Physics 506   Midterm exam   March 4, 2004

1.

20 points

Consider a waveguide with a square cross section of sidelength *a*. A monochromatic field with angular frequency ω is present the guide.

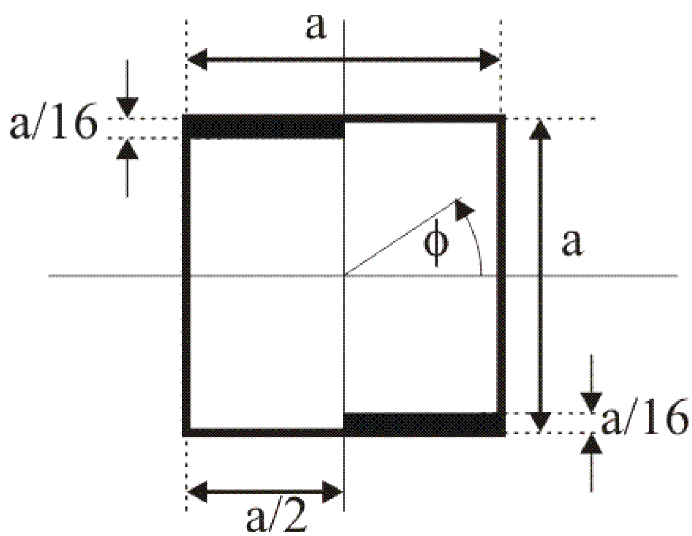
a) What are the three basic types of waveguide modes? Which ones occur in the given guide?

b) For the given guide, determine the lowest cutoff frequency. What is the degree of degeneracy of the lowest cutoff frequency, i.e. how many guide modes have that cutoff frequency? Determine all *E* and *H*-field components of these modes. Include all spatial and temporal dependences in your answers for the fields.

c) We assume $a = 5cm$ and $\omega = 2.4 \times 10^{10}\, rad/s$. A small experimental setup that has no effect of the waveguide field is located at the middle of the guide. How much power needs to be injected to obtain an amplitude of the transverse electric field of 100 *V/cm* in the setup? If you found degeneracy in part b, assume that only one of the degenerate modes is excited.

d) Assume that the wave travels through 1000*m* of guide before it reaches the experiment. How much power needs to be injected in order to achieve the condition of part c? (wall conductivity $\sigma = (2 \times 10^{-8}\, \Omega m)^{-1}$).

e) The shape perturbation indicated in the drawing is applied to the guide. Using perturbation theory and the mode(s) of part b, determine the *k*-numbers and polarization angles ϕ of the transverse electric fields of the perturbed modes (all walls have $\sigma = \infty$).



You may use all applicable results stated in the homework solutions or in the textbook without losing credit. You may receive partial credit for derivations in case an answer is wrong. **Derivations are required for results not found in the textbook/homework.**

2.

20 points

A current $I = \mathrm{Re}\, I_o \mathrm{e}^{-\mathrm{i}\omega t}$ is flowing in the x-y plane on a circle with radius *a* centered to the origin.

a) Find all spherical-multipole radiation coefficients $a_E(l,m)$ and $a_M(l,m)$ **without** using any small-source approximation.

   Which coefficients are zero, and which ones are not?

b) Knowing the coefficients, how would you obtain the **E** and **H**-fields in the radiation zone, the angular distribution of the radiated power, $\frac{dP}{d\Omega}$, and the total radiated power P ? *It is not required to work out the expressions for the fields,* $dP/d\Omega$ *and P.*

c) Using the small-source approximation, *ka*<<1, and assuming that the observation point is in the radiation zone, determine the $\hat{r}$, $\hat{\vartheta}$- and $\hat{\varphi}$- components of the **E** and **H**-fields of the lowest-order non-vanishing spherical-multipole radiation field. What are the distribution of the radiated power, $\frac{dP}{d\Omega}$, and the total radiated power *P?*

3.

