HUBER RINGS

RANKEYA DATTA

ABSTRACT. These notes discuss basic properties of Huber rings, and were prepared for the Arithmetic Geometry Learning Seminar at the University of Michigan. The goal of the seminar was to understand [Hub93]. The notes contain much more than what was covered in the lectures. In particular, a section on uniform Huber rings was later added once it became obvious that some results discussed in the course [Bha17] can be generalized to uniform Tate rings. We borrowed basic terminology from [Bha17].

CONTENTS

1. INTRODUCTION

Just as the local algebraic objects in the theory of schemes are commutative rings, the local algebraic objects in Huber's theory of adic spaces are *Huber rings*, which are topological rings satisfying some other axioms. The goal of this note is to define Huber rings, give examples, and study some properties of these rings.[1] The basic reference is

(1) *R. Huber, Continuous Valuations, Mathematische Zeitschrift, 212, 455-477 (1993),*

[1] I thank the seminar participants for helpful feedback. I am the sole culprit for all inaccuracies, and comments and corrections are highly appreciated. Please send them to rankeya@umich.edu. Special thanks to TeX guru Takumi Murayama for providing the code for diagram 4.3.3.1, and saving me many hours of frustration in the process.

although we will also use

(2) T. Wedhorn, *Adic Spaces*, available online.

(3) P. Scholze, *Lectures on p-adic geometry Fall 2014*, Berkeley course notes by Jared Weinstein, available online.

2. Notation and conventions

Unless otherwise specified, A will denote a topological commutative ring. For an element $a \in A$, the map

$$\ell_a : A \to A$$

denotes left-multiplication by a. This is a continuous map that maps $0 \mapsto 0$, and is a homeomorphism if and only if a is a unit. Note also that

$$\_ + a : A \to A;\ x \mapsto x + a$$

is a homeomorphism with inverse

$$\_ + (-a) : A \to A.$$

Given a subsets U, V of A, UV or $U \cdot V$ denotes the subgroup of $(A, +)$ generated by elements uv for $u \in U$ and $v \in V$. In particular, the notation U^n is self-explanatory. We use $U(n)$ to denote the set

$$U(n) := \{u_1 u_2 \dots u_n : u_i \in U\}.$$

Thus, U^n is generated as a group by $U(n)$.

A local ring $(R, \mathfrak{m})$ is a ring with a unique maximal ideal $\mathfrak{m}$, not necessarily Noetherian. Valuation rings are a source of non-Noetherian local rings. Recall that given a field K, and an ordered abelian group Γ (with group operation written multiplicatively), a **valuation** $|\cdot|$ on K with **value group** Γ is a map

$$|\ \ | : K \twoheadrightarrow \Gamma \cup \{0\}$$

satisfying

(1) $|x| = 0 \Leftrightarrow x = 0$.
(2) $|xy| = |x||y|$ (so $|\ \ |$ induces a group homomorphism $K^\times \twoheadrightarrow \Gamma$).
(3) $|x + y| \leqslant max\{|x|, |y|\}$.

Here the total ordering on Γ is extended to $\Gamma \cup \{0\}$ by declaring that $0 \leqslant \gamma$ for all $\gamma \in \Gamma$, and multiplication is extended by defining $\gamma \cdot 0 = 0 = 0 \cdot \gamma$, for all $\gamma \in \Gamma \cup \{0\}$.

Given a valuation $|\cdot|$ on a field K, it easy to verify that the set

$$R_{|\cdot|} := \{x \in K : |x| \leqslant 1\}$$

is a local ring with maximal ideal

$$\mathfrak{m}_{|\cdot|} = \{x \in K : |x| < 1\}.$$

The ring $R_{|\cdot|}$ is the **valuation ring** corresponding to the valuation $|\cdot|$, satisfying the property that for all $x \in K$, x or x^{-1} is in $R_{|\cdot|}$. Note that $R_{|\cdot|}$ is Noetherian if and only if Γ is trivial (in which case $K^\circ = K$), or Γ is order isomorphic to $\mathbb{Z}$, so that most valuation rings will not be Noetherian.

Remark 2.0.1. Valuations are written multiplicatively instead of additively because we want to think of valuations as higher rank analogues of non-Archimedean norms (these correspond to valuation rings that have Krull dimension 1), and the term "higher rank norm" is less commonly used in the literature.

Example 2.0.2. A perfectoid field is a field equipped with a valuation of rank 1 whose value group is not a discrete subgroup of $\mathbb{R}_{\geqslant 0}$, satisfying some other conditions as defined in Bhargav's class [Bha17].

3. Huber rings- definition and examples

3.1. **Adic rings.** Before we define the main objects of study, we briefly review the notion of adic rings. Throughout this section, A will denote a topological ring.

Definition 3.1.1. A is called an **adic ring**, if there exists an ideal I of A such that $\{I^n : n \geqslant 0\}$ forms a basis of open neighborhoods of 0. The ideal I is then called **an ideal of definition**.

Example 3.1.2.

(1) One can give any commutative ring R the discrete topology. R is then adic with ideal of definition (0).
(2) Given any commutative ring R and ideal I, there exists a unique topology on R (making R into a topoological ring) such that $\{I^n : n > 0\}$ is a neighborhood basis of 0.
(3) Let K be a perfectoid field with valuation ring K°. We have a natural metric topology on K, and K° is an open subring of K under this topology (even though K° is the closed unit ball in K, it is open by the non-Archimedean property). The induced topology on K° is adic, with ideal of definiton (a), for any $a \in K^{\circ\circ}$. This example shows that ideals of definition are not unique.
(4) The completion of a local ring $(R, \mathfrak{m})$ with respect to the maximal ideal $\mathfrak{m}$, although admitting a purely algebraic definition, can also be interpreted as the topological completion of R equipped with the $\mathfrak{m}$-adic topology.

Exercise 3.1.3. What do you get when you complete the valuation ring K° of a perfectoid field K with respect to the $K^{\circ\circ}$-adic topology? (Hint: $(K^{\circ\circ})^2 = K^\circ$.)

Exercise 3.1.4. Let A be an adic ring. If I, J are ideals of definition of A, then show that

$$\sqrt{I} = \sqrt{J}.$$

Show that the converse holds if I, J are both finitely generated. Give a counter-example for general I, J? (Hint: Consider $A = K^\circ$, for a perfectoid field K.)

Exercise 3.1.5. Let A be an adic ring with ideal of definition I. Show that the topology on A is Hausdorff if and only if

$$\bigcap_{n \geqslant 1} I^n = (0).$$

Hence deduce that the $\mathfrak{m}$-adic topology on a Noetherian local ring $(R, \mathfrak{m})$ is always Hausdorff.

Lemma 3.1.6. *Let A be an adic ring with ideal of definition I. A subset S of A containing 0 is open if and only if $I^n \subseteq S$, for some $n \geqslant 0$. In particular, the set of open prime ideals of A is in bijective correspondence with $\mathsf{Spec}(A/I)$.*

Proof. Left as an exercise. □

Remark 3.1.7. The set of open prime ideals of an adic ring A is denoted $\mathsf{Spf}(A)$, and is the local object in the study for *formal schemes*. There is a fully faithful functor from the category of schemes to the category of formal schemes that carries $\mathsf{Spec}(A)$ to $\mathsf{Spf}(A)$, where A is considered as an adic ring with discrete topology.

3.2. **Huber rings.** We will now define Huber rings. our definition not the one given in [Hub93], but instead the one given in [Wed12] and [Sch14].

Definition 3.2.1. A topological ring A is called a **Huber ring** if there exists an open subring A_0 of A such that the induced topology on A_0 is adic for some finitely generated ideal I of A_0. The ring A_0 is then called **a ring of definition of** A and I **an ideal of definition of** A.

Remark 3.2.2.

(1) Huber rings are called *f-adic* rings by Huber in [Hub93]. However, following [Sch14], the term ‘Huber ring’ is becoming increasingly popular. The letter f in ‘f-adic’ stands for finite.

(2) When we say that I is an ideal of definition of a Huber ring A, we mean I is an ideal of some open subring of A whose induced topology is the I-adic topology.

The next result reconciles Huber's original definition with the definition we have chosen. Since we will work with our definition in these notes, we omit the proof.

Proposition 3.2.3. *Let A be a topological ring. Then the following are equivalent:*

(1) *There exists an open subring A_0 of A whose topology is induced by a finitely generated ideal I of A_0.*
(2) *There exists a subset U of A and a finite subset T of U such that $\{U^n : n \in \mathbb{N}\}$ is a fundamental system of neighborhoods of 0 in A, and $T \cdot U = U^2 \subseteq U$.*

Proof. The proof of this proposition will not be very important for us, and can be found in [Wed12, Proposition and Definition 6.1]. Note the hard part is (2) $\Rightarrow$ (1). $\square$

Example 3.2.4.

(1) An adic with a finitely generated ideal of definition is a Huber ring. It is also true that if an adic ring is Huber, then it has a finitely generated ideal of definition. But this takes more work [cite].
(2) Any commutative ring A with discrete topology is a Huber ring, since it is adic with finitely generated ideal of definition (0).
(3) Let K be a perfectoid field. Note that under the topology on K induced by the non-Archimedean norm, K° is an open subring of K whose topology is (π)-adic, for any non-zero $\pi \in K^{\circ\circ}$.
(4) Let K be a perfectoid field. Let A be a non-Archimedean normed K-algebra. In other words, A comes equipped with a function
$$||\cdot|| : A \to \mathbb{R}_{\geqslant 0},$$
satisfying
   (a) $||x|| = 0 \Leftrightarrow x = 0$.
   (b) $||x + y|| \leqslant \max\{||x||, ||y||\}$.
   (c) $||\lambda \cdot x|| = |\lambda| \cdot ||x||$, for all $\lambda \in K$, where $|\cdot|$ is the norm on K.
   (d) $||xy|| \leqslant ||x|| \cdot ||y||$.
   (e) $||1|| \leqslant 1$.

   Then A is a Huber ring in the metric topology induced by the norm, with ring of definition
$$A^\circ = \{x \in A : ||x|| \leqslant 1\},$$
   and ideal of definition πA°, for any $\pi \in K^{\circ\circ}$. In particular, this shows that the Tate algebra $K\langle T_1, \dots, T_n\rangle$ is a Huber ring.

Exercise 3.2.5. Let A be a Huber ring. Show that the topology on A is discrete if and only if any ideal of definition is nilpotent (i.e. some power of the ideal is (0)).

We next discuss an extended example, put together from material in [Wed12].

Valuation topology: Let K be a field with a valuation $|\cdot|$ on it that has value group Γ. Then K has a natural *valuation* topology which makes it into a topological field, where a basis of open neighborhoods of 0 is given by sets of of the form
$$K_{<\gamma} := \{x \in K : |x| \leqslant \gamma\},$$
for $\gamma \in \Gamma$. **The topology on a valued field K will always be the valuation topology, unless otherwise specified**. Note that the sets $K_{<\gamma}$ are totally order by inclusion (since Γ is totally ordered), and that the valuation ring $R_{|\cdot|}$, being equal to $K_{\leqslant 1}$, is an open subring of K. The question we hope to answer is the following:

When is K with the valuation topology a Huber ring with ring of definition $R_{|\cdot|}$?

