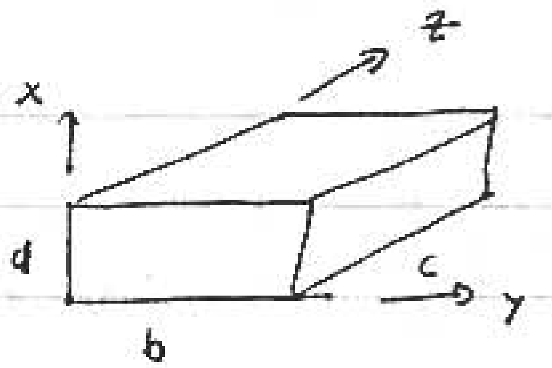
1a)



We can choose any side to be the z side. But we pick the above.

TM

$$E_z = \psi \cos\frac{p\pi z}{c}$$
$$\vec{E}_t = -\frac{p\pi}{c\gamma^2}\sin\frac{p\pi z}{d}\,\vec{\nabla}_t\psi$$
$$\vec{H}_t = \frac{i\epsilon\omega}{\gamma^2}\cos\frac{p\pi z}{d}\,\hat{z}\times\vec{\nabla}_t\psi$$

since $[\nabla_t^2+\gamma^2]\psi = 0 \qquad \psi|_s = 0$

$$\psi = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \gamma = \pi\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

$\rightarrow$

$$\begin{cases} E_z = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c} \\ E_x = -\frac{p\pi}{c\gamma^2}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{c} \\ E_y = -\frac{p\pi}{c\gamma^2}\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c} \end{cases}$$

$$\begin{cases} H_x = -\frac{i\epsilon\omega}{\gamma^2}\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{c} \\ H_y = \frac{i\epsilon\omega}{\gamma^2}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c} \end{cases}$$

$m,n \geq 1 \quad p \geq 0$

TE

$$H_z = \psi\sin\frac{p\pi z}{c}$$
$$\vec{E}_t = -\frac{i\omega\mu}{\gamma^2}\sin\frac{p\pi z}{c}\,\hat{z}\times\vec{\nabla}_t\psi$$
$$\vec{H}_t = \frac{p\pi}{c\gamma^2}\cos\frac{p\pi z}{c}\,\vec{\nabla}_t\psi$$

with $[\nabla_t^2+\gamma^2]\psi = 0 \qquad \frac{\partial\psi}{\partial n}\Big|_s = 0$

$$\rightarrow \qquad \psi = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

The fields are similar

$$\begin{cases} E_x = -\frac{i\omega\mu}{\gamma^2}\frac{n\pi}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{c} \\ E_y = \frac{i\omega\mu}{\gamma^2}\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c} \end{cases}$$

$$\begin{cases} H_z = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c} \\ H_x = -\frac{p\pi}{c\gamma^2}\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{c} \\ H_y = -\frac{p\pi}{c\gamma^2}\frac{n\pi}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c} \end{cases}$$

$m+n \geq 1$
$m, n \geq 0 \quad p \geq 1$

In both cases, the resonant frequency is

$$\boxed{\omega_{mnp}^2 = \frac{\pi^2}{\mu\epsilon}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2\right]}$$

Note that in all cases E_i has cos in the i^{th} coordinate
H_i has sin in the i^{th} coordinate
this ensures the boundary conditions $E_{\parallel} = H_{\perp} = 0$

By appropriate linear combinations, we can enforce
TE & TM in any plane (not just x-y)

The lowest modes are TM_{110} and TE_{101} TE_{011}

b) For Q we use the expressions in the textbook when appropriate

TM mode

Stored energy $\quad U = \frac{c}{4}\epsilon\left[1 + \left(\frac{p\pi}{\gamma c}\right)^2\right]\int_A |\psi|^2 da$

Power loss $\quad P_{loss} = \frac{\epsilon}{\sigma\delta\mu}\left[1 + \left(\frac{p\pi}{\gamma c}\right)^2\right]\left(1 + \xi\frac{Cc}{4A}\right)\int_A |\psi|^2 da$

where $A = ab$ = area $\quad C = 2(a+b)$ = circumference

$$\xi = \frac{\oint_C \frac{1}{\gamma^2}|\partial\psi/\partial n|^2 dl}{\frac{C}{A}\int_A |\psi|^2 da}$$

