Physics 506 **Winter 2006**

Homework Assignment #10 — Solutions

Textbook problems: Ch. 12: 12.15, 12.16, 12.19, 12.20

12.15 Consider the Proca equations for a localized steady-state distribution of current that has' only a static magnetic moment. This model can be used to study the observable effects of a finite photon mass on the earths magnetic field. Note that if the magnetization is $\vec{\mathcal{M}}(\vec{x}\,)$ the current density can be written as $\vec{J} = c(\vec{\nabla} \times \vec{\mathcal{M}})$.

a) Show that if $\vec{\mathcal{M}} = \vec{m} f(\vec{x}\,)$, where $\vec{m}$ is a fixed vector and $f(\vec{x}\,)$ is a localized scalar function, the vector potential is

$$\vec{A}(\vec{x}\,) = -\vec{m} \times \vec{\nabla} \int f(\vec{x}\,') \frac{e^{-\mu|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} d^3x'$$

In the static limit, the Proca equation

$$[\partial^\lambda \partial_\lambda + \mu^2] A_\mu = \frac{4\pi}{c} J_\mu$$

takes the form

$$[\nabla^2 - \mu^2] A_\mu = -\frac{4\pi}{c} J_\mu$$

This admits a time independent Greens' function solution

$$A_\mu(x) = \frac{1}{c} \int J_\mu(x') G(x, x') d^3x'$$

where

$$G(x, x') = \frac{e^{-\mu|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|}$$

Taking $\vec{J} = c(\vec{\nabla} \times \vec{\mathcal{M}})$ with $\vec{\mathcal{M}} = \vec{m} f(\vec{x}\,)$ gives

$$\vec{J} = c\vec{\nabla} \times (\vec{m} f(\vec{x}\,)) = c\vec{\nabla} f \times \vec{m} = -c\vec{m} \times \vec{\nabla} f$$

Then

$$\vec{A} = -\vec{m} \times \int \vec{\nabla}' f(\vec{x}\,') \frac{e^{-\mu|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} d^3x'$$

Integration by parts (assuming the surface term vanishes since the source is localized) gives

$$\begin{aligned}
\vec{A} &= \vec{m} \times \int f(\vec{x}\,') \vec{\nabla}' \left(\frac{e^{-\mu|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} \right) d^3x' \\
&= -\vec{m} \times \int f(\vec{x}\,') \vec{\nabla} \left(\frac{e^{-\mu|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} \right) d^3x' \\
&= -\vec{m} \times \vec{\nabla} \int f(\vec{x}\,') \frac{e^{-\mu|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} d^3x'
\end{aligned}$$

where we made use of the fact that $\vec\nabla' G(x,x') = -\vec\nabla G(x,x')$.

b) If the magnetic dipole is a point dipole at the origin $[f(\vec x\,) = \delta(\vec x\,)]$, show that the magnetic field away from the origin is

$$\vec B(\vec x\,) = [3\hat r(\hat r\cdot\vec m) - \vec m]\left(1+\mu r+\frac{\mu^2r^2}{3}\right)\frac{e^{-\mu r}}{r^3} - \tfrac{2}{3}\mu^2\vec m\frac{e^{-\mu r}}{r}$$

For $f(\vec x\,) = \delta(\vec x\,)$ the resulting vector potential is

$$\vec A = -\vec m\times\vec\nabla\left(\frac{e^{-\mu r}}{r}\right) = (1+\mu r)\frac{e^{-\mu r}}{r^3}\vec m\times\vec r$$

The magnetic field is then

$$\begin{aligned}\vec B = \vec\nabla\times\vec A = \vec\nabla\left((1+\mu r)\frac{e^{-\mu r}}{r^3}\right)\times(\vec m\times\vec r) + (1+\mu r)\frac{e^{-\mu r}}{r^3}\vec\nabla\times(\vec m\times\vec r)\\
&= -(3+3\mu r+\mu^2r^2)\frac{e^{-\mu r}}{r^3}\hat r\times(\vec m\times\hat r)\\
&\qquad + (1+\mu r)\frac{e^{-\mu r}}{r^3}(\vec m(\vec\nabla\cdot\vec r) - (\vec m\cdot\vec\nabla)\vec r)\\
&= -(3+3\mu r+\mu^2r^2)\frac{e^{-\mu r}}{r^3}(\vec m - \hat r(\hat r\cdot\vec m\,)) + (2+2\mu r)\frac{e^{-\mu r}}{r^3}\vec m\\
&= (3\hat r(\hat r\cdot\vec m) - \vec m)\left(1+\mu r+\frac{\mu^2r^2}{3}\right)\frac{e^{-\mu r}}{r^3} - \tfrac{2}{3}\mu^2\vec m\frac{e^{-\mu r}}{r}\end{aligned}$$