We will see that we can give complete characterization of which valued fields are Huber rings, thereby generalizing Example 3.2.4(3).

First note that the valuation topology on K is Hausdorff because

$$\bigcap_{\gamma \in \Gamma} \mathsf{K}_{<\gamma} = (0).$$

Thus the induced topology on $\mathsf{R}_{|\cdot|}$ is also Hausdorff. Since finitely generated ideals of a valuation ring are principal, $\mathsf{R}_{|\cdot|}$ being a ring of definition is then equivalent to the existence of a non-zero element $a \in \mathsf{R}_{|\cdot|}$ such that

$$\bigcap_{n \geqslant 1} (a)^n = (0).$$

The next result characterizes those valuation rings where such elements a exist, thereby giving a characterization of valued fields that are Huber.

Proposition 3.2.6. *Let K be a field with a valuation $|\cdot|$, and valuation ring $\mathsf{R}_{|\cdot|}$ that does not equal K. Then the following are equivalent:*

(1) *The valuation topology on K makes it a Huber ring with ring of definition $\mathsf{R}_{|\cdot|}$.*
(2) *There exists a non-zero element $a \in \mathsf{R}_{|\cdot|}$ such that $\bigcap_{n \geqslant 1} (a)^n = (0)$.*
(3) *$\mathsf{R}_{|\cdot|}$ has a prime ideal of height 1.*
(4) *If Σ be the set of non-zero prime ideals of $\mathsf{R}_{|\cdot|}$, then $\bigcap_{\mathsf{P} \in \Sigma} \mathsf{P} \neq (0)$.*

In particular, any valued field whose valuation has finite rank (i.e. $\mathsf{R}_{|\cdot|}$ has finite Krull dimension), is a Huber ring in the valuation topology.

Proof. We already showed why (1) and (2) are equivalent. Using the fact that the set of prime ideals of a valuation ring is totally ordered by inclusion, it is easy to see that (3) and (4) are equivalent. For the proof of (2) $\Rightarrow$ (3) we claim that

$$\sqrt{(a)}$$

is the prime ideal of height 1 in $\mathsf{R}_{|\cdot|}$. That $\sqrt{(a)}$ is a non-zero prime ideal is clear because in a valuation ring, all radical ideals are prime (a radical ideal is the intersection of the prime ideals containing it, which are totally ordered for $\mathsf{R}_{|\cdot|}$). Thus, it suffices to show that if P is a non-zero prime ideal of $\mathsf{R}_{|\cdot|}$, then

$$\sqrt{(a)} \subseteq \mathsf{P}.$$

If $\sqrt{(a)} \nsubseteq \mathsf{P}$, then $\mathsf{P} \subsetneq \sqrt{(a)}$ (since any two ideals of a valuation ring are comparable). In particular, a and its powers are not in P. So for all $n \geqslant 1$,

$$\mathsf{P} \subsetneq (a)^n,$$

that is

$$(0) \neq \mathsf{P} \subseteq \bigcap_{n \geqslant 1} (a)^n.$$

But this contradicts the hypothesis of (2), which shows that $\sqrt{(a)} \subseteq \mathsf{P}$, that is $\sqrt{(a)}$ is the prime ideal of height 1.

It remains to show that (3) $\Rightarrow$ (4). The proof becomes quite easy once we use the following general fact about intersection of powers of an ideal of a valuation ring:

Lemma 3.2.7. [FS00, Chapter 2, Lemma 1.3(c)] *Let V be a valuation domain, and I a proper ideal of V. Then $\bigcap_{n \geqslant 1} I^n$ is a prime ideal of V.*

Using this lemma, note that if P is the prime ideal of $\mathsf{R}_{|\cdot|}$ of height 1, then for any non-zero $a \in \mathsf{P}$,

$$\bigcap_{n \geqslant 1} (a)^n = (0).$$

This follows because $\bigcap_{n \geqslant 1} (a)^n$ is a prime ideal contained in P, but cannot equal P. And the reason for the latter is that either $(a) \subsetneq \mathsf{P}$ to begin with, or $(a) = \mathsf{P}$, in which case if $\bigcap_{n \geqslant 1} (a)^n = \mathsf{P}$ then we would have $(a) = (a^2)$ which cannot happen for a proper ideal of a domain. $\square$

Remark 3.2.8.

(1) You do not really need the full strength of Lemma 3.2.7 in the proof of Proposition 3.2.6 above. You only need that if V is a valuation domain, and $a \in V$ is not a unit, then

$$\bigcap_{n>0} a^n V$$

is a prime ideal of V. This is much easier to see. In particular, since proving that an ideal of a valuation ring is prime is equivalent to proving it is radical, it is easy to show that $\bigcap_{n>0} a^n V$ is radical (use the valuation to see this).

(2) The previous proposition is consistent with Example 3.2.4(3), because the valuation ring of a perfectoid field K has Krull dimension 1, and so the maximal ideal $K^{\circ\circ}$ is the prime ideal of height 1.

Remark 3.2.9. [Wed12] calls valuations/valuation rings satisfying the equivalent assertions of Proposition 3.2.6 ***microbial***, but we will refrain from using this terminology. So far we have classified only those Huber valued fields whose valuation rings are rings of definition. However, it is also true that if a valued field with the valuation topology is Huber, then its valuation ring has to be a ring of definition. We will see this soon (see Remark 4.2.3(2)).

3.3. **Tate rings.**

Definition 3.3.1. An element a of a topological ring A is called **topologically nilpotent** if $a^n \to 0$ as $n \to \infty$.

Example 3.3.2.

(1) Let $(K, |\cdot|)$ be a field with a valuation which is also a Huber ring. Then any non-zero element in the prime ideal of $R_{|\cdot|}$ of height 1 (see Proposition 3.2.6) is a topologically nilpotent elements of K which is also a unit.
(2) If A is an adic ring with ideal of definition I, then $a \in A$ is topologically nilpotent if and only if $a \in \sqrt{I}$.

Exercise 3.3.3. Let A be a Huber ring. Show the following:

(1) Every nilpotent element of A is topologically nilpotent (need A to only be topological for this).
(2) If I is an ideal of definition of some ring of definition of A, then every element of I is topologically nilpotent.
(3) The ideal $\mathcal{N}$ of A generated by the topologically nilpotent elements is an open ideal, hence so is $\sqrt{\mathcal{N}}$.
(4) The topology on A is discrete if and only if $\mathcal{N} = \mathrm{nil}(A)$, where $\mathrm{nil}(A)$ is the nilradical of A.
(5) A is discrete and reduced if and only if it has no non-zero topologically nilpotent element. Hence show that $A/\sqrt{\mathcal{N}}$ has no non-zero topologically nilpotent elements in the quotient topology.
(6) $\mathcal{N} = A$ if and only if A is the only open ideal of A.

Lemma 3.3.4. *Let A be a topological ring and $a \in A$ such that $a^n \to 0$ as $n \to \infty$. Then for any $b \in A$, $ba^n \to 0$ as $n \to \infty$.*

Proof. By hypothesis,

$$\ell_b : A \to A$$

is continuous. Let U be an open neighborhood of 0. We need to show that $ba^n \in U$ for all $n >> 0$. By continuity, $\ell_b^{-1}(U)$ is an open neighborhood of 0, and by hypothesis, $a^n \in \ell_b^{-1}(U)$, for all $n >> 0$. It follows that $ba^n = \ell_b(a^n) \in U$ for all $n >> 0$. $\square$

Definition 3.3.5. A Huber ring is called a **Tate ring** if it has a topologically nilpotent unit.

Example 3.3.6.

(1) Every valued field which is a Huber ring is also a Tate ring. In particular, all valued fields whose valuation rings have finite Krull dimension are Tate rings (this includes perfectoid fields).
(2) If K is a perfectoid field, then the Tate algebra $K\langle T_1, \ldots, T_n\rangle$ is a Tate ring with topologically nilpotent unit any non-zero $\pi \in K^{\circ\circ}$.
(3) In fact, any non-Archimedean normed K algebra is a Tate ring for the same reason that $K\langle T_1, \ldots, T_n\rangle$ is.
(4) Let A be any commutative ring, and $g \in A$ a non-zero divisor. Then $A[\frac{1}{g}]$ with topology induced by $\{g^n A : n \in \mathbb{Z}\}$ as a neighborhood basis of 0 is clearly a Tate ring with ring of definition A (the non-zero divisor assumption ensures A can be identified as a subring of $A[\frac{1}{g}]$).

Example 3.3.6(4) is not too far from how most Tate rings look with respect to their rings of definition. Here is the general result:

Proposition 3.3.7. *Let A be a Tate ring with ring of definition B, and topologically nilpotent unit g.*

(a) $g^n \in B$*, for all* $n >> 0$.
(b) *If* $g^n \in B$*, then the topology on* B *is the* $g^n B$*-adic topology.*
(c) $A = B[\frac{1}{g^n}]$*, for* $g^n \in B$.

Proof. (a) follows from the definition of topological nilpotence and the fact that B is open in A. For (b), let I be an ideal of definition of B. Since I is open in A, we know that $g^{nm} \in I$, for some large m, i.e., $(g^n B)^m \subseteq I$. To show that the topology on B is $g^n B$-adic, it suffices to show there exists $j >> 0$ such that

$$I^j \subseteq g^n B.$$

However, $g^n B$ is an open neighborhood of 0 in A since left-multiplication by g^n is a homeomorphism on A, and $g^n B$ is the image of the open set B under this homeomorphism. Thus, $I^j \subseteq g^n B$, for some $j >> 0$.

Finally, for (c) note that since g^n is a unit in A, by the universal property of localization, there exists an injective map

$$B\left[\frac{1}{g^n}\right] \hookrightarrow A.$$

Surjectivity of this map will follow if we can show that for any $a \in A$, $g^n a \in B$ for n large enough. However, this is now an easy consequence of Lemma 3.3.4 using openness of B. ☐

Remark 3.3.8. We already saw a special case of Proposition 3.3.7(c) in [Bha17, Lecture 4], with $A = \mathrm{Frac}(K^{\circ\flat})$ (for a perfectoid field K), $B = K^{\circ\flat}$, and the topologically nilpotent unit of A being any non-zero $t \in K^{\circ\flat}$ such that

$$t^\sharp \in K^{\circ\circ},$$

where

$$\cdot^\sharp : K^{\circ\flat} \to K^\circ$$

is the map of multiplicative monoids constructed in [Bha17, Lecture 2].