20 points

Scattering of light with wavelength $\lambda = 632nm$ is used to study spherical non-absorbing and non-permeable nanoparticles with 20*nm* diameter and refractive index *n*=1.5. The incident light is a plane wave propagating in *z*-direction.

a) Find the differential scattering cross section of a single particle for the case that the incident light is linearly polarized ($\varepsilon_0 = \hat{\mathbf{x}}$, no selectivity with respect to outgoing polarization). Provide a numerical result.
b) Find the differential scattering cross section for the case that the incident light is circularly polarized with negative helicity ($\varepsilon_0 = \varepsilon_-$, no selectivity with respect to outgoing polarization). Provide a numerical result.
c) A small-diameter laser beam with circular polarization and $10W$ power is traversing through a $1mm$ thick dust layer filled with such particles at a density of $10^{11} cm^{-3}$. The particles are in a state of total spatial disorder. A detector with an area of $1cm^2$ is located at a distance of 50*cm* from the intersection volume between the laser beam and the dust layer, at a scattering angle of ϑ. What is the power measured by the detector? Make reasonable approximations and provide a numerical result.

Phys. 506 Electricity and Magnetism Winter 2004

Prof. G. Raithel

Midterm exam

50 Points = 100%

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1. Problem **20 Points** (4 on each part)

a): Waveguides with multiple surfaces have TE, TM and TEM modes. The given one has only one surface and therefore only supports TE and TM modes.

b): According to Section 8.4 of Jackson, the lowest mode is the TE_{10}-mode. Since the cross section is square, the TE_{10} mode is degenerate with TE_{01} (i.e. the degree of degeneracy is two). The cutoff frequency is

$$\omega_{10} = \omega_{01} = \frac{\pi}{\sqrt{\mu\epsilon}a} = \frac{\pi c}{a}$$

and $k = \frac{1}{c}\sqrt{\omega^2 - \omega_{01}^2}$ (not required). The fields are given by Eq. 8.46 of Jackson,

TE_{10}:

$$\begin{aligned}
H_z &= H_0 \cos\left(\frac{\pi x}{a}\right) \exp(\mathrm{i}kz - \mathrm{i}\omega t) \\
H_x &= -\frac{\mathrm{i}ka}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) \exp(\mathrm{i}kz - \mathrm{i}\omega t) \\
E_y &= \frac{\mathrm{i}\omega a\mu}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) \exp(\mathrm{i}kz - \mathrm{i}\omega t)
\end{aligned}$$

TE_{01}:

$$\begin{aligned}
H_z &= H_0 \cos\left(\frac{\pi y}{a}\right) \exp(\mathrm{i}kz - \mathrm{i}\omega t) \\
H_y &= -\frac{\mathrm{i}ka}{\pi} H_0 \sin\left(\frac{\pi y}{a}\right) \exp(\mathrm{i}kz - \mathrm{i}\omega t) \\
E_x &= -\frac{\mathrm{i}\omega a\mu}{\pi} H_0 \sin\left(\frac{\pi y}{a}\right) \exp(\mathrm{i}kz - \mathrm{i}\omega t)
\end{aligned}$$

where the extra $-$ in the last line follows from the fact that $\mathbf{S} = \frac{1}{2}\mathbf{E}_t \times \mathbf{H}_t^*$ must point in the positive z-direction.

c): Pick, for instance, TE_{10}. It is

$$P = \frac{1}{2}\int_A \mathrm{Re}\left[\hat{\mathbf{z}} \cdot \mathbf{E}_t \times \mathbf{H}_t^*\right] da = -\frac{1}{2}\int_A \mathrm{Re}\left[-E_y H_x^*\right] da = \frac{a^2}{4}\frac{ka^2\omega\mu}{\pi^2}\left|H_0\right|^2$$

Since also the amplitude of the transverse electric field, $E_y = H_0\frac{\omega a\mu}{\pi}$, it is

$$P = E_0^2 \frac{a^2 k}{4\omega\mu}$$

We also use $k = \frac{1}{c}\sqrt{\omega^2 - \left(\frac{\pi c}{a}\right)^2}$, which gives $k = 49.61m^{-1}$. The numerical result for P is

$$P = 102.8W$$

Alternate method: Use Eq. 8.51 with $\psi = H_0 \cos\left(\frac{\pi x}{a}\right)$ with $H_0 = \frac{\pi E_0}{\omega a \mu} = 20.83\frac{A}{m}$.

