Physics 506 **Winter 2006**

Homework Assignment #11 — Solutions

Textbook problems: Ch. 13: 13.1, 13.2, 13.3, 13.9

13.1 If the light particle (electron) in the Coulomb scattering of Section 13.1 is treated classically, scattering through an angle θ is correlated uniquely to an incident trajectory of impact parameter b according to

$$b = \frac{ze^2}{pv} \cot\frac{\theta}{2}$$

where $p = \gamma m v$ and the differential scattering cross section is $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$.

a) Express the invariant momentum transfer squared in terms of impact parameter and show that the energy transfer $T(b)$ is

$$T(b) = \frac{2z^2e^4}{mv^2}\frac{1}{b^2 + b_{\text{min}}^{(c)\,2}}$$

where $b_{\text{min}}^{(c)} = ze^2/pv$ and $T(0) = T_{\text{max}} = 2\gamma^2\beta^2 mc^2$.

The invariant momentum transfer squared is defined as $Q^2 = -(p-p')^2$ where p^μ and $p^{\mu\,\prime}$ are the initial and final 4-momenta of the electron. Expanding this out, and using $p^2 = p'^2 = m^2c^2$ gives

$$Q^2 = 2(p^\mu p'_\mu - m^2c^2) = 2(EE'/c^2 - m^2c^2 - \vec{p}\cdot\vec{p}\,') \tag{1}$$

Now consider the center of mass frame, where the heavy particle is essentially stationary and the electron undergoes scattering by an angle θ. Since this is an elastic scattering, we use conservation of energy to write $E = E' = \sqrt{m^2c^4 + |\vec{p}\,|^2c^2}$. In addition the scattering angle is related to 3-momentum transfer according to $\vec{p}\cdot\vec{p}\,' = |\vec{p}\,|^2\cos\theta$. Inserting this into (1) for the Q^2 invariant gives

$$Q^2 = 2|\vec{p}\,|^2(1-\cos\theta) = 4|\vec{p}\,|^2\sin^2\frac{\theta}{2} = 4p^2\sin^2\frac{\theta}{2}$$

where in the final expression we simply use p to denote the magnitude of the 3-momentum $\vec{p}$. Rewriting $\sin^2\theta/2$ in terms of $\cot^2\theta/2$ according to

$$\sin^2\tfrac{\theta}{2} = \frac{1}{1+\cot^2\frac{\theta}{2}}$$

and inserting the relation between b and θ given above results in

$$Q^2 = \left(\frac{2ze^2}{v}\right)^2 \frac{1}{b^2 + b_{\text{min}}^{(c)\,2}} \tag{2}$$

with $b_{\rm min}^{(c)} = ze^2/pv$.

We now examine the kinetic energy transfer in the lab frame. In this frame, the electron is initially at rest. Hence $E = mc^2$ and $\vec{p} = 0$. Inserting this into (1) gives

$$Q^2 = 2m(E' - mc^2) = 2mT$$

where $T \equiv E' - mc^2$ is the kinetic energy transfer. Finally, using this relation $Q^2 = 2mT$ in (2) gives

$$T = \frac{2z^2e^4}{mv^2}\frac{1}{b^2 + b_{\rm min}^{(c)\,2}} \tag{3}$$

b) Calculate the small transverse impulse Δp given to the (nearly stationary) light particle by the transverse electric field (11.152) of the heavy particle $q = ze$ as it passes by at large impact parameter b in a (nearly) straight line path at speed v. Find the energy transfer $T \approx (\Delta p)^2/2m$ in terms of b. Compare with the exact classical result of part *a*). Comment.

The transverse electric field of (11.152) is given by

$$E_\perp = \frac{q\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

We now calculate the impulse according to

$$\Delta p_\perp = \int F_\perp\, dt = e\int E_\perp\, dt = ze^2\gamma b\int \frac{dt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

where we used $q = ze$ for the charge of the heavy particle. This integral can be performed by trig substitution $t = (b/\gamma v)\tan\theta$ with the result

$$\Delta p = \frac{ze^2}{bv}\int_{-\pi/2}^{\pi/2} \sin\theta\, d\theta = \frac{2ze^2}{bv}$$

As a result, the energy transfer is approximately

$$T \approx \frac{(\Delta p)^2}{2m} = \frac{2z^2e^4}{mv^2}\frac{1}{b^2} \tag{4}$$

This is similar to the exact classical result (3), with the exception that the $b_{\rm min}^{(c)\,2}$ term is missing. That this term is missing is actually not surprising, because we have assumed the particle passes by at large impact parameter. This is essentially the limit $b \gg b_{\rm min}^{(c)}$, and it corresponds to having almost no deflection from the straight line path. When the impact parameter gets too small, the electron suffers a large deflection, and the straight line approximation breaks down. Thus instead of going to infinity as this approximate result does, the exact result (3) remains finite as $b \to 0$.

