Physics 506 **Winter 2006**

Homework Assignment #12 — Solutions

Textbook problems: Ch. 14: 14.2, 14.4, 14.6, 14.12

14.2 A particle of charge e is moving in nearly uniform nonrelativistic motion. For times near $t = t_0$, its vectorial position can be expanded in a Taylor series with fixed vector coefficients multiplying powers of $(t - t_0)$.

a) Show that, in an inertial frame where the particle is instantaneously at rest at the origin but has a small acceleration $\vec{a}$, the Liénard-Wiechert electric field, correct to order $1/c^2$ inclusive, at that instant is $\vec{E} = \vec{E}_v + \vec{E}_a$, where the velocity and acceleration fields are

$$\vec{E}_v = e\frac{\hat{r}}{r^2} + \frac{e}{2c^2 r}[\vec{a} - 3\hat{r}(\hat{r}\cdot\vec{a}\,)]; \qquad \vec{E}_a = -\frac{e}{c^2 r}[\vec{a} - \hat{r}(\hat{r}\cdot\vec{a}\,)]$$

and that the total electric field to this order is

$$\vec{E} = e\frac{\hat{r}}{r^2} - \frac{e}{2c^2 r}[\vec{a} + \hat{r}(\hat{r}\cdot\vec{a}\,)]$$

The unit vector $\hat{r}$ points from the origin to the observation point and r is the magnitude of the distance. Comment on the r dependences of the velocity and acceleration fields. Where is the expansion likely to be valid?

Expanding the position around a time t_0 gives

$$\vec{r}(t') = \vec{r} + \vec{v}(t' - t_0) + \tfrac{1}{2}\vec{a}(t' - t_0)^2 + \cdots$$

However, we work in the instantaneous rest frame with the particle at the origin. Hence it is sufficient to consider

$$\vec{r}(t') = \tfrac{1}{2}\vec{a}(t' - t_0)^2 + \cdots, \qquad \vec{\beta}(t') = \tfrac{1}{c}\vec{a}(t' - t_0) + \cdots, \qquad \dot{\vec{\beta}}(t') = \tfrac{1}{c}\vec{a} + \cdots$$

To proceed, we would like to develop a relation between observer time t and retarded time t'. The exact expression is of course $t = t' + |\vec{x} - \vec{r}(t')|/c$. However, since we wish to expand at time $t' \approx t_0$, it is sufficient to write $t = t' + x/c + \cdots$ where $x = |\vec{x}\,|$. The omitted terms turn out to be of higher order in $1/c^2$. We now write down the electric field at observer time $t = t_0$. This corresponds to a retarded time $t' = t_0 - x/c$. As a result, the various expressions showing up in the velocity and acceleration fields are given (up to order $1/c^2$) by

$$\vec{r} = \frac{x^2}{2c^2}\vec{a}, \qquad \vec{\beta} = -\frac{x}{c^2}\vec{a}, \qquad \dot{\vec{\beta}} = \frac{1}{c}\vec{a}$$

as well as

$$\vec{R} = \vec{x} - \vec{r} = \vec{x} - \frac{x^2}{2c^2}\vec{a} \qquad \Rightarrow \qquad R = x(1 - \frac{x}{2c^2}\hat{x}\cdot\vec{a}\,),\ \hat{n} \equiv \frac{\vec{R}}{R} = \hat{x} - \frac{x}{2c^2}[\vec{a} - \hat{x}(\hat{x}\cdot\vec{a}\,)] \tag{1}$$

We also note that $1/\gamma^2 = 1 - \beta^2 = 1 + \mathcal{O}(1/c^4) = 1 + \cdots$ to the order of interest. This yields the fields