d): We need to find the damping constant, which we can calculate, for instance, for the TE$_{10}$-mode:

$$\beta = \frac{1}{2P}\frac{1}{2\sigma\delta}\oint |\mathbf{H}|^2 dl = \frac{1}{2P}\frac{1}{2\sigma\delta}\oint \left(|H_z|^2 + |H_x|^2\right) dl$$

Along the $y-$sides, which have $x = 0$ or $x = a$, it is $H_x = 0$ and $H_z = H_0$, and thus

$$\int_{x=0\ or\ a} |H_z|^2 dl = 2a|H_0|^2$$

Along the $x-$sides, which have $y = 0$ or $y = a$, it is

$$\int_{y=0\ or\ a} \left(|H_z|^2 + |H_x|^2\right) dl = 2\int_{x=0}^{a} |H_0|^2 \left[\cos^2\left(\frac{\pi x}{a}\right) + \frac{k^2 a^2}{\pi^2}\sin^2\left(\frac{\pi x}{a}\right)\right] dx = a|H_0|^2 \left[1 + \frac{k^2 a^2}{\pi^2}\right]$$

and the sum over all sides,

$$\oint |\mathbf{H}|^2 dl = a|H_0|^2 \left[3 + \frac{k^2 a^2}{\pi^2}\right]$$

With the result from c), $P = \frac{a^2}{4}\frac{ka^2\omega\mu}{\pi^2}|H_0|^2$, we find

$$\beta = \sqrt{\frac{1}{2\sigma\mu\omega}}\frac{3\pi^2 + k^2 a^2}{ka^3} = 3.32 \times 10^{-3} m^{-1}$$

The injected power is the result of c) times $\exp(2\beta \times 1000m) = 765$,

$$P = 78.8kW$$

Alternate method to find β: use Eq. 8.63 with the information in the next two paragraphs.

e): Since there are two degenerate TE-modes, we use the result of Problem 8.13a. Since the modes will be normalized during the process, we can use just

$$\psi_{10} = \cos\left(\frac{\pi x}{a}\right) \quad \text{and} \quad \psi_{01} = \cos\left(\frac{\pi y}{a}\right)$$

The normalization integrals are then

$$N_{10} = N_{01} = \int_A |\psi_{any}|^2 da = \frac{a^2}{2}$$

The deformation is $\delta = a/16$ on the lower side for $a/2 < x < a$ and on the upper side for $0 < x < a/2$. Otherwise $\delta = 0$. Also, since we are dealing with TE-modes, all single-normal-derivative terms in the equation for Δ_{ij} are zero. On the lower side,

$$\psi_{10} = \cos\left(\frac{\pi x}{a}\right) \quad \text{and} \quad \psi_{01} = 1 \quad \text{and} \quad \frac{\partial^2 \psi_{10}}{\partial n^2} = \left.\frac{\partial^2 \psi_{10}}{\partial y^2}\right|_{y=0} = 0 \quad \text{and} \quad \frac{\partial^2 \psi_{01}}{\partial n^2} = \left.\frac{\partial^2 \psi_{01}}{\partial y^2}\right|_{y=0} = -\frac{\pi^2}{a^2}$$

On the upper side,

$$\psi_{10} = \cos\left(\frac{\pi x}{a}\right) \quad \text{and} \quad \psi_{01} = -1 \quad \text{and} \quad \frac{\partial^2 \psi_{10}}{\partial n^2} = \left.\frac{\partial^2 \psi_{10}}{\partial (-y)^2}\right|_{y=a} = 0 \quad \text{and} \quad \frac{\partial^2 \psi_{01}}{\partial n^2} = \left.\frac{\partial^2 \psi_{01}}{\partial (-y)^2}\right|_{y=a} = \frac{\pi^2}{a^2}$$