13.2 Time-varying electromagnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ of finite duration act on a charged particle of charge e and mass m bound harmonically to the origin with natural frequency ω_0 and small damping constant Γ. The fields may be caused by a passing charged particle or some other external source. The charge's motion in response to the fields is nonrelativistic and small in amplitude compared to the scale of spatial variation of the fields (dipole approximation). Show that the energy transferred to the oscillator in the limit of very small damping is

$$\Delta E = \frac{\pi e^2}{m}|\vec{E}(\omega_0)|^2$$

where $\vec{E}(\omega)$ is the symmetric Fourier transform of $\vec{E}(0,t)$:

$$\vec{E}(0,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(\omega)e^{-i\omega t}d\omega, \qquad \vec{E}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(0,t)e^{i\omega t}dt$$

The classical dynamics of the charged particle is given by $\vec{F} = m\vec{a}$

$$m\ddot{\vec{x}} = -m\omega_0^2\vec{x} - m\Gamma\dot{\vec{x}} + e\vec{E}(\vec{x},t) + \frac{e}{c}\dot{\vec{x}}\times\vec{B}(\vec{x},t)$$

In general, the Lorentz force terms are non-linear in displacement $\vec{x}(t)$. However for small amplitudes we may replace $\vec{E}(\vec{x},t) \approx \vec{E}(0,t)$ and $\vec{B}(\vec{x},t) \approx \vec{B}(0,t)$ on the right hand side. This gives the equation

$$\ddot{\vec{x}} + \Gamma\dot{\vec{x}} + \omega_0^2\vec{x} = \frac{e}{m}\vec{E}(t) + \frac{e}{mc}\dot{\vec{x}}\times\vec{B}(t)$$

Note that this equation is still rather awkward to solve because of the magnetic field coupling. Fortunately, this $\dot{\vec{x}}\times\vec{B}$ term can also be dropped at the same linearized level of approximation. This is because it can be treated as a perturbation: if $\vec{x}$ is first order in the external fields, the $\dot{\vec{x}}\times\vec{B}$ will be second order. As a result, we have the familiar damped driven harmonic oscillator

$$\ddot{\vec{x}} + \Gamma\dot{\vec{x}} + \omega_0^2\vec{x} = \frac{e}{m}\vec{E}(t)$$

with frequency domain solution

$$\vec{x}(\omega) = \frac{e/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}\vec{E}(\omega) \tag{5}$$

The energy transfer is then obtained by integrating the power

$$\Delta E = \int_{-\infty}^{\infty}\vec{F}(t)\cdot\dot{\vec{x}}(t)\,dt = e\int_{-\infty}^{\infty}\vec{E}(t)\cdot\dot{\vec{x}}(t)\,dt$$

This may be converted into the frequency domain using Parseval's relation (or the convolution theorem)

$$\Delta E = e \int_{-\infty}^{\infty} \dot{\vec{x}}(\omega) \cdot \vec{E}^*(\omega)\, d\omega$$

Substituting in (5) and expressing d/dt in frequency space then gives

$$\begin{aligned}
\Delta E &= \frac{e^2}{m} \int_{-\infty}^{\infty} \frac{-i\omega}{\omega_0^2 - \omega^2 - i\omega\Gamma} |\vec{E}(\omega)|^2 d\omega \\
&= \frac{e^2}{m} 2\Re \int_0^{\infty} \frac{-i\omega}{\omega_0^2 - \omega^2 - i\omega\Gamma} |\vec{E}(\omega)|^2 d\omega \\
&= \frac{2e^2}{m} \int_0^{\infty} \frac{\omega^2\Gamma}{(\omega^2 - \omega_0^2)^2 + \omega^2\Gamma^2} |\vec{E}(\omega)|^2 d\omega
\end{aligned}$$