$$\begin{aligned}
\vec{E}_v(\vec{x}, t_0) = e\frac{\hat{n} - \vec{\beta}}{\gamma^2 R^2 (1 - \vec{\beta}\cdot\hat{n})^3} &= e\frac{\hat{x} - \frac{x}{2c^2}[\vec{a} - \hat{x}(\hat{x}\cdot\vec{a}\,)] + \frac{x}{c^2}\vec{a}}{x^2(1 - \frac{x}{2c^2}\hat{x}\cdot\vec{a}\,)^2(1 + \frac{x}{c^2}\hat{x}\cdot\vec{a}\,)^3} \\
&= e\frac{\hat{x} + \frac{x}{2c^2}([\vec{a} + \hat{x}(\hat{x}\cdot\vec{a}\,)]}{x^2(1 + \frac{2x}{c^2}(\hat{x}\cdot\vec{a}\,))} \\
&= \frac{e\hat{x}}{x^2} + \frac{e}{2c^2 x}[\vec{a} - 3\hat{x}(\hat{x}\cdot\vec{a}\,)]
\end{aligned} \tag{2}$$

which agrees with the desired result (although we have used x and $\hat{x}$ instead of r and $\hat{r}$). The result for the acceleration field is even more straightforward, as the leading term is already of order $1/c^2$

$$\begin{aligned}
\vec{E}_a(\vec{x}, t_0) = \frac{e}{c}\frac{\hat{n}\times[(\hat{n} - \vec{\beta})\times\dot{\vec{\beta}}]}{R(1 - \vec{\beta}\cdot\hat{n})^3} = \frac{e}{c}\frac{\hat{x}\times(\hat{x}\times\frac{1}{c}\vec{a}\,)}{x} &= \frac{e}{c^2}\frac{\hat{x}\times(\hat{x}\times\vec{a}\,)}{x} \\
&= -\frac{e}{c^2 x}[\vec{a} - \hat{x}(\hat{x}\cdot\vec{a}\,)]
\end{aligned} \tag{3}$$

Adding (2) and (3) gives

$$\vec{E} = e\frac{\hat{x}}{x^2} - \frac{e}{2c^2 x}[\vec{a} + \hat{x}(\hat{x}\cdot\vec{a}\,)]$$

Note that the velocity field contains the static Coulomb term $e\hat{x}/x^2$ along with an acceleration term, which is perhaps unusual for a 'velocity' field. The latter only falls of as $1/x$ for large x, which is also surprising, as the velocity field ordinarily is thought of as a $1/R^2$ field. The acceleration field is as expected, however, as it depends on acceleration and exhibits the proper $1/R$ behavior. The resolution to this apparent discrepancy is the fact that our expansion is only valid for 'small' values of x, namely $x \ll c^2/a$, where the retarded time approximation is valid (corresponding to the $1/c^2$ term in (1) being small compared to the leading term). Roughly this is similar to being in the near zone (and not the radiation zone).

b) What is the result for the instantaneous magnetic induction $\vec{B}$ to the same order? Comment.

The magnetic induction is given by

$$\begin{aligned}
\vec{B} = \hat{n}\times\vec{E} &= \left(\vec{x} - \frac{x}{2c^2}[\vec{a} - \hat{x}(\hat{x}\cdot\vec{a}\,)]\right)\times\left(\frac{e\hat{x}}{x^2} - \frac{e}{2c^2 x}[\vec{a} + \hat{x}(\hat{x}\cdot\vec{a}\,)]\right) \\
&= -\frac{e}{2c^2 x}(\vec{a}\times\hat{x} + \hat{x}\times\vec{a}\,) = 0
\end{aligned}$$

In other words, the instantaneous $\vec{B}$ vanishes (to this level of approximation). This should not be surprising, because the particle is instantaneously at rest (and a static particle does not generate a magnetic field).

c) Show that the $1/c^2$ term in the electric field has zero divergence and that the curl of the electric field is $\vec{\nabla}\times\vec{E} = e(\hat{r}\times\vec{a})/c^2r^2$. From Faraday's law, find the magnetic induction $\vec{B}$ at times near $t=0$. Compare with the familiar elementary expression.