Norms defined by Hausdorff Tate rings: (Compare with [Bha17, Lecture 8, Example 8.9]) Let A be a Tate ring with ring of definition A_0 and topologically nilpotent unit $g \in A_0$. Suppose also that the topology on A is Hausdorff, so that

$$\bigcap_{n>0} g^n A_0 = (0).$$

We can then define a norm $|\cdot| : A \to \mathbb{R}_{\geqslant 0}$ as follows– for $a \in A$, define

$$|a| = \inf \{2^{-n} : a \in g^n A_0, n \in \mathbb{Z}\}.$$

Note this is really a norm because $|a| = 0 \Leftrightarrow a \in g^n A_0$, for $n >> 0 \Leftrightarrow a \in \bigcap_{n>0} g^n A_0 = (0)$. This norm will also not be multiplicative in general, but you can check it is sub-multiplicative. Thus, if A is in addition complete in the $g^n A_0$-adic topology, then A is a **Banach ring** [2].

[2] A *Banach ring* is a complete, normed topological ring (i.e. the topology is defined by the norm).

Note also that for $a \in A$, $|a| \leqslant 1$ if and only if there exists $n \geqslant 0$ such $a \in g^n A_0$, and the latter is clearly equivalent to a being an element of A_0 (remember we picked g so that $g \in A_0$). Thus, we see that under the above norm, the subring of elements of norm $\leqslant 1$ is precisely A_0. It is important to observe that the norm $|\cdot|$ is not unique, and depends on the ring of definition A_0.

4. Further properties of Huber rings

4.1. **Boundedness and power boundedness.**

Definition 4.1.1. Let A be a topological ring.

(1) A subset $S \subseteq A$ is **bounded**, if for every open neighborhood U of 0, there exists an open neighborhood V of 0 such that the set
$$\{vs : v \in V, s \in S\} \subseteq U.$$
Put differently, we say S is bounded, if for any open neighborhood U of 0, the set
$$\bigcap_{s \in S} \ell_s^{-1}(U),$$
which is an intersection of open neighborhoods of 0, contains an open neighborhood of 0.

(2) We say an element $a \in A$ is **power bounded**, if the set $\{a^n : n > 0\}$ is a bounded set. The set of power bounded elements of A is denoted A°. The set of topologically nilpotent elements of A is denoted $A^{\circ\circ}$.

Example 4.1.2. Let $(K, |\cdot|)$ be a valued field with the valuation topology. Then the valuation ring $R_{|\cdot|}$ is a bounded subring of K.

Remark 4.1.3. If A is a Huber ring with ideal of definition I, a subset $S \subseteq A$ is bounded if and only if there exists some I^m such that $I^m \cdot S \subset I$.

Here are some basic properties of bounded sets, collected in Lemma:

Lemma 4.1.4 (**Properties of boundedness and power boundedness**). *Let A be a topological ring.*

(1) *Any finite subset of A is bounded. In particular, any nilpotent element of A is power bounded.*
(2) *Any subset of a bounded set is bounded.*
(3) *A finite union of bounded sets is bounded.*
(4) *If A is adic, then A itself is bounded.*
(5) *If A is a Huber ring, then any ring of definition of A is bounded.*
(6) *Any topologically nilpotent element of A is power bounded.*
(7) *If A is a Tate ring with a ring of definition A_0, and a topologically nilpotent unit $g \in A_0$, then a subset S of A is bounded if and only if $S \subseteq g^n A_0$, for some $n \in \mathbb{Z}$.*
(8) *If A is Huber, a subset $S \subseteq A$ is bounded if and only if the subgroup generated by S is bounded.*
(9) *If A_1, A_2 are bounded subrings of a Huber ring A, then so is $A_1 \cdot A_2$ (the subring of A generated by A_1 and A_2).*
(10) *The collection of bounded open subrings of a Huber ring is a directed set under inclusion.*
(11) *If B is a bounded subring of a Huber ring A, and $x \in A$ is power bounded, then $B[x]$ is a bounded subring of A, which is open if B is.*
(12) *If A is Huber, $a \in A$ is power bounded, and $t \in A$ is topologically nilpotent, then at is topologically nilpotent.*
(13) *If A is a Huber ring and $a_1, a_2 \in A$ are topologically nilpotent, then $a_1 + a_2$ is topologically nilpotent.*

Proof. We leave the proofs of (1)-(5) to the reader, as they are not hard to deduce from the definition of boundedness and from each other.

(6) Suppose $a \in A$ is a topologically nilpotent element. Let I be an ideal of definition of A (as a reminder, I is an ideal of some open subring of A). Since a is topologically nilpotent, there exists n_0 such that for all $n > n_0$, $a^n \in I$. Thus, $\{a_n : n \geqslant n_0\}$ is bounded, being a subset of I which is bounded. Furthermore, since $\{a, a^2, \dots, a^{n_0-1}\}$ is a finite set, it is bounded, and so the union $\{a_n : n \geqslant n_0\} \cup \{a, a^2, \dots, a^{n_0-1}\} = \{a^n : n > 0\}$ is also bounded.

(7) Note the topology on A_0 is gA_0-adic by Proposition 3.3.7(b). Suppose S is bounded in A. Then there exists $n \geqslant 1$ such that

$$g^n A_0 \cdot S \subseteq A_0.$$

Since g^n is a unit, we get

$$S \subseteq g^{-n} A_0.$$

The converse is easy to establish, since the ideals $g^m A_0$ $(m \geqslant 1)$ form a basis of open neighborhoods of A_0 by Proposition 3.3.7(b), and boundedness can be checked against such ideals.

(8) Since S is contained in the subgroup it generates, it follows that it is bounded if the subgroup it generates is. On the other hand, suppose S is bounded. Then for any open subgroup U of A, there exists an open neighborhood V of 0 such that the group $V \cdot S$ is contained in U. If $\langle S \rangle$ is the subgroup generated by S (so elements of $\langle S \rangle$ are finite sums elements of S and their additive inverses), then it is easy to see that $V \cdot S = V \cdot \langle S \rangle$. This shows $\langle S \rangle$ is also bounded.

(9) Let U be an open subgroup of A. Then there exist open subgroups V_1, V_2 such that $V_1 \cdot A_1 \subseteq V_2$ and $V_2 \cdot A_2 \subseteq U$. Then

$$V_1 \cdot (A_1 \cdot A_2) = (V_1 \cdot A_1) \cdot A_2 \subseteq V_2 \cdot A_2 \subseteq U.$$

Thus, $A_1 \cdot A_2$ is bounded. *Note we did not need A_1, A_2 to be subrings here. Just assuming they are bounded sets would have sufficed.*

(10) Note that any subring of a topological ring is open if (and only if) it contains an open neighborhood of 0. Thus, if A_1, A_2 are open subrings of A, so is $A_1 \cdot A_2$. Then the directedness assertion follows from (9).

(11) The proof of the boundedness of $B[x]$ is very similar to that of (9), and is left to the reader. If B is open, then so is $B[x]$.

(12) Let I be an ideal of definition of A. Then at is topologically nilpotent if for all $m > 0$, there exists $n_m > 0$ such that

$$(at)^{n_m}, (at)^{n_m+1}, (at)^{n_m+2}, \dots \in I^m.$$

Since a is power bounded, there exists $j > 0$ such that

$$I^j \cdot \{a^n : n > 0\} \subseteq I^m.$$

By topological nilpotence of t, there exists $n_m > 0$ such that for all $n \geqslant n_m$, $t^n \in I^j$. Then

$$(at)^{n_m}, (at)^{n_m+1}, (at)^{n_m+2}, \dots \in I^m.$$

(13) Let I be an ideal of definition, as in (12). We have to show that for $n >> 0$, $(a_1 + a_2)^n$ is in I^m, for any $m > 0$. The elements a_1, a_2 are both power bounded by (6). Thus, there exists $n_0 > 0$ such that

$$I^{n_0} \cdot \{a_1^n : n > 0\}, I^{n_0} \cdot \{a_2^n : n > 0\} \subseteq I^m.$$

Now pick n_m such that for all $n \geqslant n_m$,

$$a_1^n, a_2^n \in I^{n_0}$$

(such a choice follows from topological nilpotence of a_1, a_2). Expanding $(a_1 + a_2)^n$ using the binomial formula, it follows that for $n \geqslant 2n_m$,

$$(a_1 + a_2)^n \in I^m.$$

□

Proposition 4.1.5. *Let A be a Huber ring.*

(1) A° *is the filtered direct limit of open, bounded subrings of* A*, hence is itself also an open subring.*
(2) $A^{\circ\circ}$ *is a radical ideal of* A°*.*
(3) A° *is integrally closed in* A*.*

Proof.

(1) Since a ring is closed under multiplication, every element of a bounded subring of A is power bounded. Thus, any bounded subring of A (open or not) is contained in A°. The set of open, bounded subrings of A is a directed set under inclusion by Lemma 4.1.4(10), so the direct limit, which is the union of these subrings, is contained in A°. That the direct limit equals A° follows from Lemma 4.1.4(11). Thus, A° is a ring.

(2) That $A^{\circ\circ}$ is an ideal of A° follows from Lemma 4.1.4(12),(13). We need to show it is radical. So let a be a power bounded element such that a^n is topologically nilpotent. Given any ideal of definition I, by power boundedness of a, for any $m > 0$, one can pick I^j such that

$$I^j \cdot \{a^r : r > 0\} \subseteq I^m.$$

Topological nilpotence of a^n allows us to choose k such that

$$a^{nk} \in I^j.$$

Then $a^{nk+1}, a^{nk+2}, a^{nk+3}, \cdots \in I^m$. Thus, a is also topologically nilpotent, i.e., $a \in A^{\circ\circ}$.

(3) Let $x \in A$ be integral over A°, satisfying the monic polynomial

$$x^n + a_{n-1}x^{n+1} + \cdots + a_0 = 0,$$

with coefficients $a_i \in A^\circ$. Since A° is the union of the open bounded subrings of A, one can pick an open bounded subring B of A such that $a_0, a_1, \ldots, a_{n-1} \in B$. Then x is integral over B, and we have

$$B[x] = B + Bx + \cdots + Bx^{n-1}.$$

It is easy to see that since B is bounded, so is Bx^i for any i. Thus, $B + Bx + \cdots + Bx^{n-1}$ is also bounded, i.e., $B[x]$ is an open, bounded subring of A. This shows that $x \in B[x] \subseteq A^\circ$, completing the proof. $\square$

Exercise 4.1.6. For a perfectoid field K, we introduced the notation K° for the valuation ring and $K^{\circ\circ}$ for the maximal ideal. Show that K° is the set of power bounded elements and $K^{\circ\circ}$ the set of topologically nilpotent elements.

More generally, if $(K, |\cdot|)$ is an arbitrary valued field (with a non-trivial valuation) which is Huber in the valuation topology, and $R_{|\cdot|}$ is the valuation ring, then show the following:

(i) If q is the height 1 prime of $R_{|\cdot|}$, then identifying $(R_{|\cdot|})_q$ as a subring of K show that $q(R_{|\cdot|})_q = q$ as sets.

(ii) $(R_{|\cdot|})_q$ is bounded in K, and $K^\circ = (R_{|\cdot|})_q$. *Hint: Use that there are no subrings B of K such that $(R_{|\cdot|})_q \subsetneq B \subsetneq K$ to see $K^\circ = (R_{|\cdot|})_q$.*

(iii) Show that $K^{\circ\circ} = q(R_{|\cdot|})_q = q$.