Since $\psi = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

$$\int_A |\psi|^2 da = \int_0^a dx\int_0^b dy\ \sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} = \frac{ab}{4} = \frac{A}{4}$$

each averages to $\frac{1}{2}$

$$\oint_C |\partial\psi/\partial n|^2 dl = 2\int_0^a dx\left(\frac{n\pi}{b}\right)^2 \sin^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}\Big|_{y=0}$$
$$+2\int_0^b dy\left(\frac{m\pi}{a}\right)^2\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}\Big|_{x=0}$$
$$= 2\left(\frac{n\pi}{b}\right)^2\int_0^a dx\ \sin^2\frac{m\pi x}{a}$$
$$+2\left(\frac{m\pi}{a}\right)^2\int_0^b dy\ \sin^2\frac{n\pi y}{b}$$
$$= a\left(\frac{n\pi}{b}\right)^2 + b\left(\frac{m\pi}{a}\right)^2$$

Hence

$$\xi = \frac{1}{\gamma^2}\ \frac{\pi^2\left[a\left(\frac{n}{b}\right)^2 + b\left(\frac{m}{a}\right)^2\right]}{2(a+b)/4} = \frac{2\left[a\left(\frac{n}{b}\right)^2 + b\left(\frac{m}{a}\right)^2\right]}{(a+b)\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]}$$

$$Q^{TM} = \frac{\omega U}{P_{loss}} = \frac{\omega c\,\sigma\delta\mu}{4}\left[1+\xi\frac{Cc}{4A}\right]^{-1}$$
$$= \frac{c}{2\delta}\frac{\mu}{\mu_c}\left[1+\frac{2\left[a\left(\frac{n}{b}\right)^2+b\left(\frac{m}{a}\right)^2\right]}{(a+b)\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]}\ \frac{c}{2}\frac{(a+b)}{ab}\right]^{-1}$$

$$\boxed{Q^{TM} = \frac{\mu}{\mu_c}\frac{c}{\delta}\ \frac{1}{2}\left[1+\frac{(c/a)(m/a)^2+(c/b)(n/b)^2}{(m/a)^2+(n/b)^2}\right]^{-1}}$$

$(c\to 2c \text{ if } p=0)$

<u>TE mode</u>

Stored energy $\quad U = \frac{c}{4}\mu\left[1+\left(\frac{p\pi}{\gamma c}\right)^2\right]\int_A|\psi|^2 da$

For the power loss

$$P_{loss} = \frac{1}{2\sigma\delta}\left[\oint_C dl\int_0^c dz\,|\hat{n}\times\vec{H}|^2_{sides} + 2\int_A da\,|\hat{n}\times\vec{H}|^2_{ends}\right]$$

we have to consider both terms

<u>ends</u> $\quad |\hat{n}\times\vec{H}|^2 = |\hat{z}\times\vec{H}|^2 = \left(\frac{p\pi}{c\gamma^2}\right)^2\cos^2\frac{p\pi z}{c}\,|\hat{z}\times\vec{\nabla}_t\psi|^2$

↳ 1 at ends

$$= \left(\frac{p\pi}{c\gamma^2}\right)^2|\vec{\nabla}_t\psi|^2$$

This may be integrated

$$\int_A da\, |\hat{n}\times\vec{H}|^2_{\text{ends}} = \left(\frac{p\pi}{c\gamma^2}\right)^2 \int_A |\vec{\nabla}_t \psi|^2 da$$
$$= \left(\frac{p\pi}{c\gamma}\right)^2 \int_A |\psi|^2 da$$

sides $\quad |\hat{n}\times\vec{H}|^2 = |\underbrace{\hat{n}\times\hat{z}}_{\text{in transverse dir}}\, \psi \sin\frac{p\pi z}{c} + \frac{p\pi}{c\gamma^2}\cos\frac{p\pi z}{c}\,\underbrace{\hat{n}\times\vec{\nabla}\psi}_{\text{in } \hat{z} \text{ dir}}|^2$

$$= \sin^2\frac{p\pi z}{c}|\psi|^2 + \left(\frac{p\pi}{c\gamma^2}\right)^2\cos^2\frac{p\pi z}{c}\,|\hat{n}\times\vec{\nabla}\psi|^2$$

integrating over the z direction gives

$$\int_0^c dz\, |\hat{n}\times\vec{H}|^2_{\text{sides}} = \frac{c}{2}\left[|\psi|^2 + \left(\frac{p\pi}{c\gamma^2}\right)^2|\hat{n}\times\vec{\nabla}\psi|^2\right]$$