We compute the divergence as follows

$$\begin{aligned}\vec{\nabla}\cdot\left(\frac{\vec{a}+\hat{x}(\hat{x}\cdot\vec{a})}{x}\right) &= \vec{\nabla}\cdot\left(\frac{\vec{a}}{x}\right)+\vec{\nabla}\cdot\left(\frac{\vec{x}(\vec{x}\cdot\vec{a})}{x^3}\right)\\ &= \vec{\nabla}\left(\frac{1}{x}\right)\cdot\vec{a}+\vec{\nabla}\left(\frac{1}{x^3}\right)\cdot\vec{x}(\vec{x}\cdot\vec{a})+\frac{1}{x^3}\vec{\nabla}\cdot\big(\vec{x}(\vec{x}\cdot\vec{a})\big)\\ &= -\frac{1}{x^2}\hat{x}\cdot\vec{a}-\frac{3}{x^2}\hat{x}\cdot\vec{a}+\frac{4}{x^2}\hat{x}\cdot\vec{a}=0\end{aligned}$$

This demonstrates that the $1/c^2$ term has zero divergence. For the curl, we obtain

$$\begin{aligned}\vec{\nabla}\times\vec{E} &= \vec{\nabla}\times\left(\frac{e\vec{x}}{x^3}\right)-\frac{e}{2c^2}\vec{\nabla}\times\left(\frac{\vec{a}}{x}+\frac{\vec{x}(\vec{x}\cdot\vec{a})}{x^3}\right)\\ &= -\frac{e}{2c^2}\left(\vec{\nabla}\left(\frac{1}{x}\right)\times\vec{a}+\frac{1}{x^2}\vec{\nabla}(\vec{x}\cdot\vec{a})\times\hat{x}\right)\\ &= -\frac{e}{2c^2x^2}\big(-\hat{x}\times\vec{a}+\vec{a}\times\hat{x}\big)=\frac{e}{c^2x^2}\hat{x}\times\vec{a}\end{aligned}$$

Faraday's law states $\vec{\nabla}\times\vec{E}+(1/c)\partial\vec{B}/\partial t=0$. Hence

$$\frac{\partial\vec{B}}{\partial t}=-c\vec{\nabla}\times\vec{E}=-\frac{e}{c}\frac{\hat{x}\times\vec{a}}{x^2}$$

Integrating this for times near t_0 gives

$$\vec{B}=-\frac{e}{c}\frac{\hat{x}\times[\vec{a}(t-t_0)]}{x^2}=-\frac{e}{c}\frac{\hat{x}\times\vec{v}(t)}{x^2}=\frac{e}{c}\frac{\vec{v}(t)\times\hat{x}}{x^2}$$

This reproduces the elementary Biot-Savart law for the magnetic field.

14.4 Using the Liénard-Wiechert fields, discuss the time-averaged power radiated per unit solid angle in nonrelativisic motion of a particle with charge e, moving

a) along the z axis with instantaneous position $z(t)=a\cos\omega_0 t$.

In the non-relativisitic limit, the radiated power is given by

$$\frac{dP(t)}{d\Omega}=\frac{e^2}{4\pi c}|\hat{n}\times\dot{\vec{\beta}}|^2 \tag{4}$$

In the case of harmonic motion along the z axis, we take

$$\vec{r} = \hat{z} a \cos \omega_0 t, \qquad \vec{\beta} = -\hat{z}\frac{a\omega_0}{c}\sin\omega_0 t, \qquad \dot{\vec{\beta}} = -\hat{z}\frac{a\omega_0^2}{c}\cos\omega_0 t$$

By symmetry, we assume the observer is in the x-z plane tilted with angle θ from the vertical. In other words, we take

$$\hat{n} = \hat{x}\sin\theta + \hat{z}\cos\theta$$

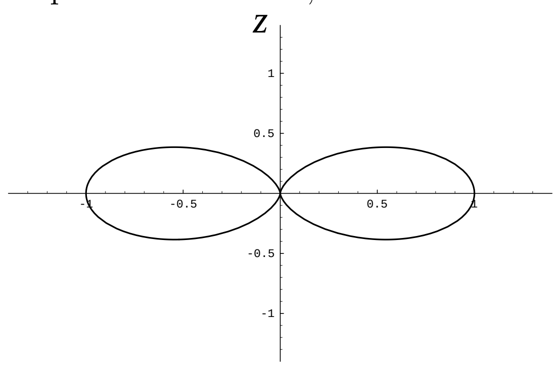
This provides enough information to simply substitute into the power expression (4)

$$\hat{n} \times \dot{\vec{\beta}} = \hat{y}\frac{a\omega_0^2}{c}\sin\theta\cos\omega_0 t \qquad \Rightarrow \qquad \frac{dP(t)}{d\Omega} = \frac{e^2a^2\omega_0^4}{4\pi c^3}\sin^2\theta\cos^2\omega_0 t$$