c) The result of part *b*) shows that at fixed $r = R$ (on the surface of the earth), the earth's magnetic field will appear as a dipole angular distribution, plus an added constant magnetic field (an apparently external field) antiparallel to $\vec m$. Satellite and surface observations lead to the conclusion that this "external" field is less than 4×10^{-3} times the dipole field at the magnetic equator. Estimate a lower limit on μ^{-1} in earth radii and an upper limit on the photon mass in grams from this datum.

At the magnetic equator we have $\hat r\cdot\vec m = 0$. Hence

$$\vec B_{\text{dipole}} = -\vec m(1+\mu R+\frac{\mu^2R^2}{3})\frac{e^{-\mu R}}{R^3},\qquad \vec B_{\text{external}} = -\vec m(\tfrac{2}{3}\mu^2R^2)\frac{e^{-\mu R}}{R^3}$$

Setting $|\vec B_{\text{dipole}}|/|\vec B_{\text{external}}| < 4\times10^{-3}$ gives

$$\tfrac{2}{3}(\mu R)^2 < 4\times10^{-3}(1+\mu R+\tfrac{1}{3}(\mu R)^2)$$

or $\mu R < 0.08$. The lower limit on μ^{-1} is then

$$\mu^{-1} > 12.5R = 8.0\times10^9\ \text{cm}$$

where we have used the radius of the earth $R = 6.38 \times 10^8$ cm. This corresponds to an upper limit on the photon mass

$$m = \frac{\mu\hbar}{c} = \frac{1.05 \times 10^{-27}\,\mathrm{erg\,s}}{(8.0 \times 10^9\,\mathrm{cm})(3 \times 10^{10}\mathrm{cm/s})} = 4.4 \times 10^{-48}\,\mathrm{gm}$$

12.16 *a*) Starting with the Proca Lagrangian density (12.91) and following the same procedure as for the electromagnetic fields, show that the symmetric stress-energy-momentum tensor for the Proca fields is

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\gamma}F_{\gamma\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\lambda\nu}F^{\lambda\nu} + \mu^2\left(A^\alpha A^\beta - \frac{1}{2}g^{\alpha\beta}A_\lambda A^\lambda\right)\right]$$

The Proca Lagrangian density is

$$\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8\pi}\mu^2 A_\mu A^\mu$$

Since

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\mathcal{A}_\lambda}\partial^\nu A_\lambda - \eta^{\mu\nu}\mathcal{L}$$

we find

$$T^{\mu\nu} = -\frac{1}{4\pi}F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{16\pi}\eta^{\mu\nu}F^2 - \frac{1}{8\pi}\mu^2\eta^{\mu\nu}A^2$$

where we have used a shorthand notation $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$ and $A^2 \equiv A_\mu A^\mu$. In order to convert this canonical stress tensor to the symmetric stress tensor, we write $\partial^\nu A_\lambda = F^\nu{}_\lambda + \partial_\lambda A^\nu$. Then

$$\begin{aligned} T^{\mu\nu} &= -\frac{1}{4\pi}[F^{\mu\lambda}F^\nu{}_\lambda - \tfrac{1}{4}\eta^{\mu\nu}F^2 + \tfrac{1}{2}\mu^2\eta^{\mu\nu}A^2] - \frac{1}{4\pi}F^{\mu\lambda}\partial_\lambda A^\nu \\ &= -\frac{1}{4\pi}[F^{\mu\lambda}F^\nu{}_\lambda - \tfrac{1}{4}\eta^{\mu\nu}F^2 + \tfrac{1}{2}\mu^2\eta^{\mu\nu}A^2 - (\partial_\lambda F^{\mu\lambda})A^\nu] - \frac{1}{4\pi}\partial_\lambda(F^{\mu\lambda}A^\nu) \end{aligned}$$

Using the Proca equation of motion $\partial_\lambda F^{\lambda\mu} + \mu^2 A^\mu = 0$ then gives

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda S^{\lambda\mu\nu}$$

where

$$\Theta^{\mu\nu} = -\frac{1}{4\pi}\left[F^{\mu\lambda}F^\nu{}_\lambda - \tfrac{1}{4}\eta^{\mu\nu}F^2 - \mu^2(A^\mu A^\nu - \tfrac{1}{2}\eta^{\mu\nu}A^2)\right] \qquad (1)$$

is the symmetric stress tensor and $S^{\lambda\mu\nu} = (1/4\pi)F^{\lambda\mu}A^\nu$ is antisymmetric on the first two indices.