We do the circumference integral by hand

$$\psi = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$
$$\oint_c dl\, |\psi|^2 = 2\int_0^a dx \cos^2\frac{m\pi x}{a} + 2\int_0^b dy\cos^2\frac{n\pi y}{b}$$
$$= a+b \qquad (a\to 2a \text{ for } m=0\text{, etc})$$
$$\oint_c dl\, |\hat{n}\times\vec{\nabla}\psi|^2 = 2\int_0^a dx\, |\partial_x\psi|^2_{y=0} + 2\int_0^b dy\, |\partial_y\psi|^2_{x=0}$$
$$= 2\left(\frac{m\pi}{a}\right)^2\int_0^a dx\sin^2\frac{m\pi x}{a} + 2\left(\frac{n\pi}{b}\right)^2\int_0^b dy\sin^2\frac{n\pi y}{b}$$
$$= a\left(\frac{m\pi}{a}\right)^2 + b\left(\frac{n\pi}{b}\right)^2$$

So

$$\oint_c dl\int_0^c dz\,|\hat{n}\times\vec{H}|^2_{\text{sides}} = \frac{c}{2}\left[(a+b) + \left(\frac{p\pi}{c\gamma^2}\right)^2\left(a\left(\frac{m\pi}{a}\right)^2 + b\left(\frac{n\pi}{b}\right)^2\right)\right]$$

Then

$$P_{loss} = \frac{1}{2\sigma\delta}\left[\frac{c}{2}(a+b) + \frac{c}{2}\left(\frac{p\pi}{c\gamma}\right)^2\frac{a(m/a)^2+b(n/b)^2}{(m/a)^2+(n/b)^2} + 2\left(\frac{p\pi}{c\gamma}\right)^2\frac{ab}{4}\right]$$

so

$$Q^{TE} = \frac{\omega U}{P_{loss}} = \frac{\omega c\mu\, 2\sigma\delta}{4}\,\frac{\left[1+\left(\frac{p\pi}{\gamma c}\right)^2\right]\frac{ab}{4}}{\frac{c}{2}\left[a+b+\left(\frac{p\pi}{c\gamma}\right)^2\left(\frac{a(m/a)^2+b(n/b)^2}{(m/a)^2+(n/b)^2}+\frac{ab}{c}\right)\right]}$$

$$\boxed{Q^{TE} = \frac{\mu}{\mu_c}\frac{c}{\delta}\;\frac{\frac{ab}{2c}\left[1+\left(\frac{p\pi}{\gamma c}\right)^2\right]}{\left[a+b+\left(\frac{p\pi}{c\gamma}\right)^2\left(\frac{a(m/a)^2+b(n/b)^2}{(m/a)^2+(n/b)^2}+\frac{ab}{c}\right)\right]}}$$

$a\to 2a$ if $m=0$ etc

2a) Current density $\vec{J} = \frac{I_0}{r}\delta(r-a)\,\delta(\cos\theta)\,\hat{\phi}$ in spherical coords

↳ so $\vec{J}\,d^3x = \frac{I_0}{r}\delta(r-a)\,\delta(\cos\theta)\,\hat{\phi}\; r^2 dr\, d\cos\theta\, d\phi$

$= I_0\,\hat{\phi}\,\underbrace{r\,d\phi}_{\text{length element along wire}}\Big|_{r=a,\cos\theta=0}$

This is a current and no magnetization. Hence

$$a_E(l,m) = \frac{k^2}{i\sqrt{l(l+1)}}\int Y_{lm}^*\left[c\rho\frac{\partial}{\partial r}[r j_l(kr)] + ik(\vec{r}\cdot\vec{J})\,j_l(kr)\right]d^3x$$

note $\rho = \frac{1}{i\omega}\vec{\nabla}\cdot\vec{J} = \frac{1}{i\omega}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}J_\phi = 0$

$\vec{r}\cdot\vec{J} = r J_r = 0$

Hence $\boxed{a_E(l,m) = 0}$

For the magnetic coefficient

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}}\int Y_{lm}^*\,\vec{\nabla}\cdot(\vec{r}\times\vec{J})\,j_l(kr)\,d^3x$$

$\vec{r}\times\vec{J} = r\,\hat{r}\times\vec{J} = -I_0\,\delta(r-a)\,\delta(\cos\theta)\,\hat{\theta}$

$\vec{\nabla}\cdot(\vec{r}\times\vec{J}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,[\vec{r}\times\vec{J}]_\theta$