Taking a time average ($\cos^2\omega_0 t \to \frac{1}{2}$) gives

$$\frac{dP}{d\Omega} = \frac{e^2a^2\omega_0^4}{8\pi c^3}\sin^2\theta$$

This is a familiar dipole power distribution, which looks like



Integrating over angles gives the total power

$$P = \frac{e^2a^2\omega_0^4}{3c^3}$$

b) in a circle of radius R in the x-y plane with constant angular frequency ω_0.

Sketch the angular distribution of the radiation and determine the total power radiated in each case.

Here we take instead

$$\vec{r} = R(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t) \qquad \rightarrow \qquad \vec{\beta} = \frac{R\omega_0}{c}(-\hat{x}\sin\omega_0 t + \hat{y}\cos\omega_0 t)$$

$$\dot{\vec{\beta}} = -\frac{R\omega_0^2}{c}(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t)$$

Then

$$\hat{n} \times \dot{\vec{\beta}} = -\frac{R\omega_0^2}{c}[\hat{y}\cos\theta\cos\omega_0 t + (\hat{z}\sin\theta - \hat{x}\cos\theta)\sin\omega_0 t]$$

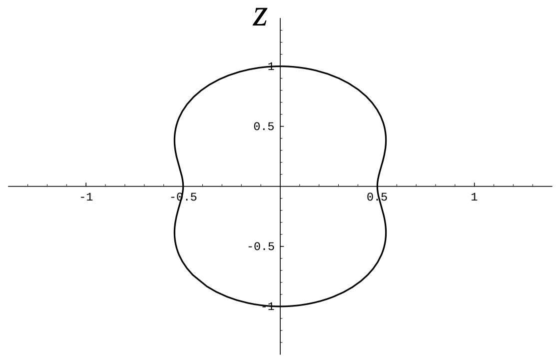
which gives

$$\frac{dP(t)}{d\Omega} = \frac{e^2R^2\omega_0^4}{4\pi c^3}(\cos^2\theta\cos^2\omega_0 t + \sin^2\omega_0 t)$$

Taking a time average gives

$$\frac{dP}{d\Omega} = \frac{e^2R^2\omega_0^4}{8\pi c^3}(1+\cos^2\theta)$$

This distribution looks like



The total power is given by integration over angles. The result is

$$P = \frac{2e^2R^2\omega_0^4}{3c^3}$$

14.6 *a*) Generalize the circumstances of the collision of Problem 14.5 to nonzero angular momentum (impact parameter) and show that the total energy radiated is given by

$$\Delta W = \frac{4z^2e^2}{3m^2c^3}\left(\frac{m}{2}\right)^{1/2}\int_{r_{\min}}^{\infty}\left(\frac{dV}{dr}\right)^2\left(E - V(r) - \frac{L^2}{2mr^2}\right)^{-1/2} dr$$

where $r_{\min}$ is the closest distance of approach (root of $E - V - L^2/2mr^2$), $L = mbv_0$, where b is the impact parameter, and v_0 is the incident speed ($E = mv_0^2/2$).

In the non-relativistic limit, we may use Lamour's formula written in terms of $\dot{\vec{p}}$

$$P(t) = \frac{2z^2e^2}{3m^2c^3}\left|\frac{d\vec{p}}{dt}\right|^2 = \frac{2z^2e^2}{3m^2c^3}\left(\frac{dV(r)}{dr}\right)^2 \tag{5}$$

where we noted that the central potential gives a force $d\vec{p}/dt = \vec{F} = -\hat{r}dV/dr$. The radiated energy is given by integrating power over time

$$\Delta W = \int_{-\infty}^{\infty} P(t)\,dt$$

However, this can be converted to an integral over the trajectory. By symmetry, we double the value of the integral from closed approach to infinity

$$\Delta W = 2\int_{r_{\min}}^{\infty} \frac{P}{dr/dt} dr \tag{6}$$

The radial velocity dr/dt can be obtained from energy conservation

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) \qquad \Rightarrow \qquad \frac{dr}{dt} = \sqrt{\frac{2}{m}}\left(E - V(r) - \frac{L^2}{2mr^2}\right)^{1/2}$$