Thus,

$$\begin{aligned}
\Delta_{10,10} &= \oint \delta(x)\, \psi_{10}^* \frac{\partial^2 \psi_{10}^*}{\partial n^2} dl = 0 \\
\Delta_{01,10} &= \oint \delta(x)\, \psi_{01}^* \frac{\partial^2 \psi_{10}^*}{\partial n^2} dl = 0 \\
\Delta_{01,01} &= \oint \delta(x)\, \psi_{01}^* \frac{\partial^2 \psi_{01}^*}{\partial n^2} dl = 2 \times \int_{x=0}^{a/2} \frac{a}{16}\left(-\frac{\pi^2}{a^2}\right) dx = -\frac{\pi^2}{16} \\
\Delta_{10,01} &= \oint \delta(x)\, \psi_{10}^* \frac{\partial^2 \psi_{01}^*}{\partial n^2} dl = 2 \times \int_{x=0}^{a/2} \frac{a}{16}\frac{\pi^2}{a^2} \cos\left(\frac{\pi x}{a}\right) dx = \frac{\pi^2}{8a}\left[\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\right]_0^{a/2} = \frac{\pi}{8}
\end{aligned}$$

Thus, the equation to be solved is

$$\begin{pmatrix} x\frac{a^2}{2} & \frac{\pi}{8} \\ 0 & x\frac{a^2}{2} - \frac{\pi^2}{16} \end{pmatrix} \begin{pmatrix} a_{10} \\ a_{01} \end{pmatrix} = 0$$

where we use $x = \gamma^2 - \gamma_0^2$. The determinant is zero if $x = 0$ or $x = \frac{\pi^2}{8a^2}$. The perturbed solutions then are

<u>For $x = 0$</u>: $a_{10} = 1$ and $a_{01} = 0$. This is the original unperturbed TE_{10}-mode. Since $x = 0$, this mode also retains its unperturbed values of γ and k, and it is $k = k_0 = 49.61 m^{-1}$. The electric-field polarization is $\phi = \pi/2$ (i.e. polarized in y-direction).

<u>For $x = \frac{\pi^2}{8a^2}$</u>: Since $x = \gamma^2 - \gamma_0^2 = k_0^2 - k^2 = \frac{\pi^2}{8a^2}$, the perturbed k-value is

$$k = \sqrt{k_0^2 - \frac{\pi^2}{8a^2}} = 44.36 m^{-1}$$

Inserting $x = \frac{\pi^2}{8a^2}$ into the above equation for a_{10} and a_{01}, we find $a_{10} = -\frac{2}{\pi} a_{01}$ and an unnormalized perturbed solution $\psi = -\frac{2}{\pi}\psi_{10} + \psi_{01}$. Noting further that the unperturbed field modes differ by a minus in the electric-field components, we see that for an electric field of 1 in some unit in the x-direction, coming from the TE_{01}-part, the electric field in the y direction, coming from the TE_{10}-part, is $+\frac{2}{\pi}$. The polarization angle thus is $\phi = \arctan\left(\frac{2}{\pi}\right) = +32^o$.

2. Problem **20 Points** (8 on a, 6 on b, 6 on c)

a): Use Eqs. 9.167f with $\rho = 0$, $\mathbf{M} = 0$ and

$$\mathbf{J}(\mathbf{x}) = \frac{I_0}{a}\delta(r-a)\delta(\cos\theta)\hat{\phi}$$

No proof required, but to see that this is correct integrate over a plane of constant ϕ:

$$\int_{r=0}^{\infty} \hat{\phi}\cdot\mathbf{J}(\mathbf{x}) r d\theta dr = \int_{r=0}^{\infty} \frac{I_0 r}{a}\delta(r-a)\delta(\cos\theta) d\theta dr = I_0$$

Since in the present case $\mathbf{r}\cdot\mathbf{J}$, it is $\underline{a_E(l,m){=}0}$.

To determine $a_M(l,m)$, we require $\nabla\cdot(\mathbf{r}\times\mathbf{J})$:

$$\begin{aligned}
\mathbf{r}\times\mathbf{J} &= \frac{I_0}{a}\delta(r-a)\delta(\cos\theta) r(\hat{\mathbf{r}}\times\hat{\phi}) = -I_0\delta(r-a)\delta(\cos\theta)\hat{\theta} \\
\nabla\cdot(\mathbf{r}\times\mathbf{J}) &= -\frac{I_0}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\delta(r-a)\delta(\cos\theta)\right) \\
&= -\frac{I_0\delta(r-a)}{r\sin\theta}\left(\cos\theta\delta(\cos\theta) + \sin\theta\frac{\partial}{\partial\theta}\delta(\cos\theta)\right) \\
&= -\frac{I_0\delta(r-a)}{r\sin\theta}\left(0 + \sin\theta\left(\frac{d\cos\theta}{d\theta}\right)\frac{d}{d\cos\theta}\delta(\cos\theta)\right) \\
&= -\frac{I_0\delta(r-a)}{r}\delta(\cos\theta)\frac{d}{d\cos\theta}\sin\theta
\end{aligned}$$