$= \frac{I_0}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\delta(r-a)\,\delta(\cos\theta)$

the θ derivative acts on $\sin\theta$ and $\delta(\cos\theta)$

only the latter is non-vanishing

$\vec{\nabla}\cdot(\vec{r}\times\vec{J}) = \frac{I_0}{r\sin\theta}\sin\theta\,\delta(r-a)\,\delta'(\cos\theta)\,(-\sin\theta)$

$= -\frac{I_0}{r}\sin\theta\,\delta(r-a)\,\delta'(\cos\theta)$

$= -\frac{I_0}{r}\delta(r-a)\,\delta'(\cos\theta)$

Then

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}}\int Y_{lm}^*\,j_l(kr)\left(-\frac{I_0}{r}\right)\delta(r-a)\,\delta'(\cos\theta)\,r^2dr\,d\cos\theta\,d\phi$$

perform the simple r & ϕ integrals

↳ gives $m=0$ only

$$a_M(l,0) = \frac{-I_0 k^2}{i\sqrt{l(l+1)}}\, a\, j_l(ka)\;\; 2\pi \int_{-1}^{1} Y_{l0}^*(\theta)\, \delta'(\cos\theta)\, d\cos\theta$$

$$= \frac{-I_0 k^2}{i\sqrt{l(l+1)}}\; 2\pi a\, j_l(ka) \sqrt{\frac{2l+1}{4\pi}}\; \underbrace{\int_{-1}^{1} P_l(\cos\theta)\, \delta'(\cos\theta)\, d\cos\theta}$$

$$- P_l'(\cos\theta)\,\Big|_{\cos\theta = 0}$$

$$= -P_l'(0)$$

So

$$\boxed{a_M(l,0) = \frac{I_0 k^2 a\, j_l(ka)}{i} \sqrt{\frac{\pi(2l+1)}{l(l+1)}}\; P_l'(0)}$$

b) The dominant mode is magnetic dipole

$$a_M(1,0) = -iI_0 k^2 a\, j_1(ka) \sqrt{\frac{3\pi}{2}}$$

$$(\, P_1'(0) = 1 \,)$$

for $ka \ll 1$ $\quad j_1(ka) \sim \frac{1}{3}(ka)$

hence

$$\boxed{a_M(1,0) \simeq -i I_0 k^3 a^2 \sqrt{\frac{\pi}{6}}}$$

The total radiated power is

$$P = \frac{Z_0}{2k^2} \sum \left[|a_E|^2 + |a_M|^2 \right]$$

$$= \frac{Z_0}{2k^2} |I_0|^2 k^6 a^2 \frac{\pi}{6} \quad = \boxed{\frac{Z_0 k^4 a^2 \pi}{12} |I_0|^2}$$

3a) Uniform dielectric sphere $\epsilon_r \simeq 1$ radius a

In the Born approximation

$$\frac{d\sigma}{d\Omega} = |\hat{\epsilon}^* \cdot \vec{f}\,|^2$$

where $\hat{\epsilon}^* \cdot \vec{f} = \frac{k^2}{4\pi} \int d^3x \; e^{i\vec{q}\cdot\vec{x}} \left[(\hat{\epsilon}^* \cdot \hat{\epsilon}_0) \frac{\delta\epsilon}{\epsilon_0} + (\hat{n}\times\hat{\epsilon}^*)\cdot(\hat{n}_0 \times \hat{\epsilon}_0) \frac{\delta\mu}{\mu_0} \right]$

$$\vec{q} = \vec{k}_0 - \vec{k}$$

$$\frac{\delta\epsilon}{\epsilon_0} = \frac{\epsilon - \epsilon_0}{\epsilon_0} = \epsilon_r - 1 \qquad \frac{\delta\mu}{\mu_0} = 0 \quad \text{(non permeable)}$$

so

$$\hat{\epsilon}^* \cdot \vec{f} = \frac{k^2}{4\pi} (\epsilon_r - 1)(\hat{\epsilon}^* \cdot \hat{\epsilon}_0) \underbrace{\int d^3x \; e^{i\vec{q}\cdot\vec{x}}}_{\text{Fourier transform (just over sphere)}}$$

in spherical coords, lined up as $\vec{q}\cdot\vec{x} = qr\cos\theta$

$$\int d^3x \; e^{i\vec{q}\cdot\vec{x}} = \int_0^a r^2 dr \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \; e^{iqr\cos\theta}$$

$$= 2\pi \int_0^a r^2 dr \frac{1}{iqr}\left[e^{iqr} - e^{-iqr}\right]$$

$$= \frac{2\pi}{iq} \int_0^a dr \; r\left[e^{iqr} - e^{-iqr}\right] = \frac{4\pi}{q}\int_0^a dr \; r\sin qr$$