Substituting $P(t)$ from (5) as well as dr/dt into (6) then yields

$$\Delta W = \frac{4z^2e^2}{3m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\left(\frac{dV}{dr}\right)^2\left(E - V(r) - \frac{L^2}{2mr^2}\right)^{-1/2} dr \tag{7}$$

b) Specialize to a repulsive Coulomb potential $V(r) = zZe^2/r$. Show that ΔW can be written in terms of impact parameter as

$$\Delta E = \frac{2zmv_0^5}{Zc^3}\left[-t^{-4} + t^{-5}\left(1 + \frac{t^2}{3}\right)\tan^{-1} t\right]$$

where $t = bmv_0^2/zZe^2$ is the ratio of twice the impact parameter to the distance of closest approach in a head-on collision.

Substituting

$$V(r) = \frac{zZe^2}{r}, \qquad L = mbv_0, \qquad E = \frac{1}{2}mv_0^2$$

into (7) gives

$$\begin{aligned}
\Delta W &= \frac{4z^4Z^2e^6}{3m^2c^3v_0}\int_{r_{\min}}^{\infty} r^{-4}\left(1 - 2\frac{zZe^2}{mv_0^2 r} - \frac{b^2}{r^2}\right)^{-1/2} dr \\
&= \frac{4zmv_0^5}{3Zc^3t^3}\int_{r_{\min}}^{\infty}\left(\frac{b}{r}\right)^2\left(1 - 2\frac{b}{tr} - \frac{b^2}{r^2}\right)^{-1/2}\frac{b\,dr}{r^2} \\
&= \frac{4zmv_0^5}{3Zc^3t^3}\int_0^{x_{\max}} \frac{x^2}{\sqrt{1 - 2(x/t) - x^2}}dx \\
&= \frac{4zmv_0^5}{3Zc^3t^3}\int_0^{x_+} \frac{x^2}{\sqrt{(x - x_-)(x_+ - x)}}dx
\end{aligned} \tag{8}$$

where we used $t = bmv_0^2/zZe^2$, and the variable substitution $x = b/r$. In the last line x_+ and x_- are the roots of the quadratic equation

$$x_\pm = -\frac{1}{t} \pm \sqrt{\frac{1}{t^2} + 1}$$

The x integral can be performed by Euler substitution. We use the indefinite integral

$$\int \frac{x^2}{\sqrt{(x-x_-)(x_+-x)}}dx = -\frac{1}{4}\sqrt{(x-x_-)(x_+-x)}\big(2x+3(x_++x_-)\big) + \frac{1}{4}\big(3(x_++x_-)^2-4x_+x_-\big)\tan^{-1}\sqrt{\frac{x-x_-}{x_+-x}}$$

Putting in limits gives

$$\begin{aligned}\int_0^{x_+} \frac{x^2}{\sqrt{(x-x_-)(x_+-x)}}dx &= \tfrac{3}{4}\sqrt{-x_+x_-}(x_++x_-) \\ &\quad + \tfrac{1}{4}\big(3(x_++x_-)^2-4x_+x_-\big)\tan^{-1}\sqrt{\frac{x_+}{-x_-}} \\ &= -\frac{3}{2t}+\left(\frac{3}{t^2}+1\right)\tan^{-1}\left(-\frac{1}{t}+\sqrt{\frac{1}{t^2}+1}\right)\end{aligned}$$

The arctan term can be simplified by double angle relations to give

$$\int_0^{x_+} \frac{x^2}{\sqrt{(x-x_-)(x_+-x)}}dx = -\frac{3}{2t}+\frac{1}{2}\left(\frac{3}{t^2}+1\right)\tan^{-1}t$$

Inserting this into (8) finally gives

$$\Delta W = \frac{2zmv_0^5}{Zc^3}\left(-\frac{1}{t^4}+\frac{1}{t^5}\left(1+\frac{t^2}{3}\right)\tan^{-1}t\right)$$

14.12 As in Problem 14.4a), a charge e moves in simple harmonic motion along the z axis, $z(t')=a\cos(\omega_0 t')$.

a) Show that the instantaneous power radiated per unit solid angle is

$$\frac{dP(t')}{d\Omega} = \frac{e^2c\beta^4}{4\pi a^2}\frac{\sin^2\theta\cos^2(\omega_0 t')}{(1+\beta\cos\theta\sin\omega_0 t')^5}$$

where $\beta = a\omega_0/c$.