Exercise 4.1.7. Let A be a topological ring, and B be an open subring of A which is integrally closed (in A). Then show that $A^{\circ\circ} \subseteq B$, i.e., any open integrally closed subring of a topological ring contains all topologically nilpotent elements.

4.2. **Rings and ideals of definition of a Huber ring.** Throughout this subsection, A will denote a Huber ring. As was pointed out in the previous subsection, a ring of definition of A is an open, bounded subring. We will now show that the converse is true as well.

Proposition 4.2.1. *Let A be a Huber ring and A_0 a subring of A. Then the following are equivalent:*

(1) *A_0 is a ring of definition of A.*
(2) *A_0 is an open and bounded subring of A.*

Thus, A° is the union of all rings of definition of A.

Proof. Once the equivalence of (1) and (2) are shown, it follows from Proposition 4.1.5 that A° is the union of the rings of definition of A.

We left (1) $\Rightarrow$ (2) as an exercise for the reader. Let's us prove (2) $\Rightarrow$ (1). Let B be a ring of definition of A with finitely generated ideal of definition I (i.e. the topology on B is I-adic). If A_0 is an open, bounded subring of A, then there exists $m > 0$ such that

$$I^m \subseteq A_0.$$

Note, this does not imply that I^m is an ideal of A_0. However, it is still true that $\{I^{m+j} : j \geqslant 0\}$ is a basis of open neighborhoods of 0 in A_0, because the same is true for the larger ring A. Since A_0 is bounded, there exists $n > m$ such that

$$I^n \cdot A_0 \subseteq I^m.$$

Consider the ideal J of A_0 generated by a finite set of generators $\{x_1, \ldots, x_r\}$ of I^n as an ideal of B. It follows from the above inclusion that $J \subseteq I^m$. We also have

$$I^{m+n} = I^m I^n = I^m(Bx_1 + \cdots + Bx_r) = I^m x_1 + \cdots + I^m x_r \subseteq A_0 x_1 + \ldots A_0 x_n = J.$$

Thus, we have showed that $I^{m+n} \subseteq J \subset I^m$, which proves that the topology on A_0 is J-adic. □

We isolate a useful result proved while establishing Proposition 4.2.1.

Lemma 4.2.2. *Let A be a Huber ring. Let A_0 be an open, bounded subring of A. Let B be a ring of definition of A with ideal of definition I. Then there exists $n > 0$, such that $I^n \subseteq A_0$, and if $\{x_1, \ldots, x_r\}$ is a generating set of I^n as an ideal of B, then the ideal of A_0 generated by $\{x_1, \ldots, x_r\}$ is an ideal of definition of A_0.*

Proof. See the proof of Proposition 4.2.1. □

Remark 4.2.3.

(1) In many cases, it will turn out that A° is itself bounded, and so is a ring of definition of A. In particular A° will be a ring of definition for all perfectoid algebras. However, one can construct quite simple examples where A° is unbounded. For instance, for a prime p, consider $A = \mathbb{Q}_p[t]/(t^2)$. Then one can verify that A° for this ring is $\mathbb{Z}_p \oplus \mathbb{Q}_p t$, and that A° is not bounded.

(2) As an application of Proposition 4.2.1, note that if a valued field K is Huber in the valuation topology, then the valuation ring is a ring of definition since it is bounded and open in the valuation topology on K.

Proposition 4.2.1 has many nice consequences for rings of definition, that we now collect.

Corollary 4.2.4. *Let A be a Huber ring.*

(1) *If A_1, A_2 are rings of definition of A, then so are $A \cap A_2$ and $A_1 \cdot A_2$.*
(2) *If A is a topological ring, then A is Huber if and only if any open subring of A is also a Huber.* [3] *In particular, any open subring of a Huber ring contains a ring of definition.*
(3) *Any bounded subring of A is contained in a ring of definition.*
(4) *A is adic if and only if it is bounded.*

Proof. Translate (1) into a statement about open, bounded subrings and use Lemma 4.1.4.

(2) It is clear that if an open subring of a topological ring is Huber, then the ring itself is also Huber. So suppose that A is a Huber ring and C an open subring of A. Let A_0 be a ring of definition of A. Then $A_0 \cap C$ is an open subring of C which is bounded because A_0. Thus, $A_0 \cap C$ is a ring of definition of A, hence also of C, and so C is Huber.

(3) Let B be a bounded subring of A. Let A_0 be a ring of definition of A. Then $B \cdot A_0$ is bounded (since B and A_0 both are), and $B \cdot A_0$ is an open subring of A since it contains A_0. Thus, $B \cdot A_0$ is a ring of definition of A (by Proposition 4.2.1) containing B.

[3] Note an open subring of a Tate ring need not be Tate. As an example, consider a perfectoid field ...

(4) Adic rings are clearly bounded. If A is bounded, it is a ring of definition of itself by Proposition 4.2.1, hence adic. $\square$

Exercise 4.2.5. Generalize Corollary 4.2.4(3) to show that if A is a Huber ring, B a bounded subring of A, and C is an open subring of A containing B, then there exists a ring of definition A_0 such that

$$B \subseteq A_0 \subseteq C.$$

Lemma 4.2.6. *Let A be a Huber ring. Let $A_0 \subset A_1$ be an inclusion of rings of definition of A. If I a finitely generated ideal of A_0, then I is an ideal of definition of A_0 if and only if IA_1 is an ideal of definition of A_1.*

Proof. $\Leftarrow$ I is an ideal of definition of A_0 if and only if there exists $n > 0$ such that I^n is an ideal of definition of A_0. Since IA_1 is an ideal of definition of A_1, by Lemma 4.2.2, there exists $n > 0$ such that

$$I^n A_1 = (IA_1)^n \subseteq A_0,$$

and the ideal J of A_0 generated by any finite generating set of $(IA_1)^n$ is an ideal of definition of A_0. However, $(IA_1)^n$ and I^n have the same generating sets as ideals of their respective rings. Thus, $J = I^n$, and so I is an ideal of definition of A_0.

$\Rightarrow$ Now suppose that I is an ideal of definition of A_0. Since A_1 is a ring of definition, let J' be an ideal of definition of A_1. Because I is an open neighborhood of 0 and is contained in IA_1, the latter is open. Thus, there exists $n > 0$ such that $(J')^n \subseteq IA_1$. On the other hand, J' is also an open neighborhood of 0, and so there exists $m > 0$ such that $I^m \subset J'$, and since J' is an ideal of A_1, this implies that $(IA_1)^m = I^m A_1 \subseteq J'$. Thus, IA_1 is also an ideal of definition of A_1. $\square$

4.3. **Adic homomorphisms.** So far we have only discussed Huber rings, but not morphisms between Huber rings compatible with the underlying topological structure. This leads to the notion of an adic homomomorphism.

Definition 4.3.1. Let A, B Huber rings. An **adic homomorphism**

$$f : A \to B$$

is a continuous ring homomorphism for which there exists a ring of definition A_0 of A with ideal of definition I, and a ring of definition B_0 such that $f(A_0) \subseteq B_0$, and $f(I)B_0$ is an ideal of definition of B_0.

Example 4.3.2.

(i) Let $(R, \mathfrak{m}_R)$ and $(S, \mathfrak{m}_S)$ be Noetherian local rings with $\mathfrak{m}_R$ and $\mathfrak{m}_S$-adic topologies respectively. If $\phi : R \to S$ is any integral ring homomorphism, then ϕ is adic. This is because by the integrality hypothesis, $\mathfrak{m}_R S$ is $\mathfrak{m}_S$-primary, and so defines the same topology as the latter on S.

(ii) For any Huber ring A of characteristic $p > 0$, Frobenius

$$\mathrm{Frob}_A : A \to A$$

is an adic homomorphism.

(iii) (non-example) Let R be commutative ring with discrete topology, and S a Huber ring whose topology is not discrete. Then any ring homomorphism $i : R \to S$ is continuous, but not adic. We will give some less trivial non-examples as soon as we study some consequences of the adic condition.

Proposition 4.3.3. *Let $f : A \to B$ be an adic homomorphisms of Huber rings.*

1. *f is **bounded**, that is, for all bounded sets $C \subseteq A$, $f(C)$ is a bounded subset of B. In particular, $f(A^\circ) \subseteq B^\circ$.*
2. *$f(A^{\circ\circ}) \subseteq B^{\circ\circ}$, that is, f maps topologically nilpotent elements to topologically nilpotent elements.*

(3) *If* A_0 *and* B_0 *are any rings of definition of* A, B *respectively such that* $f(A_0) \subseteq B_0$ *and* I *is an ideal of definition of* A_0, *then* $f(I)B_0$ *is an ideal of definition of* B_0.
(4) *If* A_0 *is a ring of definition of* A, *then there exists a ring of definition* B_0 *of* B *such that* $f(A_0) \subseteq B_0$.
(5) *Suppose* f *is only a continuous homomorphism. If* B_0 *is a ring of definition of* B, *then there exists a ring of definition* A_0 *of* A *such that* $f(A_0) \subseteq B_0$.
(6) *If* I *is an open ideal of* A, *then* $f(I)B$ *is an open ideal of* B.
(7) *Suppose* f *is now only assumed to be a ring homomorphism, not necessarily continuous. If* A' *and* B' *are open subrings of* A, B *such that* $f(A') \subseteq B'$, *then* $f : A \to B$ *is adic if and only if* $f|_{A'} : A' \to B'$ *is adic.*

Proof. (1) Let A_0 be a ring of definition of A with ideal of definition I and B_0 a ring of definition of B such that $f(A_0) \subseteq B_0$ and $J := f(I)B_0$ is an ideal of definition of B_0. We need to show that for any $m > 0$, there exists $n > 0$ such that

$$J^n \cdot f(C) \subseteq J^m.$$

By boundedness of C, there exists I^n such that

$$I^n \cdot C \subseteq I^m.$$

Thus, $f(I^n) \cdot f(C) = f(I^n \cdot C) \subseteq f(I^m) \subseteq J^m$. Since J^m is an ideal of B_0, it follows that

$$J^n \cdot f(C) = (f(I^n)B_0) \cdot f(C) \subseteq J^m,$$

and so $f(C)$ is bounded. It then follows that f maps power bounded elements to power bounded elements, and so $f(A^\circ) \subseteq B^\circ$.

(2) We leave this proof to the reader. Note (2) holds for any continuous ring homomorphism that is not necessarily adic.