This general expression simplifies in the limit $\Gamma \to 0$ where the fraction in the integrand becomes a delta function

$$\lim_{\Gamma\to 0} \frac{\omega^2\Gamma}{(\omega^2 - \omega_0^2)^2 + \omega^2\Gamma^2} = \frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

This gives

$$\lim_{\Gamma\to 0} \Delta E = \frac{\pi e^2}{m} |\vec{E}(\omega_0)|^2 \tag{6}$$

13.3 The external fields of Problem 13.2 are caused by a charge ze passing the origin in a straight-line path at speed v and impact parameter b. The fields are given by (11.152).

a) Evaluate the Fourier transforms for the perpendicular and parallel components of the electric field at the origin and show that

$$E_\perp(\omega) = \frac{ze}{bv}\left(\frac{2}{\pi}\right)^{1/2} \xi K_1(\xi), \qquad E_\parallel(\omega) = -i\frac{ze}{\gamma bv}\left(\frac{2}{\pi}\right)^{1/2} \xi K_0(\xi)$$

where $\xi = \omega b/\gamma v$, and $K_\nu(\xi)$ is the modified Bessel function of the second kind and order ν.

The external fields for a charge ze are given by

$$E_\parallel = \frac{-ze\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad E_\perp = \frac{ze\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{7}$$

Before evaluating the Fourier transforms, we recall that the modified Bessel functions K_0 and K_1 may be defined by

$$K_0(x) = \int_0^{\infty} \frac{\cos(xt)}{(t^2+1)^{1/2}} dt, \qquad K_1(x) = \int_0^{\infty} \frac{t\sin(xt)}{(t^2+1)^{1/2}} dt$$

Based on symmetry/antisymmetry, these may be extended to the entire real line

$$K_0(x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ixt}}{(t^2+1)^{1/2}}dt, \qquad K_1(x) = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{te^{ixt}}{(t^2+1)^{1/2}}dt$$

Comparing these expressions to (7), we see some similarities. However, the denominators in (7) are raised to the 3/2 power. This suggests that we integrate by parts to obtain the transform expressions

$$\int_{-\infty}^{\infty} \frac{te^{ixt}}{(t^2+1)^{3/2}}dt = 2ixK_0(x), \qquad \int_{-\infty}^{\infty} \frac{e^{ixt}}{(t^2+1)^{3/2}}dt = 2xK_1(x)$$

We are now ready to evaluate the Fourier transforms. For $E_\perp$, we have

$$\begin{aligned} E_\perp(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{ze\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}}e^{i\omega t}dt \\ &= \frac{ze\gamma}{b^2\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(1+(\gamma vt/b)^2)^{3/2}}dt \\ &= \frac{ze}{bv\sqrt{2\pi}}\int_{\infty}^{\infty} \frac{e^{i\xi t'}}{(1+t'^2)^{3/2}}dt' = \frac{ze}{bv}\sqrt{\frac{2}{\pi}}\xi K_1(\xi) \end{aligned} \tag{8}$$

where we made the change of variables $t' = \gamma vt/b$ and introduced the parameter $\xi = \omega b/\gamma v$. The transform for $E_\parallel$ is similar

$$\begin{aligned} E_\parallel(\omega) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{ze\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}e^{i\omega t}dt \\ &= -\frac{ze\gamma v}{b^3\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{te^{i\omega t}}{(1+(\gamma vt/b)^2)^{3/2}}dt \\ &= -\frac{ze}{\gamma bv\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{t'e^{i\xi t'}}{(1+t'^2)^{3/2}}dt' = -i\frac{ze}{\gamma bv}\sqrt{\frac{2}{\pi}}\xi K_0(\xi) \end{aligned} \tag{9}$$

b) Using the result of Problem 13.2, write down the energy transfer ΔE to a harmonically bound charged particle. From the limiting forms of the modified Bessel functions for small and large argument, show that your result agrees with the appropriate limit of $T(b)$ in Problem 13.1 on the one hand and the arguments at the end of Section 13.1 on the adiabatic behavior for $b \gg \gamma v/\omega_0$ on the other.