For one-dimensional motion, the relativistic radiated power expression simplifies

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]|^2}{(1-\vec{\beta}\cdot\hat{n})^5} = \frac{e^2}{4\pi c}\frac{|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})|^2}{(1-\vec{\beta}\cdot\hat{n})^5} = \frac{e^2}{4\pi c}\frac{|\hat{n}\times\dot{\vec{\beta}}|^2}{(1-\vec{\beta}\cdot\hat{n})^5} \quad (9)$$

We use the same setup as Problem 14.2a), namely

$$\vec{r} = \hat{z}a\cos\omega_0 t', \qquad \vec{\beta} = -\hat{z}\frac{a\omega_0}{c}\sin\omega_0 t', \qquad \dot{\vec{\beta}} = -\hat{z}\frac{a\omega_0^2}{c}\cos\omega_0 t'$$

The observer is located at a point

$$\vec{x} = x(\hat{x}\sin\theta + \hat{z}\cos\theta)$$

which gives rise to

$$\vec{R} = \vec{x} - \vec{r} = \hat{x}x\sin\theta + \hat{z}(x\cos\theta - a\cos\omega_0 t')$$

or

$$R = x\left(1 - \frac{2a}{x}\cos\theta\cos\omega_0 t' + \frac{a^2}{x^2}\cos^2\omega_0 t'\right)^{1/2}, \qquad \hat{n} = \frac{\vec{R}}{R}$$

This rather complicated expression actually simplifies in the radiation zone ($x \to \infty$), which is the only region we are interested in. In this case, $R = x$ and $\hat{n} = \vec{x}/x = \hat{x}\sin\theta + \hat{z}\cos\theta$. Noting that

$$1 - \vec{\beta}\cdot\hat{n} = 1 + \frac{a\omega_0}{c}\cos\theta\sin\omega_0 t'$$

we simply evaluate (9) to obtain

$$\frac{dP(t')}{d\Omega} = \frac{e^2 a^2 \omega_0^4}{4\pi c^3}\frac{\sin^2\theta\cos^2\omega_0 t'}{(1 + \frac{a\omega_0}{c}\cos\theta\sin\omega_0 t')^5}$$

Making the substitution $\beta = a\omega_0/c$ then results in

$$\frac{dP(t')}{d\Omega} = \frac{e^2 c\beta^4}{4\pi a^2}\frac{\sin^2\theta\cos^2\omega_0 t'}{(1 + \beta\cos\theta\sin\omega_0 t')^5} \tag{10}$$

b) By performing a time averaging, show that the average power per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{e^2 c\beta^4}{32\pi a^2}\left[\frac{4 + \beta^2\cos^2\theta}{(1 - \beta^2\cos^2\theta)^{7/2}}\right]\sin^2\theta$$

To time average (10), we need to perform the integral

$$I(a) = \int_0^{2\pi}\frac{\cos^2\alpha}{(1 + a\sin\alpha)^5}d\alpha$$

This may be performed by complex variables techniques. Defining $z = e^{i\alpha}$ converts this to a contour integral

$$I(a) = \frac{8}{a^5}\oint_{|z|=1}\frac{z^2(1+z^2)^2}{(z^2 + 2iz/a - 1)^5}dz = \frac{8}{a^5}\oint_{|z|=1}\frac{z^2(1+z^2)^2}{(z - z_-)^5(z - z_+)^5}dz$$

where z_+ and z_- are the roots

$$z_\pm = -\frac{i}{a} \pm i\sqrt{\frac{1}{a^2} - 1}$$

It is easy to see that only z_+ lies inside the unit circle, provided $0 < a < 1$. (Since $I(-a) = I(a)$, we can extend the result to $|a| < 1$.) As a result, the value of $I(a)$ comes from the residue at z_+

$$I(a) = \frac{16\pi i}{a^5} \frac{1}{4!} \frac{d^4}{dz^4} \left(\frac{z^2(1+z^2)^2}{(z-z_-)^5} \right) \bigg|_{z=z_+}$$