(3) Since f is adic, one can pick rings of definitions A_1 of A and B_1 of B, and an ideal of definition J of A_1 such that $f(A_1) \subseteq B_1$, and $f(J)B_1$ *is an ideal of definition of* B_1. Note, $A_0 \cdot A_1$ and $B_0 \cdot B_1$ are then rings of definitions of A, B respectively. We have the following diagram [4]

(4.3.3.1)
$$\begin{tikzcd}
A_0 \arrow[rrr, bend left] \arrow[dr, phantom, sloped, "\subseteq"] & & A_1 \arrow[rrr, bend left] \arrow[dl, phantom, sloped, "\supseteq"] & B_0 \arrow[dr, phantom, sloped, "\subseteq"] & & B_1 \arrow[dl, phantom, sloped, "\supseteq"] \\
& A_0 \cdot A_1 \arrow[rrr] & & & B_0 \cdot B_1 &
\end{tikzcd}$$

I $\quad$ J $\quad$ $f(I)B_0$ $\quad$ $f(J)B_1$

$I(A_0 \cdot A_1)$ $\quad$ $J(A_0 \cdot A_1)$ $\quad$ $f(I)(B_0 \cdot B_1) = f(I(A_0 \cdot A_1))(B_0 \cdot B_1)$ $\quad$ $f(J)(B_0 \cdot B_1) = f(J(A_0 \cdot A_1))(B_0 \cdot B_1)$

Here the arrows indicate that A_0 is mapping into B_0, A_1 is mapping into B_1, and consequently $A_0 \cdot A_1$ is mapping into $B_0 \cdot B_1$ via f.

By Lemma 4.2.6, $I(A_0 \cdot A_1)$ and $J(A_0 \cdot A_1)$ are ideals of definitions of $A_0 \cdot A_1$. Thus, powers of $I(A_0 \cdot A_1)$ and $J(A_0 \cdot A_1)$ are cofinal, hence the same is true for powers of $f(I)(B_0 \cdot B_1)$ and $f(J)(B_0 \cdot B_1)$. Therefore, $f(I)(B_0 \cdot B_1)$ is an ideal of definition of $B_0 \cdot B_1$ if and only if $f(J)(B_0 \cdot B_1)$ is. Since

$$f(J)(B_0 \cdot B_1) = f(J)B_1(B_0 \cdot B_1),$$

and $f(J)B_1$ is an ideal of definition of B_1 by choice, Lemma 4.2.6 implies $f(J)(B_0 \cdot B_1)$ is an ideal of definition of $B_0 \cdot B_1$. Again, because

$$f(I)(B_0 \cdot B_1) = f(I)B_0(B_0 \cdot B_1),$$

a third application of Lemma 4.2.6 tells us that $f(I)B_0$ is an ideal of definition of B_0, and we win!

(4) follows (1) and the fact that any bounded subring of a Huber ring is contained in a ring of definition (Corollary 4.2.4(3)).

[4]Thanks to Takumi Murayama for this diagram.

(5) Note B_0 is open in B. Thus, $f^{-1}(B_0)$ is an open subring of A. It therefore contains a ring of definition of A since any open subring of a Huber ring is Huber (Corollary 4.2.4(2)).

(6) First observe that an ideal of a Huber ring is open if and only if it contains some ideal of definition of some ring of definition of the Huber ring. Since I is open, pick an ideal of definition J of some ring of definition A_0 such that $J \subset I$. Furthermore, by (4) one may pick a ring of definition B_0 such that $f(A_0) \subseteq B_0$, and then from (3) it will follows that $f(J)B_0$ is an ideal of definition of B_0. Since $J \subseteq I$, $f(J)B \subseteq f(I)B$. But $f(J)B_0 \subseteq f(J)B$, and so $f(I)B$ is open.

(7) is left as an exercise for the reader. You may need Corollary 4.2.4(2). Also note that to establish that f is adic using the fact that $f|_{A'} : A' \to B'$ is adic, one must also establish that f is continuous, since we have only assumed in the hypothesis for (7) that f is a ring homomorphism. $\square$

As the above properties show, requiring a continuous homomorphism of Huber rings to be adic is quite restrictive. It is easy to give examples of non-adic continuous homomorphisms:

Example 4.3.4.

(1) (non-example) Let R be any Noetherian ring of dimension $\geqslant 1$. Pick prime ideals $\mathfrak{p} \subsetneq \mathfrak{q}$ of R, and consider the identity map

$$\mathrm{id} : R \to R$$

where the first copy of R is given the $\mathfrak{p}$-adic topology and the target copy the $\mathfrak{q}$-adic topology. This map is easily seen to be continuous, but not adic using Proposition 4.3.3(3).

(2) (non-example) Let K be a perfectoid field. Consider the canonical inclusion

$$i : K^\circ \hookrightarrow K^\circ[[x]],$$

where $K^\circ[[x]]$ is considered as an adic ring with ideal of definition $(t, x)K^\circ[[x]]$, for a fixed $t \in K^{\circ\circ}$. Then i is again continuous, but the ideal of definition tK° expanded to $K^\circ[[x]]$ does not have the same radical as $(t, x)K^\circ[[x]]$. Therefore i is not adic by Proposition 4.3.3(3).

Corollary 4.3.5. *Let $f : A \to B$, $g : B \to C$ be continuous homomorphisms of adic rings.*

1. *If f, g are adic, so is $g \circ f$.*
2. *If $g \circ f$ is adic, so is g.*

Proof.

(1) is obvious from Proposition 4.3.3 parts (3), (4).

(2) Let C_0 be a ring of definition of C with ideal of definition J. Then $g^{-1}(C_0)$ is an open subring of B with open ideal $g^{-1}(J)$. Let B_0 be a ring of definition of B contained in $g^{-1}(C_0)$. Then

$$I := g^{-1}(J) \cap B_0$$

is an open ideal of B_0. By continuity of f, $f^{-1}(B_0)$ is an open subring of A with open ideal $f^{-1}(I)$. Let A_0 be a ring of definition of A contained in $f^{-1}(B_0)$, and let I' be an ideal of definition of A_0 contained in the open ideal $f^{-1}(B_0) \cap A_0$. By construction,

$$g \circ f(A_0) \subseteq C_0 \text{ and } g(B_0) \subseteq C_0.$$

Since $g \circ f$ is adic, by Proposition 4.3.3(3), $g \circ f(I')C_0$ is an ideal of definition of C_0. However, again by construction we have

$$g \circ f(I')C_0 \subseteq g(I)C_0 \subseteq J.$$

Thus, $g(I)C_0$ is an ideal of C_0 nested in between two ideals of definition, hence is itself an ideal of definition. $\square$

Although adic homomorphisms are rare among continuous homomorphisms of Huber rings, the situation is vastly different for continuous homomorphisms of Tate rings.

Proposition 4.3.6. *Let $f : A \to B$ be a continuous ring homomorphism of Huber rings (not necessarily adic).*

(1) *If A is a Tate ring, then B is a Tate ring.*
(2) *f is adic.*

Proof. (1) It is easy to see that a continuous ring hommormophism maps topologically nilpotent elements to toplogically nilpotent elements, and ring homomorphisms map units to units. Thus B is a Tate ring if A is.

(2) By Proposition 4.3.3(5), choose a ring of definition B_0 of B (since B is Tate), and a ring of definition A_0 of A such that $f(A_0) \subseteq B_0$. Let $g \in A_0$ be a topologically nilpotent unit of A. Then $f(g)$ is a topologically nilpotent unit of B. Proposition 3.3.7(b) implies that the topology on A_0 is gA_0-adic, and the topology on B_0 is $f(g)B_0$-adic. But, $f(g)B_0 = f(gA_0)B_0$. Thus, f is adic. □

4.4. **Uniform Huber rings.** Given a Huber ring A, the subring of power bounded elements A° is the union of all open and bounded subrings of A. Alternatively, A° is the union of all rings of definition of A, courtesy Proposition 4.2.1. However, it may not be the case that A° is bounded (see Remark 4.2.3(1) for an example). Huber rings for which the subring of power bounded elements is bounded are known by a special name.

Definition 4.4.1. A Huber ring A is **uniform** if A° is bounded in A.

Remark 4.4.2.

(i) If A° is bounded in A, then A° is a ring of definition of A– in fact, it is the largest ring of definition of A. In particular, a uniform Huber ring has a canonical ring of definition.

(ii) If A is a uniform Tate ring (for example, a uniform K-Banach algebra [Bha17, Definition 8.7, 9.3] over a perfectoid field K) with topologically nilpotent unit $t \in A^\circ$, then by Proposition 3.3.7

$$A = A^\circ[1/t].$$

For the purpose of applying the theory of Huber rings to perfectoid algebras, not much is lost by only dealing with uniform Huber rings since perfectoid algebras are uniform K-Banach algebras.

Proposition 4.4.3. *Let A be a uniform, Hausdorff Tate ring. Then A is reduced. In particular, if K is a perfectoid field (or any non-Archimedean field), then any uniform normed K-algebra is reduced. Thus, all perfectoid algebras over a perfectoid field are reduced.*

Proof. By hypothesis, A° is a ring of definition of A. Let $g \in A^\circ$ be a topologically nilpotent unit of A. By Proposition 3.3.7, the induced topology on A° is gA°-adic. Furthermore, since A° is Hausdorff, we also have

$$\bigcap_{n>0} g^n A^\circ = (0).$$

Note any nilpotent element of A is clearly power bounded, and so also an element of A°. Let $a \in A$ be nilpotent. Then for all $n > 0$,

$$ag^{-n}$$

is nilpotent, hence lies in A°. Thus, $a \in \bigcap_{n>0} g^n A^\circ$, which implies $a = 0$, showing that A is reduced.

Any normed K-algebra $(R, |\cdot|)$ is a Tate ring with topologically nilpotent unit any non-zero element of $K^{\circ\circ}$, and ring of definition

$$R_{\leqslant 1} = \{x \in R : |x| \leqslant 1\}.$$

Also, R is Hausdorff since its topology is given by a metric. Therefore, R is a uniform, Hausdorff Tate ring, hence reduced. Moreover, any perfectoid K-algebra is a uniform, normed K-algebra, thus also reduced. □

We have seen in Proposition 4.1.5 that for an arbitrary Huber ring A, A° is integrally closed in A. For uniform Huber rings, one can in fact, say something stronger. For this, we need to introduce the notion of complete/total integral closure.

Definition 4.4.4. Given an extension of rings $A \subseteq B$, we say $b \in B$ is *completely/totally integrally closed* over A if the powers of b are all contained in a finitely generated A-submodule of B. We say A is *completely/totally integrally closed* in B if every element of B that is completely integrally closed over A is contained in A.

Remark 4.4.5.

(i) If A is totally integrally closed in B, then A is integrally closed in B. The converse holds when A is Noetherian.

(ii) The elements of B totally integrally closed over A form a subring of B containing A. However, this ring may not be totally integrally closed in B, in contrast with the integral closure of A in B which is integrally closed in B.