The energy transfer ΔE is approximately given by (6)

$$\Delta E = \frac{\pi e^2}{m}|\vec{E}(\omega_0)|^2$$

Substituting in $E_\perp$ and $E_\parallel$ from (8) and (9) gives

$$\Delta E = \frac{2z^2e^4}{mv^2}\frac{\xi_0^2[K_1(\xi_0)^2 + \gamma^{-2}K_0(\xi_0)^2]}{b^2}$$

where $\xi_0 = \omega_0 b/\gamma v$.

Note that the adiabatic regime is governed by the scale of b compared to $b_{\rm max}^{(c)} \equiv \gamma v/\omega_0$. In particular, since $\xi_0 = b/b_{\rm max}^{(c)}$ the two regimes of interest (small and large impact parameter) correspond directly to small and large argument of the modified Bessel functions. In the small impact parameter regime $b \ll b_{\rm max}^{(c)}$ we expand

$$K_0(\xi) = -\ln\left(\frac{\xi e^\gamma}{2}\right) + \cdots, \qquad K_1(\xi) = \frac{1}{\xi} + \cdots$$

Thus

$$\Delta E \approx \frac{2z^2e^4}{mv^2}\frac{1 + (\gamma^{-1}\xi\ln(\xi e^\gamma/2))^2}{b^2} \approx \frac{2z^2e^4}{mv^2}\frac{1}{b^2} \qquad (\xi \to 0)$$

This agrees with the large (but not so large as to be in the adiabatic regime) impact parameter limit expression (4) of the previously computed energy transfer. (Note that Problem 13.1 concerned a free electron, namely $\omega_0 \to 0$ or $b_{\rm max}^{(c)} \to \infty$.) Of course this expression breaks down for zero impact parameter for the same reason that (4) breaks down. Finally, for large impact parameters $b \gg b_{\rm max}^{(c)}$, we use the asymptotic expansion

$$K_\nu(\xi) \sim \sqrt{\frac{\pi}{2\xi}}e^{-\xi}$$

In this case, we obtain

$$\Delta E \sim \frac{\pi z^2e^4}{mv^2}\frac{(1+\gamma^{-2})e^{-2b/b_{\rm max}^{(c)}}}{b\, b_{\rm max}^{(c)}}$$

This vanishes exponentially as $e^{-2b/b_{\rm max}^{(c)}}$, which agrees with the notion that there is no significant energy transfer in the adiabatic limit (corresponding to $b > b_{\rm max}^{(c)}$).

13.9 Assuming that Plexiglas or Lucite has an index of refraction of 1.50 in the visible region, compute the angle of emission of visible Cherenkov radiation for electrons and protons as a function of their kinetic energies in MeV. Determine how many quanta with wavelengths between 4000 and 6000 Å are emitted per centimeter of path in Lucite by a 1 MeV electron, a 500 MeV proton, and a 5 GeV proton.

The Cherenkov angle is given by $\cos\theta_c = 1/n\beta$. To obtain the velocity β from the kinetic energy, we note that $T = E - mc^2 = (\gamma - 1)mc^2$. Solving this for β yields