This is to be read as a distribution, i.e. the derivative needs to be applied on all θ-dependent functions that will show up under the integral. Thus, following Eq. 9.168 of Jackson

$$\begin{aligned}
a_M(l,m) &= -\frac{I_0 k^2}{\mathrm{i}\sqrt{l(l+1)}}\int \frac{\delta(r-a)}{r} j_l(kr)\delta(\cos\theta)\frac{d}{d\cos\theta}\left(\sin\theta Y_{lm}^*(\theta,\phi)\right) r^2 d\cos\theta d\phi \\
&= \frac{\mathrm{i}I_0 k^2 a j_l(ka)}{\sqrt{l(l+1)}}\int \delta(\cos\theta)\frac{d}{d\cos\theta}\left(\sin\theta Y_{lm}^*(\theta,\phi)\right) d\cos\theta d\phi \\
&= \frac{\mathrm{i}I_0 k^2 a j_l(ka)}{\sqrt{l(l+1)}} 2\pi\delta_{m,0}\int \delta(\cos\theta)\frac{d}{d\cos\theta}\left(\sin\theta Y_{l0}^*(\theta,0)\right) d\cos\theta \\
&= \frac{\mathrm{i}I_0 k^2 a j_l(ka)}{\sqrt{l(l+1)}} 2\pi\delta_{m,0}\int \delta(\cos\theta)\frac{d}{d\cos\theta}\left(\sin\theta Y_{l0}^*(\theta,0)\right) d\cos\theta
\end{aligned}$$

Denoting $x = \cos\theta$, we have

$$\begin{aligned}
a_M(l,m) &= \frac{\mathrm{i}I_0 k^2 a j_l(ka)}{\sqrt{l(l+1)}} 2\pi\delta_{m,0}\sqrt{\frac{2l+1}{4\pi}}\int_{-1}^{1}\delta(x)\frac{d}{dx}\left(\sqrt{1-x^2}P_l(x)\right) dx \\
&= \frac{\mathrm{i}I_0 k^2 a j_l(ka)\sqrt{\pi(2l+1)}}{\sqrt{l(l+1)}}\delta_{m,0}\int_{-1}^{1}\delta(x)\left(\frac{d}{dx}P_l(x)\right) dx \\
&= \frac{\mathrm{i}I_0 k^2 a j_l(ka)\sqrt{\pi(2l+1)}}{\sqrt{l(l+1)}}\delta_{m,0}\left.\frac{d}{dx}P_l(x)\right|_{x=0} \qquad (1)
\end{aligned}$$

It is $a_M(l,m) \neq 0$ for $m = 0$ and l odd.

b): Copy Eqs. 9.149, 9.150 and 9.155 of Jackson for our case,

$$\begin{aligned}
\mathbf{H} &= \frac{\exp(\mathrm{i}kr - \mathrm{i}\omega t)}{kr} \sum_{l\,\mathrm{odd}} (-\mathrm{i})^{l+1} a_M(l,0)\hat{\mathbf{n}} \times \mathbf{X}_{l,0} \\
\mathbf{E} &= Z_0 \mathbf{H} \times \hat{\mathbf{n}} \\
\frac{dP}{d\Omega} &= \frac{Z_0}{2k^2} \left| \sum_{l\,\mathrm{odd}} (-\mathrm{i})^{l+1} a_M(l,0) \mathbf{X}_{l,0} \right|^2 \\
P &= \frac{Z_0}{2k^2} \sum_{l\,\mathrm{odd}} |a_M(l,0)|^2
\end{aligned}$$

Note $\hat{\mathbf{n}} = \hat{\mathbf{r}}$.

