applications [Sch09, ST12, BS13, ST14, BST15].

As an interesting consequence of the results in [Dat17(a)], I was also able to give a new proof of *Izumi's theorem* in prime characteristic for regular centers (see also the more general work of [RS14]).

Theorem 2.2.4. [Dat17(a), Corollary C] *Suppose K is a function field over a perfect field k of prime characteristic. Let v and w be non-trivial, real-valued Abhyankar valuations of K/k, centered on a regular local ring A which is essentially of finite type over k with fraction field K. Then there exists a real number $C > 0$ such that for all non-zero $x \in A$,*

$$v(x) \leq Cw(x).$$

In particular, if two different real-valued, Abhyankar valuations share a common center on a regular point of a variety, then the valuation topologies on the center induced by these valuations are *linearly equivalent*.

Future research: Theorem 2.2.2 is related to the fundamental notion of *volumes of graded families of ideals*, introduced in [ELS03], and further developed by [Mus02, JM12, Cut13, Cut14, BdFFU15]. Inspired by connections made by Li to the differential-geometric notion of K-semistability [Li15, Li17], Blum has showed that a certain *volume function* defined on the Riemann-Zariski space of valuations centered on a variety with mild (klt) singularities always attains a minimum value at some valuation [Blu16].

The analogue of a klt singularity in prime characteristic is a *strongly F-regular* singularity. There are similar relationships between other characteristic 0 singularities and F-singularities, and a general philosophy in algebraic geometry is that results about characteristic 0 singularities should have analogues for the corresponding

F-singularities. Certainly, my results (Theorems 2.2.2, 2.2.4) are inspired by this philosophy. Another recent example involving singular varieties is a finiteness result on the local étale fundamental group of the complement of a klt singularity in characteristic 0 [Xu14], which inspired a prime characteristic analogue for the local étale fundamental group of the complement of a strongly F-regular singularity [CST16].

In a similar vein, I am attempting to understand if Blum's work has an analogue for varieties with strongly F-regular singularities. More precisely, is a *suitably* defined volume function on the space of valuations centered on a strongly F-regular point of a variety minimized by some valuation? Moreover, can we choose the volume minimizing valuation to be Abhyankar? The second question, a conjecture of Jonsson and Mustaţă in characteristic 0 [JM12, Conjecture B], is open even for klt singularities.

2.3. **p^{-e}-linear maps, finiteness of Frobenius and test ideals.** Crucial to Schwede's reformulation of test ideals, which was used heavily in my proof of Theorem 2.2.2, is the existence of non-trivial *p^{-e}-linear maps.* Suppose $F^e : R \to R$ is the e-th iterate of the Frobenius map of a ring R of prime characteristic $p > 0$. Then the target copy of R can be considered as an R-module by restriction of scalars via F^e. When considered as an R-module in this manner, it is denoted $F^e_* R$.

Definition 2.3.1. A **p^{-e}-linear map** is an R-linear map $F^e_* R \to R$.

The easiest examples of p^{-e}-linear maps are sections $F_* R \to R$ of the Frobenius map, called *Frobenius splittings.* For example, if $R = \mathbb{F}_p[x]$, then $F_* R$ is a free R-module with basis $\{1, x, \dots, x^{p-1}\}$, and a Frobenius splitting of $\mathbb{F}_p[x]$ is given by mapping $1 \mapsto 1$ and all other basis elements to 0.

One can also define p^{-e}-linear maps globally for any scheme over $\mathbb{F}_p$, and non-trivial p^{-e}-linear maps carry a startling amount of geometric information. For example, Mehta and Ramanathan showed that Frobenius split projective varieties satisfy Kodaira vanishing [MR85], even though Kodaira vanishing fails in general in prime characteristic [Ray78]. Furthermore, an application of Grothendieck duality shows that global p^{-e}-linear maps on a normal variety X correspond to $\mathbb{Q}$-divisors Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier [BS13]. In order to guarantee the existence of non-trivial p^{-e}-linear maps, the underlying assumptions in Schwede's theory of test ideals are that rings and schemes are Noetherian and *F-finite*, where F-finite means that the Frobenius map is finite. These assumptions are satisfied, for instance, by essentially of finite type algebras over a field k such that $[k : k^p] < \infty$.