Working out the derivatives gives the result

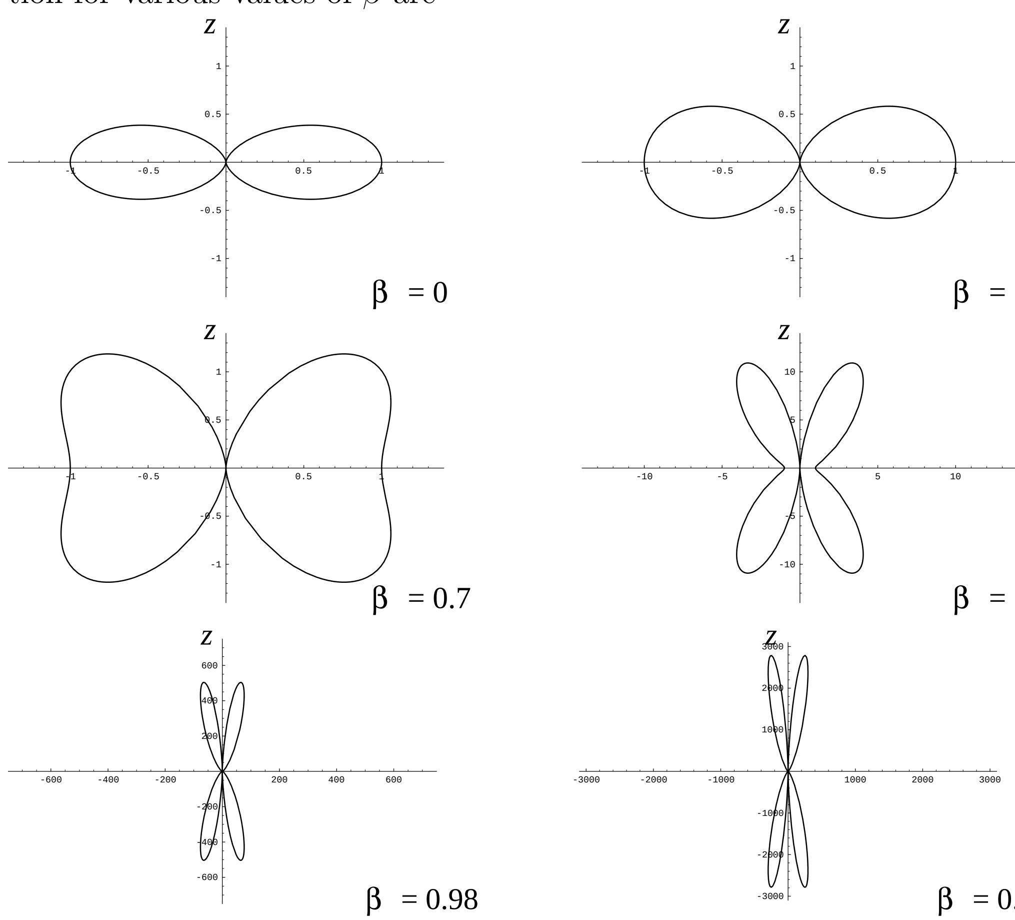
$$I(a) = \frac{\pi}{4} \frac{4+a^2}{(1-a^2)^{7/2}}$$

Using $a = \beta \cos\theta$ for time averaging (10), we find

$$\frac{dP}{d\Omega} = \frac{e^2 c \beta^4}{32\pi a^2} \frac{4+\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^{7/2}} \sin^2\theta$$

c) Make rough sketches of the angular distribution for nonrelativistic and relativistic motion.

The nonrelativistic limit yields ordinary dipole radiation. The angular distribution for various values of β are



The relativistic beaming effect (along the z axis) is clearly pronounced at large values of β.