Proposition 4.4.6. *If A is a uniform Huber ring, then A° is completely/totally integrally closed in A.*

Proof. Let $a \in A$ be completely/totally integrally closed over A°. Then there exists a finitely generated A°-submodule M of A such that

$$a^{\mathbb{N}} := \{a^n : n \in \mathbb{N}\} \subseteq M.$$

Choosing generators $x_1, \ldots, x_n$ for M, we have

$$M = A^\circ x_1 + \cdots + A^\circ x_n.$$

Since A° is bounded, each $A^\circ x_i$ is also bounded, proving boundedness of M. Thus, $a^{\mathbb{N}}$ is bounded, and so a is power bounded, i.e., $a \in A^\circ$. $\square$

Corollary 4.4.7. *Let A be a Tate ring. The following are equivalent:*

(1) *A is uniform.*
(2) *There exists a ring of definition A_0 of A that is totally/completely integrally closed in A.*

Proof. For the proof of (1) $\Rightarrow$ (2), note that given (1), A° is a ring of definition of A since it is open and bounded, and it is totally integrally closed in A by Proposition 4.4.6.

To prove the converse, first observe that rings of definitions are clearly bounded. Thus, it suffices to show that

$$A^\circ = A_0.$$

We already know $A_0 \subseteq A^\circ$, since A° is the union of all rings of definition. Let $g \in A_0$ be a topologically nilpotent unit of A. Let $f \in A^\circ$. Then $f^{\mathbb{N}}$ is bounded, and so

$$f^{\mathbb{N}} \subseteq g^n A_0,$$

for some $n \in \mathbb{Z}$ by Proposition 4.1.4(7). Thus, $f^{\mathbb{N}}$ is contained in a finitely generated A_0 submodule of A, and so $f \in A_0$ because A_0 is totally integrally closed in A. This shows $A^\circ \subseteq A_0$, completing the proof. $\square$

Remark 4.4.8. The proof of Corollary 4.4.7 shows that if any ring of definition of a Tate ring A is totally integrally closed in A, then it coincides with A°.

Theorem 4.4.9. *(compare with* [Bha17, Theorem 9.7]*) Let K be a perfectoid field. Consider the categories*

$$\mathcal{C} := \left\{ \begin{array}{l} \bullet \textit{ Objects are } \textbf{Tate rings } A \textit{ which are } K\textit{-algebras such that:} \\ \qquad - \textit{The structure map } K \hookrightarrow A \textit{ is adic.} \\ \qquad - A \textit{ is uniform.} \\ \bullet \textit{ Morphisms are continuous } K\textit{-algebra maps (automatically} \\ \textit{adic by Corollary 4.3.5(2)).} \end{array} \right\}$$

and

$$\mathcal{D}_{\mathtt{tic}} := \left\{ \begin{array}{l} \bullet \textit{ Objects are } \textbf{\textit{adic}} \ \mathsf{K}^\circ\textit{-algebras } \mathsf{B} \textit{ that satisfy the following:} \\ \quad - \textit{The structure map } \mathsf{K}^\circ \to \mathsf{B} \textit{ is adic.} \\ \quad - \mathsf{B} \textit{ is a torsion-free } (\Leftrightarrow \textit{flat}) \ \mathsf{K}^\circ\textit{-module.} \\ \quad - \mathsf{B} \textit{ is } \textbf{\textit{totally integrally closed}} \textit{ in } \mathsf{B} \otimes_{\mathsf{K}^\circ} \mathsf{K}. \\ \bullet \textit{ Morphisms are } \mathsf{K}^\circ\textit{-algebra maps (automatically continuous,} \\ \textit{hence again adic by Corollary 4.3.5(2)).} \end{array} \right\}.$$

Then $\mathcal{C}$ *and* $\mathcal{D}_{\mathtt{tic}}$ *are equivalent.*

Proof. For the rest of the proof, fix a non-zero $\mathsf{t} \in \mathsf{K}^{\circ\circ}$.

Define a functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}_{\mathtt{tic}}$ as follows: for $\mathsf{A} \in \mathsf{ob}(\mathcal{C})$,

$$\mathsf{F}(\mathsf{A}) := \mathsf{A}^\circ.$$

Since the structure map $\mathsf{K} \hookrightarrow \mathsf{A}$ is adic, we get $\mathsf{K}^\circ \hookrightarrow \mathsf{A}^\circ$, and $\mathsf{K}^{\circ\circ} \hookrightarrow \mathsf{A}^{\circ\circ}$, by Proposition 4.3.3. Thus, A° is a K°-algebra, and t is a topologically nilpotent unit of A contained in A°.

Uniformity implies A° is a ring of definition of A (since A° is then open and bounded), and A° is totally integrally closed in

$$\mathsf{A} = \mathsf{A}^\circ \otimes_{\mathsf{K}^\circ} \mathsf{K}\ [5]$$

by Proposition 4.4.6. Furthermore, the map $\mathsf{K}^\circ \hookrightarrow \mathsf{A}^\circ$ is adic as a consequence of Proposition 4.3.3(7), since it is induced by the adic map $\mathsf{K} \hookrightarrow \mathsf{A}$ by restricting it to open subsrings. This shows that F maps objects of $\mathcal{C}$ into $\mathcal{D}_{\mathtt{tic}}$.

Moreover, if $\mathsf{f} : \mathsf{A} \to \mathsf{C}$ is a morphism in $\mathcal{C}$, then $\mathsf{f}(\mathsf{A}^\circ) \subseteq \mathsf{C}^\circ$ by Proposition 4.3.3(1), since adic homorphisms map bounded sets to bounded sets. We then define $\mathsf{F}(\mathsf{f})$ to be the induced map $\mathsf{A}^\circ \to \mathsf{C}^\circ$. It it then easy to see that F is a functor.

Now consider the functor

$$\mathsf{G} : \mathcal{D}_{\mathtt{tic}} \to \mathcal{C}$$

that maps $\mathsf{B} \in \mathsf{ob}(\mathcal{D}_{\mathtt{tic}})$ to

$$\mathsf{G}(\mathsf{B}) := \mathsf{B} \otimes_{\mathsf{K}^\circ} \mathsf{K} = \mathsf{B}[1/\mathsf{t}].$$

First observe that the structure map $\mathsf{K}^\circ \to \mathsf{B}$ being adic implies that the topology on B is tB-adic (Proposition 4.3.3(3)). We consider $\mathsf{G}(\mathsf{B}) = \mathsf{B}[1/\mathsf{t}]$ as a Tate ring with topology generated by a neighborhood basis of 0 given by the collection of sets $\{\mathsf{t}^n\mathsf{B} : n \in \mathbb{Z}\}$ (check this gives $\mathsf{G}(\mathsf{B})$ the structure of a topological ring!). Then B is a ring of definition of $\mathsf{G}(\mathsf{B})$ with ideal of definition tB, and topologically nilpotent unit t. That $\mathsf{G}(\mathsf{B})$ is uniform now follows from Corollary 4.4.7, because B is a ring of definition of $\mathsf{G}(\mathsf{B})$, totally integrally closed in the latter. The map $\mathsf{K} \hookrightarrow \mathsf{G}(\mathsf{B})$ extends the adic structure map $\mathsf{K}^\circ \to \mathsf{B}$, and both K°, B are open in K, $\mathsf{G}(\mathsf{B})$ respectively. Therefore, again by Proposition 4.3.3(7), $\mathsf{K} \hookrightarrow \mathsf{G}(\mathsf{B})$ is adic. This shows that G maps objects of $\mathcal{D}_{\mathtt{tic}}$ into $\mathcal{C}$. Finally, G acts on morphisms by localization at t/tensoring by K (one needs to check this preserves continuity, but this is an easy check).

Clearly, $\mathsf{G} \circ \mathsf{F}$ is isomorphic to the identity functor on $\mathcal{C}$ by the above footnote and since for any $\mathsf{A} \in \mathsf{ob}(\mathcal{C})$, A° is a ring of definition of A with ideal of definition tA° (the latter follows because $\mathsf{K} \hookrightarrow \mathsf{A}$ is adic). That $\mathsf{F} \circ \mathsf{G}$ is isomorphic to the identity functor follows from the proof of Corollary 4.4.7, where we show that if A is a Tate ring with a totally integrally closed ring of definition A_0, then $\mathsf{A}_0 = \mathsf{A}^\circ$. Applying it to our situation, note that $\mathsf{G}(\mathsf{B})$ is Tate, with totally integrally closed ring of definition B, and so,

$$\mathsf{F}(\mathsf{G}(\mathsf{B})) = \mathsf{G}(\mathsf{B})^\circ = \mathsf{B}.$$

□

[5] Note that $\mathsf{A} = \mathsf{A}^\circ \otimes_{\mathsf{K}^\circ} \mathsf{K}$ because

$$\mathsf{A}^\circ \otimes_{\mathsf{K}^\circ} \mathsf{K} = \mathsf{A}^\circ \otimes_{\mathsf{K}^\circ} \mathsf{K}^\circ[1/\mathsf{t}] = \mathsf{A}^\circ[1/\mathsf{t}],$$

and the right-most ring equals A by Proposition 3.3.7(c).

Remark 4.4.10. The 'tic' in $\mathcal{D}_{\mathtt{tic}}$ stands for *totally integrally closed*. The definitions of the categories $\mathcal{C}$, $\mathcal{D}_{\mathtt{tic}}$ in the above theorem are a little less transparent than in [Bha17, Theorem 9.7], because the goal was to make $\mathcal{C}$, $\mathcal{D}_{\mathtt{tic}}$ explicitly independent of the choice of any $t \in K^{\circ\circ} - \{0\}$. Although implicit in the definition of $\mathcal{C}$, $\mathcal{D}_{\mathtt{tic}}$, we want to point out that by requiring the structure maps $K \to A$ and $K^\circ \to B$ to be adic, one ensures that any non-zero element of $K^{\circ\circ}$ generates an ideal of definition of both A and B, where A and B are as above.

Although we assumed our ground field is perfectoid in Theorem 4.4.9, the equivalence of categories holds for any non-trivially valued field $(K, |\cdot|)$ which is Huber. This is clear because we did not use any special property of a perfectoid field, but we nevertheless state formally:

Proposition 4.4.11. *Theorem 4.4.9 holds if instead of assuming* K *is perfectoid, we assume that* $(K, |\cdot|)$ *is a non-trivially valued field which is Huber.*

Proof. Recall that by Exercise 4.1.6, K° is the localization of the valuation ring of K at its unique height 1 prime, and $K^{\circ\circ}$ equals the height 1 prime (which also coincides with the maximal ideal of K°). In particular, K° is a valuation ring of K of Krull dimension 1, and so for any $t \in K^{\circ\circ} - \{0\}$,

$$K^\circ[1/t] = K.$$

All the arguments in the proof of Theorem 4.4.9 go through verbatim. $\square$

Our final goal for this subsection is to prove an analogue of [Bha17, Corollary 9.9] in the setting of Tate/adic rings. For this, we need the following concept from almost mathematics:

Definition 4.4.12. Let K be a perfectoid field. Let A be a torsion-free ($\Leftrightarrow$ flat) K° algebra. We define the **ring of almost elements of** A, denoted A_*, as

$$A_* := \{f \in A \otimes_{K^\circ} K : K^{\circ\circ} \cdot f \subseteq A\}.$$

Using the fact that $(K^{\circ\circ})^2 = K^{\circ\circ}$ for a perfectoid field K, it is easy to see that A_* is actually a ring. Moreover, it is also clear that we have an inclusion $A \hookrightarrow A_*$.