Given an F-finite, Noetherian domain R, the simplest type of Schwede's test ideals, the *absolute test ideal of* R, is defined to be the *smallest non-zero ideal* J of R such that for every p^{-e}-linear map ϕ, $\phi(F^e_*(J))$ is mapped back into J. In other words, ϕ induces a p^{-e}-linear map of the quotient R/J. The existence of absolute test ideals, and more generally test ideals of pairs, is not at all obvious from their definitions and follows from deep results on the existence of completely stable test elements in tight closure theory [HH94]. In addition, the following result of Karen Smith and myself explains why finiteness of Frobenius is an important hypothesis.

Theorem 2.3.2. [DS17(b), Theorem 4.0.2] *Let R be a Noetherian domain of characteristic $p > 0$ and fraction field K such that $[K : K^p] < \infty$. Then the following are equivalent :*

(1) *R is F-finite.*
(2) *R is excellent.*
(3) *There exists a non-trivial R-linear map $F_* R \to R$.*
(4) *For all $e > 0$, there exists a non-trivial R-linear map $F^e_* R \to R$.*

Our theorem shows that for generically F-finite Noetherian domains, there is a deep connection between the geometric notion of excellence and the existence of non-trivial p^{-e}-linear maps. Grothendieck formulated the notion of excellence to distill desirable properties shared by Noetherian rings arising in algebraic geometry, number theory and even analysis, with the expectation that fundamental results in commutative algebra and algebraic geometry, such as resolution of singularities, should hold for all excellent schemes.

Future research: In keeping with Grothendieck's philosophy, Schwede and others, including myself, have been striving to create a theory of test ideals for excellent schemes in arbitrary characteristic. In mixed characteristic, building on André and Bhatt's proofs of the Direct Summand Conjecture [And16, Bha16], Ma and Schwede have recently proposed a notion of *perfectoid test ideals* for excellent regular rings [MS17]. Using perfectoid test ideals, they have also proved a mixed characteristic analogue of [ELS01, HH02]. This raises the natural question of whether perfectoid test ideals and de Jong's alterations of integral schemes over discrete valuation rings [dJ96] can yield a mixed characteristic analogue of my work on Abhyankar valuation ideals (Theorem 2.2.2). The prospect provides a natural impetus for me to participate in the development of a more robust theory of test ideals in mixed characteristic.

Even in prime characteristic, a theory of test ideals for arbitrary excellent rings remains elusive. The obstructions are tied to a long-standing query of Hochster and Huneke on existence of *completely stable test elements*. Astonishingly, we also do not know if non-trivial p^{-e}-linear maps exist for arbitrary excellent rings of characteristic p. Takumi Murayama and I are trying to answer, perhaps, the simplest case of the latter existence problem – do excellent regular rings of prime characteristic admit Frobenius splittings? This basic question is open, to the best of our knowledge, even for discrete valuation rings.

2.4. ***F*-singularities and valuation rings.** My work using Frobenius techniques to study valuation rings began with the intent of investigating whether non-trivial p^{-e}-linear maps exist for interesting classes of non-Noetherian rings. More specifically, Patakfalvi, Schwede and Smith asked when valuation rings are Frobenius split, and, building on my earlier work with Smith [DS16, DS17(a)], I was able to establish the following result:

Theorem 2.4.1. [Dat17(c), Theorem A] *If K is a function field over a perfect field k of prime characteristic, and ν is an Abhyankar valuation of K/k, then the valuation ring of ν is Frobenius split.*

Frobenius splitting of divisorial valuation rings over perfect fields follows from F-finiteness of such rings and from Kunz's characterization of regularity (Theorem 1.0.1). Thus Theorem 2.4.1 extends a well-known fact about divisorial valuation rings to a class of valuation rings that behaves the most like divisorial ones.

Valuation rings share many common traits with regular local rings. For example, Frobenius is flat for valuation rings of prime characteristic, just as for regular local rings. Similarly, valuation rings are *splinters* in any characteristic [Dat17(c), Remark 5.0.7]. A *splinter* is an integral domain R such that any module finite ring extension of R admits an R-linear retraction. For instance, Hochster's Direct Summand Conjecture asserts that any regular local ring is a splinter, a result that was only recently settled in all characteristics [Hoc73, And16, Bha16, HM17].

In equicharacteristic 0, the trace map can be used to see that the property of being a splinter coincides with *normality*. In characteristic $p > 0$ the situation is more delicate, and it is conjectured that a Noetherian, F-finite domain is a splinter if and only if it is strongly F-regular (the prime characteristic analogue of a klt singularity).

Definition 2.4.2. A Noetherian, F-finite domain R is **strongly F-regular** if for any non-zero $r \in R$, there exists $e > 0$ and a p^{-e}-linear map $F^e_* R \to R$ mapping $r \mapsto 1$.