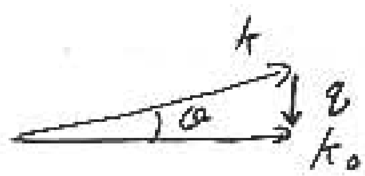
This can be integrated by parts

$$\rightarrow \int d^3x \; e^{i\vec{q}\cdot\vec{x}} = \frac{4\pi a^3}{(qa)^3}\left[\sin qa - qa\cos qa\right]$$

So $\hat{\epsilon}^* \cdot \vec{f} = k^2 a^3 (\epsilon_r - 1)(\hat{\epsilon}^* \cdot \hat{\epsilon}_0) \dfrac{\sin qa - qa\cos qa}{(qa)^3}$

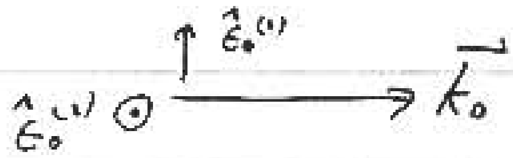
and

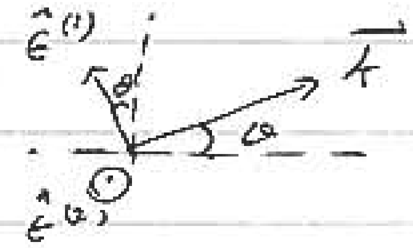
$$\frac{d\sigma}{d\Omega} = k^4 a^6 |\epsilon_r - 1|^2 (\hat{\epsilon}^* \cdot \hat{\epsilon}_0)^2 \left(\frac{\sin qa - qa\cos qa}{(qa)^3}\right)^2$$

Note that  $q = |\vec{k}_0 - \vec{k}| = 2k\sin\frac{\theta}{2}$

For the unpolarized cross section, we average over $\hat{\epsilon}_0$ and sum over $\hat{\epsilon}$

A convenient basis





Then

$$\sum (\hat{\epsilon}^* \cdot \hat{\epsilon}_0)^2 = \cos^2\theta + 0 + 0 + 1 = 1 + \cos^2\theta$$

averaging over initial polarizations gives a factor of $\frac{1}{2}$

So

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2} k^4 a^6 |\epsilon_r - 1|^2 (1+\cos^2\theta) \left(\frac{\sin qa - qa\cos qa}{(qa)^3}\right)^2}$$

$$qa = 2ka \sin\frac{\theta}{2}$$

b) For $ka \ll 1$ we expand the trig function

$$\frac{\sin x - x\cos x}{x^3} = \frac{(x - \frac{1}{6}x^3 + \cdots) - x(1 - \frac{1}{2}x^2 + \cdots)}{x^3} = \frac{1}{3} + \cdots$$

this part is isotropic

then
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{18} k^4 a^6 |\epsilon_r - 1|^2 (1 + \cos^2\theta)}$$

Note that scalar scattering is isotropic in the $ka \ll 1$ limit. The angular distribution here is purely a polarization vector issue

c) For $ka \gg 1$ note that $qa = 2ka \sin\frac{\theta}{2} \gg 1$ unless $\theta \simeq 0$

When $ka \gg 1$ the trig function can be approximated

$$\frac{\sin x - x\cos x}{x^3} \simeq -\frac{\cos x}{x^2} \rightarrow \text{bounded by } \frac{1}{x^2}$$

So
$$\left(\frac{\sin qa - qa\cos qa}{(qa)^3}\right)^2 \sim \frac{1}{2(qa)^4} = \frac{1}{32(ka)^4} \frac{1}{\sin^4(\frac{\theta}{2})}$$

average of $\cos^2 qa$

This gives

$$\frac{d\sigma}{d\Omega} \simeq \frac{a^2}{64} |\epsilon_r - 1|^2 (1+\cos^2\theta) \frac{1}{\sin^4(\frac{\theta}{2})}$$

<u>unless</u> $\boxed{|\theta| \lesssim \frac{1}{ka}}$ in which case the trig function becomes of order one

ie the cross section in this forward region gets considerably larger

Since $ka \gg 1$, the restriction $|\theta| \lesssim \frac{1}{ka}$ is a highly peaked cross section behavior.