$$\beta = \frac{\sqrt{(T/mc^2)(2 + T/mc^2)}}{1 + T/mc^2} \tag{10}$$

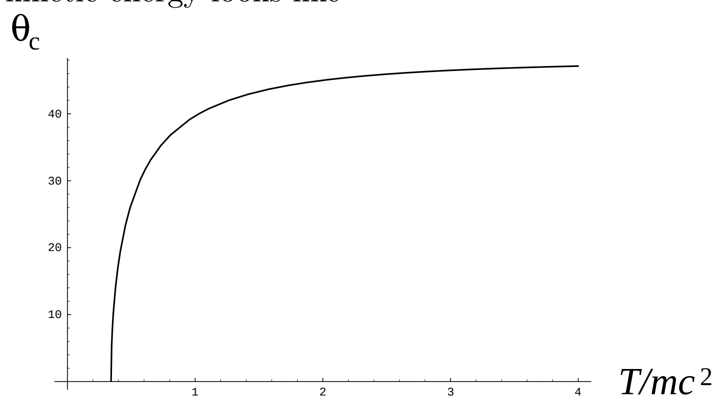
As a result

$$\cos\theta_c = \frac{1 + T/mc^2}{n\sqrt{(T/mc^2)(2 + T/mc^2)}}$$

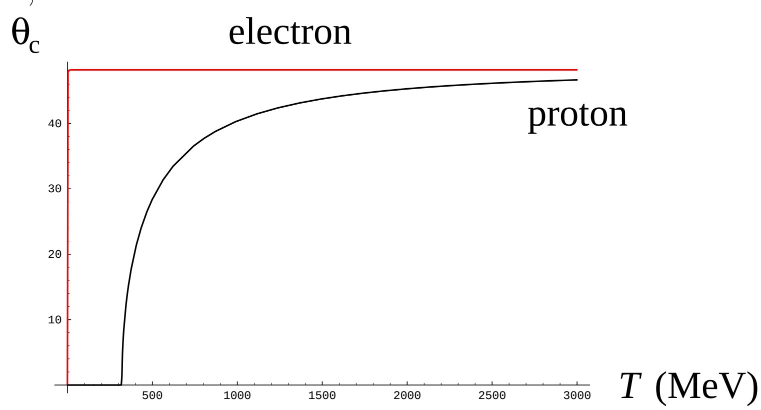
There is of course a lower threshold for kinetic energy, $T > T_{\rm min}$, where

$$T_{\rm min}/mc^2 = \frac{n}{\sqrt{n^2-1}} - 1$$

For $n = 1.5$, this lower threshold is $T_{\rm min}/mc^{\approx}0.342$, and a plot of the Cherenkov angle versus kinetic energy looks like



Note that the maximum opening angle is $\theta_{\rm max} = \cos^{-1}(1/1.50) = 48°$. For electrons (rest mass $m_e = 0.511\,{\rm MeV}$) and protons (rest mass $m_p = 938\,{\rm MeV}$), we may plot the opening angle as a function of kinetic energy in MeV units. On the same scale, the result is



We clearly see that electrons are essentially relativistic throughout the entire MeV and beyond energy range.

To compute the number of quanta $\mathcal{N}$ emitted, we note that a single photon carries a quantum unit of energy $\hbar\omega$. As a result, the Cherenkov energy expression

$$\frac{d^2E}{d\omega dx} = \frac{z^2e^2}{c^2}\omega\left(1-(n\beta)^{-2}\right)$$

directly yields

$$\frac{d^2\mathcal{N}}{d\omega dx} = \frac{z^2\alpha}{c}\left(1-(n\beta)^{-2}\right)$$

where we used the fine structure constant $\alpha = e^2/\hbar c$. This indicates that the number of quanta per unit frequency range is independent of frequency. Since the problem is interested in a range of wavelengths, we may use the relation $\omega = 2\pi c/n\lambda$ (taking into account the speed of light in a medium $v_p = c/n$) to write

$$\frac{d^2\mathcal{N}}{d\lambda dx} = \frac{2\pi z^2\alpha}{n\lambda^2}\left(1-(n\beta)^{-2}\right)$$

Assuming the index of refraction is independent of λ (at least in the range of wavelengths of interest), a simple integration gives

$$\frac{d\mathcal{N}}{dx} = \frac{2\pi z^2\alpha}{n}\left(1-(n\beta)^{-2}\right)\left(\frac{1}{\lambda_{\text{min}}}-\frac{1}{\lambda_{\text{max}}}\right)$$

For $\alpha = 1/137.036$, $z = 1$ (single electron or proton), $n = 1.50$, $\lambda_{\text{min}} = 4000\,\text{Å} = 4\times 10^{-5}\,\text{cm}$ and $\lambda_{\text{max}} = 6000\,\text{Å} = 6\times 10^{-5}\,\text{cm}$, the above expression becomes

$$\frac{d\mathcal{N}}{dx} \approx 255(1-0.444/\beta^2)\,\text{cm}^{-1}$$

where β may be obtained from (10). The number of photons emitted for the three requested cases are then

particle	β	θ_c	# photons/cm
1 MeV electron	0.941	45°	130
500 MeV proton	0.758	28°	58
5 GeV proton	0.987	47.5°	140

Finally, note that there is a slight ambiguity in specifying the wavelengths between 4000 and 6000 Å. Here we have taken the wavelengths as measured inside the Lucite. However, if the wavelengths are specified in vacuum, we would have an extra index of refraction factor according to $\lambda = \lambda_0/n$ where λ_0 is the wavelength in vacuum. This ambiguity could have easily been avoided by specifying a frequency range, although the spectrum of visible light is conventionally specified by wavelengths.