c): Use result of a) and Eq. 9.88, which says $j_1(ka) = \frac{ka}{3}$ for $ka \to 0$. Also, $\frac{d}{dx} P_1(x)\big|_{x=0} = 1$. Thus,

$$a_M(1,0) = \frac{\mathrm{i} I_0 k^3 a^2 \sqrt{\pi}}{\sqrt{6}}$$

and the power

$$P = \frac{Z_0}{2k^2} |a_M(1,0)|^2 = \frac{Z_0 I_0^2 k^4 a^4 \pi}{12}$$

Also, using Table 9.1 of Jackson, it is

$$\frac{dP}{d\Omega} = \frac{Z_0 I_0^2 k^4 a^4 \pi}{12} \frac{3}{8\pi} \sin^2\theta$$

The magnetic field

$$\begin{aligned}
\mathbf{H} &= \frac{\exp(\mathrm{i}kr - \mathrm{i}\omega t)}{kr} (-\mathrm{i})^2 a_M(1,0) \hat{\mathbf{n}} \times \mathbf{X}_{1,0} \\
&= -\frac{\exp(\mathrm{i}kr - \mathrm{i}\omega t)}{kr} \frac{\mathrm{i} I_0 k^3 a^2 \sqrt{\pi}}{\sqrt{6}} \hat{\mathbf{r}} \times \hat{\mathbf{L}} \frac{1}{\sqrt{2}} Y_{1,0} \\
&= -\frac{\exp(\mathrm{i}kr - \mathrm{i}\omega t)}{kr} \frac{\mathrm{i} I_0 k^3 a^2 \sqrt{\pi}}{\sqrt{12}} \hat{\mathbf{r}} \times \left[\frac{1}{\mathrm{i}} \left(\hat{\phi} \frac{\partial}{\partial\theta} - \hat{\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} \right) \right] \sqrt{\frac{3}{4\pi}} \cos\theta \\
&= \frac{\exp(\mathrm{i}kr - \mathrm{i}\omega t)}{kr} \frac{I_0 k^3 a^2}{4} (\hat{\mathbf{r}} \times \hat{\phi}) \sin\theta \\
&= -\hat{\theta} \frac{\exp(\mathrm{i}kr - \mathrm{i}\omega t)}{r} \frac{I_0 k^2 a^2}{4} \sin\theta
\end{aligned}$$

and the electric field

$$\mathbf{E} = Z_0 \mathbf{H} \times \hat{\mathbf{r}} = \hat{\phi} \frac{\exp(\mathrm{i}kr - \mathrm{i}\omega t)}{r} \frac{Z_0 I_0 k^2 a^2}{4} \sin\theta$$

Alternate method 1: Use Eqs. 9.171f of Jackson.

Alternate method 2: Use results of Chapter 9.3 of Jackson for a magnetic dipole $\mathbf{m} = I_0 a^2 \pi \hat{\mathbf{z}}$.

3. Problem **20 Points** (7 on a, 7 on b, 6 on c)

a): The spheres have $\mu = \mu_0$ and $\epsilon = \epsilon_r \epsilon_0$ with $\epsilon_r = n^2 = 2.25$. Since further the radius of the spheres $a = d/2 = 10nm \ll \lambda = 632nm$, this is a case of electric-dipole scattering. We can use Eq. 10.6 of Jackson,

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right)^2 |\epsilon^* \cdot \epsilon_0|^2$$

with incident polarization $\epsilon_0 = \hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Two orthonormal vectors of linear exit polarization are

$$\epsilon_1 = \hat{\theta} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \quad \text{and} \quad \epsilon_2 = \hat{\phi} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

For the given incident polarization, the differential scattering cross section summed over the exit polarizations is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right)^2 \left[|\epsilon_1^* \cdot \epsilon_0|^2 + |\epsilon_2^* \cdot \epsilon_0|^2 \right] \\ &= k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right)^2 \left[\cos^2\theta \cos^2\phi + \sin^2\phi \right] \\ &= 8.45 \times 10^{-22} \frac{m^2}{sterad} \left[\cos^2\theta \cos^2\phi + \sin^2\phi \right] \end{aligned}$$

The angular part can be written in various other forms, such as $\cos^2\theta\cos^2\phi + \sin^2\phi = \cos^2\theta + \sin^2\theta\sin^2\phi = 1 - \sin^2\theta\cos^2\phi$.