Exercise 4.4.13. Show that if K is a perfectoid field, then $(K^\circ)_* = K^\circ$.

Definition 4.4.14. Let $A \hookrightarrow B$ be an extension of rings. For a prime number $p > 0$, we say that A is **p-root closed** in B if for all $f \in B$, $f^p \in A \Rightarrow f \in A$.

Example 4.4.15.

(i) If $A \subset B$ is an extension such that A is (totally)integrally closed in B, then A is p-root closed in B.

(ii) Let A be a ring of prime characteristic $p > 0$. Then A is semi-perfect (i.e. Frobenius on A is surjective) if and only if A^p is p-root closed in A. Here A^p is the image of the Frobenius endomorphism on A.

Theorem 4.4.16. *(Compare with* [Bha17, Corollary 9.10]*) Let* K *be a perfectoid field such that the residue field has characteristic* $p > 0$*, and* $\mathcal{D}_{\mathtt{tic}}$ *be the category defined in Theorem 4.4.9. Consider the categories*

$$\mathcal{D}_{\mathtt{ic}} := \left\{ \begin{array}{l} \bullet \textit{ Objects are } \textbf{\textit{adic}}\ K^\circ\textit{-algebras } B \textit{ that satisfy the following:} \\ \quad - \textit{The structure map } K^\circ \to B \textit{ is adic.} \\ \quad - B \textit{ is a torsion-free } (\Leftrightarrow \textit{flat}) \ K^\circ\textit{-module.} \\ \quad - \textit{The inclusion } B \hookrightarrow B_* \textit{ is surjective.} \\ \quad - B \textit{ is } \textbf{\textit{integrally closed}} \textit{ in } B \otimes_{K^\circ} K. \\ \bullet \textit{ Morphisms are } K^\circ\textit{-algebra maps (automatically continuous,} \\ \textit{hence again adic by Corollary 4.3.5(2)).} \end{array} \right\}.$$

and

$$\mathcal{D}_{\mathtt{prc}} := \left\{\begin{array}{l} \bullet \textit{ Objects are } \textbf{\textit{adic}} \ \mathsf{K}^\circ\textit{-algebras } \mathsf{B} \textit{ that satisfy the following:} \\ \qquad - \textit{The structure map } \mathsf{K}^\circ \to \mathsf{B} \textit{ is adic.} \\ \qquad - \mathsf{B} \textit{ is a torsion-free } (\Leftrightarrow \textit{flat}) \ \mathsf{K}^\circ\textit{-module.} \\ \qquad - \textit{The inclusion } \mathsf{B} \hookrightarrow \mathsf{B}_* \textit{ is surjective.} \\ \qquad - \mathsf{B} \textit{ is } \mathsf{p}\textbf{\textit{-root closed}} \textit{ in } \mathsf{B} \otimes_{\mathsf{K}^\circ} \mathsf{K}. \\ \bullet \textit{ Morphisms are } \mathsf{K}^\circ\textit{-algebra maps (automatically continuous,} \\ \textit{hence again adic by Corollary 4.3.5(2)).} \end{array}\right\}.$$

Then $\mathcal{D}_{\mathtt{tic}}$, $\mathcal{D}_{\mathtt{ic}}$, $\mathcal{D}_{\mathtt{prc}}$ are the same full subcategories of the category of adic K°-algebras whose structure maps are adic.

Proof. First, let us show that $\mathcal{D}_{\mathtt{tic}}$ is a full subcategory of $\mathcal{D}_{\mathtt{ic}}$. Let $A \in \mathtt{ob}(\mathcal{D}_{\mathtt{tic}})$. It suffices to show that A is also an object in $\mathcal{D}_{\mathtt{ic}}$. Since totally integrally closed implies integrally closed, we see that A is integrally closed in $A \otimes_{K^\circ} K$. It remains to check that $A = A_*$. So let $f \in A_*$. Since A_* is a ring, we have

$$f^{\mathbb{N}} \subseteq A_*.$$

Choosing a non-zero $t \in K^{\circ\circ}$, we then get

$$t \cdot f^{\mathbb{N}} \subseteq A,$$

by definition of A_*. Therefore,

$$f^{\mathbb{N}} \subseteq t^{-1}A \subseteq A \otimes_{K^\circ} K,$$

and A is totally integrally closed in $A \otimes_{K^\circ} K$, implies $f \in A$. This completes the proof that $\mathcal{D}_{\mathtt{tic}} \subseteq \mathcal{D}_{\mathtt{ic}}$.

That $\mathcal{D}_{\mathtt{ic}}$ is a full subcategory of $\mathcal{D}_{\mathtt{prc}}$ is obvious (p-root closedness of an object B of $\mathcal{D}_{\mathtt{ic}}$ follows because B is integrally closed in $B \otimes_{K^\circ} K$). To complete the proof, it then suffices to show that every object of $\mathcal{D}_{\mathtt{prc}}$ is also an object of $\mathcal{D}_{\mathtt{tic}}$.

So let $B \in \mathtt{ob}(\mathcal{D}_{\mathtt{prc}})$. We need to show that B is totally integrally closed in $B \otimes_{K^\circ} K$. We will now use the perfectoid hypothesis. Using the isomorphism of multiplicative monoids [Bha17, Proposition 2.1.2]

$$\varprojlim_{x \mapsto x^p} K^\circ \xrightarrow{\sim} \varprojlim_{\phi} K^\circ/p,$$

where ϕ is the Frobenius map, one can choose a non-zero $t \in K^{\circ\circ}$ with a compatible choice of p-power roots t^{-p^e} to all lie in K°. Fix such a t.

Now suppose $f \in B \otimes_{K^\circ} K = B[1/t]$ such that $f^{\mathbb{N}}$ is contained in some finitely generated B-submodule of $B[1/t]$. There exists $c > 0$ such that

$$f^{\mathbb{N}} \subseteq t^{-c}B.$$

Thus, $t^c \cdot f^{\mathbb{N}} \subseteq B$, implying that for all $n > 0$,

$$t^c f^{p^n} \in B.$$

Using p-root closedness of B in $B \otimes_{K^\circ} K$, we now get that for all $n > 0$,

$$t^{c/p^n} f \in B.$$

Since $K^{\circ\circ}$ is generated by $\{t^{c/p^n} : n > 0\}$, it follows that

$$K^{\circ\circ} \cdot f \subseteq B,$$

that is, $f \in B_*$. But $B_* = B$ by hypothesis, and so f is an element of B. This is precisely what it means for B to be totally integrally closed in $B \otimes_{K^\circ} K$. $\square$

Implicit in the proof of Theorem 4.4.16 is the following purely algebraic fact, that has nothing to do with adic rings:

Proposition 4.4.17. *Let* K *be a perfectoid field whose residue field has characteristic* $p > 0$*, and* B *a torsion-free* K°*-algebra. Then the following are equivalent:*

(1) B *is totally integrally closed in* $B \otimes_{K^\circ} K$*.*
(2) B *is integrally closed in* $B \otimes_{K^\circ} K$*, and* $B_* = B$*.*
(3) B *is* p*-root closed in* $B \otimes_{K^\circ} K$*, and* $B_* = B$*.*

Proof. See the proof of Theorem 4.4.16. □

5. Completions of Huber rings

Let A be a Huber ring with ring of definition A_0, and finitely generated ideal of definition $I \subseteq A_0$ (remember this means I is an ideal of A_0 and not necessarily that of A). Note that the powers I^n are subgroups of $(A, +)$, and so we can form the inverse limit

$$\widehat{A} := \varprojlim_{n \geqslant 1} A/I^n,$$

in the category of abelian groups. However, $\widehat{A}$ can be given the natural structure of a ring. **Note coordinate-wise multiplication will not work!**

Before defining the ring structure on $\widehat{A}$ we discuss a few properties of this inverse limit as a group:

(1) $\widehat{A}$ is Hausdorff in the natural topology, where the natural topology is defined as follows– since each I^n is open in A, the quotient topology on A/I^n is discrete, and $\widehat{A}$ is topologized by giving it the subspace topology of the product topology on
$$\prod_{n \in \mathbb{N}} A/I^n.$$
Since $\prod_{n\in\mathbb{N}} A/I^n$ is Hausdorff being a product of discrete spaces, $\widehat{A}$ is also Hausdorff.

(2) The natural map
$$i : A \to \widehat{A}; \;\; a \mapsto (a + I, a + I^2, a + I^3, \dots)$$
is continuous, and the image $i(A)$ is dense in $\widehat{A}$. The ring structure on $\widehat{A}$ will make i a ring map. Note i is injective if and only if $\bigcap_{n \geqslant 1} I^n = (0)$, and the latter is equivalent to A being Hausdorff.

(3) Since I is an ideal of the subring A_0, $\widehat{A_0} = \varprojlim_{n\geqslant 1} A_0/I^n$ is a ring. There is an obvious inclusion of groups
$$j : \widehat{A_0} \hookrightarrow \widehat{A},$$
and we want the ring structure on $\widehat{A}$ to extend that on $\widehat{A_0}$.

(4) If $x = (x_n + I^n)_{n\geqslant 1}$ is an element of $\widehat{A}$, then x is in the image of j if and only if $x_1 \in A_0$ (use the fact that $x_n - x_1 \in I \subseteq A_0$, to see this).

(5) $\widehat{A_0}$ is open in $\widehat{A}$. Since $\widehat{A_0}$ is a topological ring, it suffices to show that $\widehat{A_0}$ contains an open neighborhood of 0. Consider the open neighborhood
$$U := (A_0/I \times A/I^2 \times A/I^3 \times \cdots \times A/I^n \times \dots) \cap \widehat{A}$$
of 0 in $\widehat{A}$. Using (4) we have $U = \widehat{A_0}$.

(6) Any element $x \in \widehat{A}$ can be written as $x = y + i(a)$, for some $y \in \widehat{A_0}$, and some $a \in A$. Indeed, if $x = (x_1 + I, x_2 + I, \dots)$, then take
$$y = (0, x_2 - x_1 + I^2, x_3 - x_1 + I^3, \dots), \;\; \text{and} \;\; a = x_1.$$

5.1. Multiplication on $\widehat{A}$. [6]

Construction 1: We first define $i(a) \cdot x$ for $x \in \widehat{A}$ and $a \in A$. Since

$$\ell_a : A \to A$$

is continuous, there exists $c > 0$ such that $I^c \subseteq \ell_a^{-1}(A_0)$, that is, $a \cdot I^c \subseteq A_0$. Then for all $n \geqslant 1$,

$$a \cdot I^{n+c} \subseteq I^n.$$

Suppose $x = (x_1 + I, x_2 + I^2, x_3 + I^3, \dots)$. Then

$$y = (ax_{c+1} + I, ax_{c+2} + I^2, ax_{c+3} + I^3, \dots)$$

is an element of $\widehat{A}$, because for all $n \geqslant 1$,

$$ax_{c+n+1} - ax_{c+n} = a(x_{n+c+1} - x_{n+c}) \in a \cdot I^{n+c} \subseteq I^n.$$

We define the product of x and $i(a)$ to equal y, where y is as above. It is quite easy to check that $x \cdot i(a)$ is well-defined (i.e. independent of the choices made for representatives of coordinates of x). Multiplication on $\widehat{A_0}$ is already known, and we also want the map

$$i : A \to \widehat{A}$$

to be a ring homomorphism. Then distributivity forces our hand on how to define multiplication in general. Namely, for $x_1, x_2 \in \widehat{A}$, expressing $x_1 = y_1 + i(a_1)$ and $x_2 = y_2 + i(a_2)$, for $y_1, y_2 \in \widehat{A_0}$, $a_1, a_2 \in A$, we define

$$x_1 \cdot x_2 := y_1 y_2 + y_1 \cdot i(a_2) + y_2 \cdot i(a_3) + i(a_1 a_2).$$

Of course one has to check that everything is well-defined because so many choices were made in defining multiplication, but it all works out!