Splinters coincide with strongly F-regular domains for $\mathbb{Q}$-Gorenstein, F-finite Noetherian domains by the work of Singh [Sin99]. A positive answer in general would imply one of the outstanding open problems in tight closure theory - the equivalence of weak and strong F-regularity.

Among non-Noetherian rings, splinters seem to be more abundant than those satisfying a natural non-Noetherian analogue of strong F-regularity called *F-pure regularity* [DS16, Definition 6.1.1]. For example, while valuation rings are always splinters, Smith and I show that

Theorem 2.4.3. [DS16, Section 6] *A valuation ring of prime characteristic is F-pure regular if and only if it is Noetherian. In particular, weak and F-pure regularity coincide for valuation rings, just as for regular rings.*

Future research: Drawing evidence from my work with Smith, Schwede raised the question of whether *arbitrary* integral domains that satisfy the defining property of strong F-regularity and have F-finite fraction fields are Noetherian. This question is motivated by the desire to use F-singularity techniques to prove finite generation results in prime characteristic, such as Noetherianity of section rings of divisors and symbolic Rees algebras, that have recently been established for klt pairs in characteristic 0 using Mori Theory [BCHM10, Theorem 1.1]. Schwede and I have made partial progress toward his question since we can show that monomial sub-algebras of polynomial rings obtained from *strongly convex polyhedral cones* are F-regular precisely when the cones are *rational*, that is, when the monomial sub-algebras determine *toric varieties*.

My work with Smith reveals that Frobenius properties of the valuation ring of a valuation ν of a field K of characteristic $p > 0$ are intertwined with properties of the extension of valuations ν/ν^p. Here ν^p denotes the restriction of ν to the sub-field K^p of K. For example, finiteness of Frobenius places strong restrictions on the extension ν/ν^p, which is reminiscent of properties of perfectoid fields (c.f. [Scho12, Lemma 3.2]).

Theorem 2.4.4. [DS16, DS17(a)] *Let ν be a valuation of a field K of characteristic $p > 0$ such that $[K : K^p] < \infty$. Suppose $(R_\nu, \mathfrak{m}_\nu, \kappa_\nu)$ is its valuation ring and Γ_ν the value group. Then R_ν is F-finite if and only if $F_* R_\nu$ is a free R_ν-module of rank $[K : K^p]$. Moreover, if R_ν is F-finite, then*

(1) *Γ_ν is p-divisible or $[\Gamma_\nu : p\Gamma_\nu] = p$.*
(2) *$[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p]$.*
(3) *R_ν is Noetherian if in addition Γ_ν is finitely generated.*

Using the notion of *defect* of an extension of valuations [FVK11], Theorem 2.4.4(2) says that if the valuation ring of ν is F-finite, then the unique extension of valuations ν/ν^p is defectless. For the purposes of this research statement, one can think of defect of ν/ν^p as a measure of the failure of the identity $[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p]$. Thus ν/ν^p is *defectless* when equality holds in the identity, and ν/ν^p has *maximal defect* when ν/ν^p is *totally unramified*, that is, when $[\Gamma_\nu : p\Gamma_\nu] = 1 = [\kappa_\nu : \kappa_\nu^p]$.

My work reveals deep connections between Frobenius splitting of R_ν, the defect of ν/ν^p and the notion of Abhyankar valuations. Moreover, the defect of ν/ν^p is related to a generalization of the concept of Abhyankar valuations that can be defined via a beautiful result of Abhyankar in which he establishes an analogue of (2.1.0.1) in a non-function field setting.

Theorem/Definition 1.4.4. [Abh56(b), Theorem 1] *Let ν be a valuation of a field K with value group Γ_ν and residue field κ_ν. If ν is centered on a Noetherian local ring $(A, \mathfrak{m}_A, \kappa_A)$ whose fraction field is K, then*

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/\kappa_A \leq \dim(A).$$

If equality holds in the above inequality, we call A an ***Abhyankar center*** *of ν.*

While the notion of an Abhyankar valuation of a function field is *intrinsic* to a valuation (Definition 2.1.1), whether a valuation admits an Abhyankar center depends on the Noetherian center. However, when one restricts the class of admissible centers, the property of admitting Abhyankar centers from the more restrictive class of local rings often becomes intrinsic to a valuation. For example, if K/k is a function field and $\mathcal{C}$ is the collection of local rings which are essentially of finite type over k with fraction field K, then a valuation ν of K/k possesses an Abhyankar center from $\mathcal{C}$ if and only if ν is an Abhyankar valuation of K/k. This implies that if ν admits an Abhyankar center from $\mathcal{C}$, then any other center of ν from $\mathcal{C}$ is also an Abhyankar center.