b): Use $\epsilon_0 = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - \mathrm{i}\mathbf{y}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\mathrm{i} \\ 0 \end{pmatrix}$. Then,

$$\begin{aligned} |\epsilon_1^* \cdot \epsilon_0|^2 &= \frac{1}{2}(\cos\theta\cos\phi - \mathrm{i}\cos\theta\sin\phi)(\cos\theta\cos\phi + \mathrm{i}\cos\theta\sin\phi) = \frac{1}{2}\cos^2\theta \\ |\epsilon_2^* \cdot \epsilon_0|^2 &= \frac{1}{2}(-\sin\phi - \mathrm{i}\cos\phi)(-\sin\phi + \mathrm{i}\cos\phi) = \frac{1}{2} \\ \frac{d\sigma}{d\Omega} &= k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right)^2 \left[|\epsilon_1^* \cdot \epsilon_0|^2 + |\epsilon_2^* \cdot \epsilon_0|^2 \right] = k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right)^2 \frac{1}{2} \left[1 + \cos^2\theta \right] \\ &= 8.45 \times 10^{-22} \frac{m^2}{sterad} \frac{1}{2} \left[1 + \cos^2\theta \right] \end{aligned}$$

Alternate method for a and b: With incident field $\mathbf{E}_0 = E_0 \underline{\epsilon}_0$, the induced dipole is

$$\mathbf{p} = 4\pi\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right) a^3 E_0 \underline{\epsilon}_0$$

In the far-field, it produces a scattered electric field

$$\mathbf{E}_{sc} = Z_0 \frac{ck^2}{4\pi} \frac{\exp(\mathrm{i}kr)}{r} [(\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}}] = Z_0 \epsilon_0 c a^3 E_0 k^2 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) \frac{\exp(\mathrm{i}kr)}{r} [(\hat{\mathbf{r}} \times \underline{\epsilon}_0) \times \hat{\mathbf{r}}]$$

The scattering cross section, summed over exit polarizations, is

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{r^2 \mathbf{E}_{sc}^* \cdot \mathbf{E}_{sc}}{\mathbf{E}_0^* \cdot \mathbf{E}_0} \\
&= k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 |(\hat{\mathbf{r}} \times \underline{\epsilon}_0) \times \hat{\mathbf{r}}|^2 \\
&= k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 |\hat{\mathbf{r}} \times \underline{\epsilon}_0|^2
\end{aligned}$$

Insert $\hat{\mathbf{r}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$ and the incident polarizations $\underline{\epsilon}_0$ to obtain the already given results.

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c): Since the laser beam illuminates only a small volume and the diode is quite far away, all scattering detected occurs at a practically constant scattering angle θ. Also, due to the disorder of the scatterers, the scattering is incoherent (i.e. scattering cross sections of multiple particles just add up).

Since the scattering is linear, we may assume an incident beam with constant intensity I_{in} over an area $A_{in} = P_{in}/I_{in}$. There, P_{in} is the incident power.

The number of illuminated particles is $N_P = A_{in} dn$, where d is the layer thickness and n the particle volume density.

The detector covers a solid angle $\Delta\Omega = \frac{\Delta A}{r^2}$, where ΔA is the detector area and r the detector distance.

The detected power ΔP then is

$$\begin{aligned}
\Delta P &= \frac{d\sigma}{d\Omega} \times (I_{in}\, N_P\, \Delta\Omega) \\
&= \frac{d\sigma}{d\Omega} \times \left(\frac{P_{in}}{A_{in}} A_{in}\, d\, n\, \frac{\Delta A}{r^2}\right) \\
&= \frac{d\sigma}{d\Omega} \times \left(P_{in}\, d\, n\, \frac{\Delta A}{r^2}\right)
\end{aligned}$$

Inserting the result of b) for $\frac{d\sigma}{d\Omega}$, and the given numbers ($P_{in} = 10W$, $d = 10^{-3}mm$, $n = 10^{17}m^{-3}$, $\Delta A = 10^{-4}m^{-2}$, and $r = 0.5m$), we find

$$\Delta P = 1.70 \times 10^{-10} W (1 + \cos^2\theta)$$