1 a) TEM electrostatic in 2dim → curlfree $\vec{E}$ field

say $\boxed{E_y = E_0}$

then $\vec{H} = \frac{1}{Z_0}\hat{z}\times\vec{E}$ → $\boxed{H_x = -\frac{E_0}{Z_0}}$

b) Boundary conditions

conductor $E_{\parallel} = 0$ $H_{\perp} = 0$

open sides $E_{\perp} = 0$ $H_{\parallel} = 0$ → "opposite" of conductor

This indicates we may flip $\psi|_S = 0$ and $\frac{\partial\psi}{\partial n}|_S = 0$ on the open sides

TM $\psi|_{y=0,b} = 0$ $\frac{\partial\psi}{\partial x}|_{x=0,a} = 0$

rectangular ⇒ $\psi = E_0 \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

ie

$$\left\{\begin{aligned} E_z &= E_0\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ E_x &= \frac{-ik}{\gamma^2}E_0\frac{m\pi}{a}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ E_y &= \frac{ik}{\gamma^2}E_0\frac{n\pi}{b}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ H_x &= \frac{-i\epsilon\omega}{\gamma^2}E_0\frac{n\pi}{b}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ H_y &= \frac{-i\epsilon\omega}{\gamma^2}E_0\frac{m\pi}{a}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\end{aligned}\right.$$

$\boxed{\begin{matrix} m \geq 0 \\ n > 0\end{matrix}}$

↳ explicitly verifies the boundary conditions

note $\gamma_{mn}^2 = \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$

TE "flipped" $\frac{\partial\psi}{\partial y}|_{y=0,b} = 0$ $\psi|_{x=0,a} = 0$

⇒ $\psi = H_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$

so

$$\left\{\begin{aligned} H_z &= H_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ E_x &= -\frac{i\mu\omega}{\gamma^2}H_0\frac{n\pi}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ E_y &= -\frac{i\mu\omega}{\gamma^2}H_0\frac{m\pi}{a}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ H_x &= \frac{ik}{\gamma^2}H_0\frac{m\pi}{a}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ H_y &= -\frac{ik}{\gamma^2}H_0\frac{n\pi}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\end{aligned}\right.$$

$\boxed{\begin{matrix} m > 0 \\ n \geq 0\end{matrix}}$

the cutoff frequencies are $\omega_{mn} = c\gamma_{mn}$

→ $\boxed{\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$

c) For the TM modes

<u>TM power</u>:

$$P = \frac{\omega k \epsilon_0}{2\gamma^2} \int_A |\psi|^2 \, da$$

$$= \frac{\omega k \epsilon_0}{2\gamma^2} |E_0|^2 \underbrace{\int_A \cos^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} \, da}_{\text{averages to } \frac{ab}{4}}$$

$(m=0 \Rightarrow a \to 2a)$

$$P = \frac{\omega k \epsilon_0}{2\gamma^2} |E_0|^2 \frac{ab}{4}$$

<u>TM power loss</u>:

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \oint |\hat{n} \times \vec{H}|^2 \, dl$$

$\hookrightarrow H_{\parallel}$

$$= \frac{1}{2\sigma\delta} \int_{0\ \text{top, bottom}}^{a} |H_x|^2 \, dx$$

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \, 2 \int_0^a |H_x|^2_{y=0} \, dx$$

$$= \frac{1}{\sigma\delta} \int_0^a \left(\frac{\epsilon_0 \omega}{\gamma^2}\right)^2 |E_0|^2 \left(\frac{n\pi}{b}\right)^2 \cos^2 \frac{m\pi x}{a} \, dx$$

$$= \frac{1}{\sigma\delta} \left(\frac{\epsilon_0 \omega}{\gamma^2}\right)^2 |E_0|^2 \left(\frac{n\pi}{b}\right)^2 \frac{a}{2}$$

$(m=0 \Rightarrow a \to 2a)$

So

$$\beta_{mn} = -\frac{1}{2P} \frac{dP}{dz} = \frac{\frac{1}{\sigma\delta} \frac{\epsilon_0^2 \omega^2}{\gamma^4} \left(\frac{n\pi}{b}\right)^2 \frac{a}{2}}{\frac{\omega k \epsilon_0}{\gamma^2} \frac{ab}{4}}$$

$$\boxed{\beta_{mn} = \frac{1}{\sigma\delta_\omega} \frac{\epsilon_0 \omega}{k_{mn}} \frac{2}{b} \left[\frac{(n/b)^2}{(m/a)^2 + (n/b)^2}\right]}$$

$$P = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)^{\frac{1}{2}} \frac{ab}{4} |E_0|^2$$

$$\frac{-dP}{dz} = \frac{1}{\sigma\delta} \left(\frac{\omega}{\omega_\lambda}\right)^2 \frac{1}{\mu^2 \omega_\lambda^2} |E_0|^2 \left(\frac{n\pi}{b}\right)^2 \frac{a}{2}$$

2 a) Write $\vec{J} = \hat{z} I_0 \delta(x)\delta(y)\,\Theta\left(\frac{d}{2}-|z|\right)$

In the radiation zone

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}\, e^{-i\vec{k}\cdot\vec{x}}\, d^3x$$

$$= \frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\hat{z}\int_{-d/2}^{d/2} e^{-ikz\cos\theta}\, dz$$

$$= \frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\hat{z}\frac{1}{-ik\cos\theta}\left[e^{-i\frac{kd}{2}\cos\theta} - e^{i\frac{kd}{2}\cos\theta}\right]$$

$$= \frac{\mu_0 I_0}{2\pi}\frac{e^{ikr}}{kr}\hat{z}\;\frac{\sin\left(\frac{kd}{2}\cos\theta\right)}{\cos\theta}$$