I recently established the existence of a broad class of Noetherian, local domains, even in a non-function field setting, such that the property of admitting Abhyankar centers from this class is independent of the choice of center.

Theorem 2.4.5. [Dat17(c), Theorem 4.0.3] *Let $(A, \mathfrak{m}_A, \kappa_A)$ be an excellent local domain of characteristic $p > 0$ and fraction field K such that $[K : K^p] < \infty$. Let ν be a valuation of K centered on A with valuation ring $(R_\nu, \mathfrak{m}_\nu, \kappa_\nu)$ and value group Γ_ν. Then A is an Abhyankar center of ν if and only if*

$$[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p].$$

Since the identity $[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p]$ *does not* depend on the center A, the previous theorem implies:

Corollary 2.4.6. *Let ν be a valuation of a field K of characteristic $p > 0$ such that $[K : K^p] < \infty$. The property that ν admits excellent Abhyankar centers with fraction field K is independent of the choice of such centers. In particular, this holds for excellent domains of function fields over perfect ground fields of prime characteristic.*

Corollary 2.4.6 is peculiar to prime characteristic because it fails for valuations of function fields over ground fields of characteristic 0. Indeed, one can easily construct a valuation ν of $\mathbb{C}(X, Y)/\mathbb{C}$ such that ν admits excellent centers that are Abhyankar as well as not Abhyankar [ELS03, Example 1(iv)]. Theorem 2.4.5 also recovers, as special cases, earlier results of Smith and myself.

Theorem 2.4.7. [DS17(a), Theorem 0.1 & Theorem 1] *Let ν be a non-trivial valuation of K/k, where K is a function field over a perfect field k of characteristic $p > 0$. Then ν is Abhyankar valuation of K/k if and only if the extension ν/ν^p is defectless. If the valuation ring of ν is F-finite, then ν is divisorial.*

Future research: Frobenius splitting of Abhyankar valuations (Theorem 2.4.1) and Theorem 2.4.7 imply that for valuations ν of function fields over perfect fields of characteristic p, if ν/ν^p is defectless, then its valuation ring R_ν is Frobenius split. In contrast, I also prove that R_ν *fails* to be Frobenius split when ν/ν^p has maximal defect.

Proposition 2.4.8. [Dat17(c), Section 5(14)] *Let ν be a valuation of a field K of characteristic $p > 0$ which is not perfect. Suppose Γ_ν is the value group of ν and $(R_\nu, \mathfrak{m}_\nu, \kappa_\nu)$ is its valuation ring. If ν/ν^p is totally unramified, that is, if $[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = 1$, then R_ν cannot be Frobenius split.*

Despite these results, Frobenius splitting of R_ν remains mysterious when the defect of ν/ν^p is not one of two possible extremes. In order to further understand the role of defect in Frobenius splitting, it may be fruitful to obtain a complete classification of Frobenius split valuation rings of function fields of surfaces - the first case where there exist interesting extensions ν/ν^p with defect. Perhaps the only Frobenius split valuation rings of function fields of surfaces are those associated to Abhyankar valuations.

I am also interested in the question of when a p^{-e}-linear map of a Noetherian center of a valuation lifts to its valuation ring. For instance, my proof of Frobenius splitting of Abhyankar valuations (Theorem 2.4.1) uses Knaf and Kuhlmann's local monomialization result (Theorem 2.2.3) to lift a Frobenius splitting of a suitable center of the valuation to a splitting of its valuation ring. Liftability of p^{-e}-linear maps from centers to valuations has its roots in a project I worked on as part of a 2015 AMS Math Research Community where Pérez, Canton, Schwede and I attempted to prove that if φ is a p^{-e}-linear map on a normal, F-finite affine variety R such that the corresponding pair (R, Δ_φ) is strongly F-regular, then φ lifts to only finitely many divisorial valuation rings admitting a center on R. This question has a positive answer if log resolutions exist in prime characteristic.

In a different direction, I am also examining the behavior of tight closure for ideals of valuation rings, an investigation that began in my work with Smith in [DS16]. There appears to be an intriguing connection between tight closure and Huber's notion of *f-adic* valued fields [Hub93], and these notions are also related to Schwede's *centers of F-purity*. Perhaps this interplay between tight closure and f-adic rings will allow us to use Huber's methods and Faltings's *almost mathematics* to tackle open problems in tight closure from a different perspective.

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