Construction 2: The previous construction is helpful because it defines multiplication directly on the inverse limit

$$\varprojlim_{n \geqslant 1} A/I^n.$$

However, it involves making a few choices. We now give a more choice independent construction, which is similar in spirit to the previous one. To explain this contruction, we need to think of $\widehat{A}$ as the quotient of the group $\mathcal{C}$ of Cauchy sequences by the subgroup $\mathcal{N}$ of null sequences. Here a sequence $(x_n)_n$ in A is Cauchy if for all I^j, there exists $N > 0$ such that for all $n, m \geqslant N$,

$$x_n - x_m \in I^j.$$

We leave the definition of a null sequence to the reader. Anyway, the group structure on $\mathcal{C}$ is obvious (addition is done coordinate-wise). However, $\mathcal{C}$ is also a ring; in fact, coordinate-wise multiplication gives multiplication on $\mathcal{C}$. It requires more work to see why this is true since I^n is not necessarily closed under multiplication by elements of A. However, because A is topological, for any $a \in A$, and I^j, one has

$$I^n a \subseteq I^j,$$

for all $n >> 0$, and we exploit this fact to show coodinate-wise multiplication of two Cauchy sequences is Cauchy.

Suppose $(x_n)_n$ and $(y_n)_n$ are Cauchy sequences. Then given I^j, we have to show the existence of $N > 0$ such that for all $n, m > 0$,

$$x_n y_n - x_m y_m \in I^j.$$

Pick n_0 such that for all $n \geqslant n_0$, $x_n - x_{n_0} \in I$ and $y_n - y_{n_0} \in I$. Thus, for all $n \geqslant n_0$,

$$x_n \in x_{n_0} + I \text{ and } y_n \in y_{n_0} + I.$$

Since left-multiplication by x_{n_0} and y_{n_0} are continuous, there exists I^k (with $k > j$) such that

$$x_{n_0} I^k, y_{n_0} I^k \subseteq I^j.$$

[6] Thanks to Bhargav Bhatt for explaining Construction 1, and Brian Conrad for explaining Construction 2.

Now, pick $N > n_0$ such that for all $n, m \geqslant N$, $x_n - x_m, y_n - y_m \in I^k$. Then

$$x_n y_n - x_m y_m = x_n(y_n - y_m) + y_m(x_n - x_m) \in x_n I^k + y_m I^k \subseteq (x_{n_0} + I)I^k + (y_{n_0} + I)I^k.$$

By our choice of I^k,

$$(x_{n_0} + I)I^k + (y_{n_0} + I)I^k = x_{n_0} I^k + y_{n_0} I^k + I^{k+1} \subseteq I^j,$$

and so $x_n y_n - x_m y_m \in I^j$ for all $n, m \geqslant N$.

Thus, the group $\mathcal{C}$ of Cauchy sequences is a ring. It now suffices to show that the subgroup $\mathcal{N}$ of null sequences is an ideal of $\mathcal{C}$. Thus, given a Cauchy sequence $(x_n)_n$ and a null sequence $(i_n)_n$, we need to show that $(x_n i_n)_n$ is also a null sequence. Suppose we are given I^j. As before, pick $n_0 > 0$ such that for all $n > n_0$

$$x_n \in x_{n_0} + I.$$

Again, by continuity of left-multplication by x_{n_0}, choose I^k, with $k > j$, such that $x_{n_0} I^k \subseteq I^j$. Now pick $N > n_0$ such that for all $n \geqslant N$, $i_n \in I^k$. Then for all $n \geqslant N$,

$$x_n i_n \in (x_{n_0} + I)I^k = x_{n_0} I^k + I^{k+1} \subseteq I^j.$$

Hence, $(x_n i_n)_n$ is a null sequence.

5.2. **Basic properties of completion of a Huber ring.** What we would now like is for $\widehat{A}$ to also be a Huber ring with ring of definition $\widehat{A_0}$ such that the ideal of definition of $\widehat{A_0}$ is $I\widehat{A_0}$. But things are more complex now because A_0 is NOT a Noetherian ring. However, everything works out because I is finitely generated.

Proposition 5.2.1. *Let A be a Huber ring, with ring of definition A_0, containing an ideal of definition I. Let $\widehat{A}$, $\widehat{A_0}$ be the respective I-adic completions.*

(1) *$\widehat{A}$ is a Huber ring with ring of definition $\widehat{A_0}$ and ideal of definition $I\widehat{A_0}$. Moreover, For all $n \geqslant 1$, the natural map*

$$\widehat{A_0} \to A_0/I^n$$

has kernel $I^n\widehat{A_0}$.

(2) *$A \otimes_{A_0} \widehat{A_0} \cong \widehat{A}$, that is, the square*

$$\begin{tikzcd} A_0 \arrow[r, hook] \arrow[d] & A \arrow[d, "i"] \\ \widehat{A_0} \arrow[r, hook, "j"] & \widehat{A} \end{tikzcd}$$

is Cocartesian.

Proof.

(1) We refer the reader to [Sta17, Tag 05GG] for a slick proof of this. One only needs I to be finitely generated for the proof.

(2) This is somewhat tricky, and done in detail in [Hub93, Lemma 1.6(ii)]. □

As a consequence, we have

Corollary 5.2.2. *Let A be a Huber ring with Noetherian ring of definition A_0, and ideal of definition I.*

(1) *The natural map $A \to \widehat{A}$ is flat.*
(2) *If A is a finitely generated A_0-algebra, then $\widehat{A}$ is a finitely generated $\widehat{A_0}$-algebra, hence Noetherian.*

Proof.

(1) It is well known that ideal adic completion of Noetherian ring is Noetherian, and flat over the ring. Thus,

$$\mathsf{A}_0 \to \widehat{\mathsf{A}_0}$$

is flat. Since flatness is preserved by base change, tensoring by A and using $\widehat{\mathsf{A}} \cong \mathsf{A} \otimes_{\mathsf{A}_0} \widehat{\mathsf{A}_0}$, we get the desired result.

(2) Left to the reader. □

6. Glossary

We collect the numerous definitions in these notes in one place for the convenience of the reader. The terms appear in their order of appearance in the notes.

Throughout, let A *denote a commutative, topological ring.*

• **Adic ring:** A is called an *adic ring*, if there exists an ideal I of A such that $\{I^n : n \geqslant 0\}$ forms a basis of open neighborhoods of 0. The ideal I is then called *an ideal of definition.*

• **Huber ring:** A is called a *Huber ring* [7] if there exists an open subring A_0 of A such that the induced topology on A_0 is adic for some finitely generated ideal I of A_0. The ring A_0 is then called *a ring of definition of* A and I *an ideal of definition of* A.

• **Valuation topology:** For a valued field $(K, |\cdot|)$, with value group Γ, it is the unique topology on K such that sets of the form

$$K_{<\gamma} := \{x \in K : |x| < \gamma\},$$

for $\gamma \in \Gamma$, form a basis of open neighborhoods of 0.

• **Topological nilpotence:** An element a of A is called *topologically nilpotent* if $a^n \to 0$ as $n \to \infty$. In other words, for every open neighborhood U of 0, $a^n \in U$ for all $n >> 0$. The set of topologically nilpotent elements of A is denoted $A^{\circ\circ}$.

• **Tate ring:** A Huber ring is called a *Tate ring* if it has a topologically nilpotent unit.

• **Bounded set:** A subset $S \subseteq A$ is *bounded*, if for every open neighborhood U of 0, there exists an open neighborhood V of 0 such that the set

$$\{vs : v \in V, s \in S\} \subseteq U.$$

Put differently, we say S is bounded, if for any open neighborhood U of 0, the set

$$\bigcap_{s \in S} \ell_s^{-1}(U),$$

(which is an intersection of open neighborhoods of 0) contains an open neighborhood of 0.

• **Power bounded element:** We say an element $a \in A$ is *power bounded*, if the set $\{a^n : n > 0\}$ is a bounded set. The set of power bounded elements of A is denoted A°.

• **Uniform Huber ring:** A Huber ring A is *uniform* if A° is bounded in A.

• **Ring of almost elements:** Let K be a perfectoid field. For a torsion-free ($\Leftrightarrow$ flat) $K\circ$-algebra A, the *ring of almost elements of* A is the ring

$$A_* := \{f \in A \otimes_{K^\circ} K : K^{\circ\circ} \cdot f \subseteq A\}.$$

• **p-root closed:** Fix a prime number $p > 0$. For an extension of rings $A \hookrightarrow B$, we say A is p*-root closed* in B is for all $f \in B$, $f^p \in A \Rightarrow f \in A$.

• **Adic homomorphism:** Let A, B Huber rings. An *adic homomorphism*

$$f : A \to B$$

is a continuous ring homomorphism for which there exists a ring of definition A_0 of A with ideal of definition I, and a ring of definition B_0 such that $f(A_0) \subseteq B_0$, and $f(I)B_0$ is an ideal of definition of B_0.

[7] Huber uses the term *f-adic.*

References

[Bha17] B. Bhatt, *Math 679: Perfectoid Spaces*, http://www-personal.umich.edu/%7Estevmatt/perfectoid.pdf, 2017, (notes by M. Stevenson).

[FS00] L. Fuchs and L. Salce, *Modules over non-Noetherian domains*, Mathematics surveys and monographs, vol. 84, American Mathematical Society, Providence, RI, 2000.

[Hub93] R. Huber, *Continuous valuations*, Mathematische Zeitschrift **212** (1993), 455–477.

[Sch14] P. Scholze, *Peter Scholze's lectures on p-adic geometry*, http://math.bu.edu/people/jsweinst/Math274/ScholzeLectures.pdf, 2014, (notes by J. Weinstein).

[Sta17] The Stacks Project Authors, *Stacks project*, http://stacks.math.columbia.edu, 2017.

[Wed12] T. Wedhorn, *Adic spaces*, https://www2.math.uni-paderborn.de/fileadmin/Mathematik/People/wedhorn/Lehre/AdicSpaces.pdf, 2012.

E-mail address: `rankeya@umich.edu`