Again in the radiation zone we may take

$$\vec{H} = \frac{1}{\mu_0} i\vec{k}\times\vec{A} = \frac{iI_0}{2\pi}\frac{e^{ikr}}{r}(\hat{k}\times\hat{z})\;\frac{\sin\left(\frac{kd}{2}\cos\theta\right)}{\cos\theta}$$

↳ magnitude is $\sin\theta$

For power

$$\frac{dP}{d\Omega} = \frac{Z_0 r^2}{2}|\vec{H}|^2 = \boxed{\frac{Z_0|I_0|^2}{8\pi^2}\;\frac{\sin^2\theta\,\sin^2\left(\frac{kd}{2}\cos\theta\right)}{\cos^2\theta}}$$

b) $\vec{p} = \int \vec{x}\rho\, d^3x$

$$i\omega\rho = \vec{\nabla}\cdot\vec{J} = \partial_z J_z = I_0\delta(x)\delta(y)\,\delta\left(\frac{d}{2}-|z|\right)(-\operatorname{sgn} z)$$

$$= I_0\delta(x)\delta(y)\left[\delta\left(z+\frac{d}{2}\right) - \delta\left(z-\frac{d}{2}\right)\right]$$

So $\vec{p} = \frac{I_0}{i\omega}\int \vec{x}\,\delta(x)\delta(y)\left[\delta\left(z+\frac{d}{2}\right) - \delta\left(z-\frac{d}{2}\right)\right]dx\,dy\,dz$

$$= \frac{I_0}{i\omega}\hat{z}(-d) = \boxed{\frac{i d I_0}{\omega}\hat{z}}$$

Alternatively $\vec{p} = \frac{i}{\omega}\int \vec{J}\, d^3x = \frac{iI_0}{\omega}\hat{z}\int_{-d/2}^{d/2} dz = \frac{idI_0}{\omega}\hat{z}$

Then

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4|\vec{p}|^2\sin^2\theta \qquad \text{for electric dipole}$$

$$= \frac{c^2 d^2 Z_0|I_0|^2}{32\pi^2\omega^2}k^4\sin^2\theta = \boxed{\frac{Z_0|I_0|^2}{32\pi^2}(kd)^2\sin^2\theta}$$

Compare with $(kd)\to 0$ limit of the above

$$\frac{dP}{d\Omega} = \frac{Z_0|I_0|^2}{8\pi^2}\;\frac{\sin^2\theta\left(\frac{kd}{2}\right)^2\cos^2\theta}{\cos^2\theta} = \frac{Z_0|I_0|^2}{32\pi^2}(kd)^2\sin^2\theta$$

c) For $kd \gg 1$, the exact result gives

$$\frac{dP}{d\Omega} = \frac{Z_0|I_0|^2}{8\pi^2}\frac{\sin^2\theta}{\cos^2\theta}\sin^2\left(\frac{kd}{2}\cos\theta\right)$$

highly oscillatory

except when $\theta = \frac{\pi}{2}$ $(\cos\theta = 0)$

For $\cos\theta \simeq 0$ we expand $\sin^2\left(\frac{kd}{2}\cos\theta\right)$

$$\frac{dP}{d\Omega} \simeq \frac{Z_0|I_0|^2}{32\pi^2}(kd)^2$$

large enhancement factor

but this gets rapidly smaller away from the normal plane

3a) In the long wavelength limit

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\hat{\epsilon}^*\cdot\vec{p}\right|^2 = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\epsilon_0\gamma E_0\ \hat{\epsilon}^*\cdot\hat{\epsilon}_0\right|^2$$

$$= \frac{\gamma^2 k^4}{(4\pi)^2}\left|\hat{\epsilon}^*\cdot\hat{\epsilon}_0\right|^2$$

Take

then averaging over $\epsilon_0^{(i)}$ and summing over $\epsilon^{(j)}$ gives

$$\left|\hat{\epsilon}^*\cdot\hat{\epsilon}_0\right|^2 \rightsquigarrow \frac{1}{2}\left[1+\cos^2\theta\right]$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\gamma}{4\pi}\right)^2 k^4\ \frac{1}{2}\left(1+\cos^2\theta\right)}$$

b) $\vec{F} = m\vec{a}$ gives $\ddot{\vec{x}} = -\omega_0^2\vec{x} + \frac{q}{m}\vec{E}$

harmonic $\rightarrow$ $-\omega^2\vec{x} = -\omega_0^2\vec{x} + \frac{q}{m}\vec{E}$

or $\vec{x} = \frac{q/m}{\omega_0^2-\omega^2}\vec{E}$

$\vec{p} = q\vec{x} \Rightarrow \vec{p} = \frac{q^2}{m}\frac{1}{\omega_0^2-\omega^2}\vec{E}$

so $\boxed{\gamma = \frac{q^2}{\epsilon_0 m}\frac{1}{\omega_0^2-\omega^2}}$

This gives $\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{q^2}{4\pi\epsilon_0 m c^2}\right)^2\frac{\omega^4}{(\omega_0^2-\omega^2)^2}\ \frac{1}{2}\left(1+\cos^2\theta\right)}$

c) Integrating over the angles gives the total cross section

$$\int\frac{1}{2}\left(1+\cos^2\theta\right)d\Omega = \pi\int_{-1}^{1}(1+x^2)\,dx = \frac{8\pi}{3}$$

$$\sigma = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0 m c^2}\right)^2\frac{\omega^4}{(\omega_0^2-\omega^2)^2}$$

at high frequencies $\frac{\omega^4}{(\omega_0^2-\omega^2)^2}\rightarrow 1$

hence $\boxed{\sigma = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0 m c^2}\right)^2}$