b) For these fields in interaction with the external source J^β, as in (12.91), show that the differential conservation laws take the same form as for the electromagnetic fields, namely

$$\partial_\alpha\Theta^{\alpha\beta} = \frac{J_\lambda F^{\lambda\beta}}{c}$$

Taking a 4-divergence of the symmetric stress tensor (1) gives

$$\begin{aligned}
\partial_\mu \Theta^{\mu\nu} &= -\frac{1}{4\pi}\big[\partial_\mu F^{\mu\lambda} F^\nu{}_\lambda + F^{\mu\lambda}\partial_\mu F^\nu{}_\lambda - \tfrac{1}{2} F_{\rho\lambda}\partial^\nu F^{\rho\lambda} \\
&\qquad - \mu^2(\partial_\mu A^\mu A^\nu + A^\mu \partial_\mu A^\nu - A^\lambda \partial^\nu A_\lambda)\big] \\
&= -\frac{1}{4\pi}\big[\partial_\mu F^{\mu\lambda} F^\nu{}_\lambda + \tfrac{1}{2} F_{\rho\lambda}(2\partial^\rho F^{\nu\lambda} - \partial^\nu F^{\rho\lambda}) + \mu^2 A^\lambda(\partial^\nu A_\lambda - \partial_\lambda A^\nu)\big] \\
&= -\frac{1}{4\pi}\big[(\partial_\mu F^{\mu\lambda} + \mu^2 A^\lambda) F^\nu{}_\lambda + \tfrac{1}{2} F_{\rho\lambda}(\partial^\rho F^{\nu\lambda} + \partial^\lambda F^{\rho\nu} + \partial^\nu F^{\lambda\rho})\big] \\
&= -\frac{1}{c} J^\lambda F^\nu{}_\lambda = \frac{1}{c} J_\lambda F^{\lambda\nu}
\end{aligned}$$

Note that in the second line we have used the fact that $\partial_\mu A^\mu = 0$, which is automatic for the Proca equation. To obtain the last line, we used the Bianchi identity $3\partial^{[\rho} F^{\nu\lambda]} = 0$ as well as the Proca equation of motion.

c) Show explicitly that the time-time and space-time components of $\Theta^{\alpha\beta}$ are

$$\begin{aligned}
\Theta^{00} &= \frac{1}{8\pi}[E^2 + B^2 + \mu^2(A^0 A^0 + \vec{A}\cdot\vec{A})] \\
\Theta^{i0} &= \frac{1}{4\pi}[(\vec{E}\times\vec{B})_i + \mu^2 A^i A^0]
\end{aligned}$$

Given the explicit form of the Maxwell tensor, it is straightforward to show that

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu} = -2(E^2 - B^2), \qquad A^2 \equiv A_\mu A^\mu = (A^0)^2 - \vec{A}^{\,2}$$

Thus

$$\Theta\mu\nu = -\frac{1}{4\pi}\left[F^{\mu\lambda}F^\nu{}_\lambda + \tfrac{1}{2}\eta^{\mu\nu}(E^2 - B^2) - \mu^2(A^\mu A^\nu - \tfrac{1}{2}\eta^{\mu\nu}((A^0)^2 - \vec{A}^{\,2}))\right]$$

The time-time component of this is

$$\begin{aligned}
\Theta^{00} &= -\frac{1}{4\pi}\left[F^{0i}F^0{}_i + \tfrac{1}{2}(E^2 - B^2) - \mu^2((A^0)^2 - \tfrac{1}{2}((A^0)^2 - \vec{A}^{\,2}))\right] \\
&= -\frac{1}{4\pi}\left[-\tfrac{1}{2}(E^2 + B^2) - \tfrac{1}{2}\mu^2((A^0)^2 + \vec{A}^{\,2})\right] \\
&= \frac{1}{8\pi}\left[E^2 + B^2 + \mu^2((A^0)^2 + \vec{A}^{\,2})\right]
\end{aligned}$$

Similarly, the time-space components are

$$\begin{aligned}
\Theta^{0i} &= -\frac{1}{4\pi}\left[F^0{}_j F^{ij} - \mu^2 A^0 A^i\right] = -\frac{1}{4\pi}\left[E^j(-\epsilon_{ijk}B^k) - \mu^2 A^0 A^i\right] \\
&= -\frac{1}{4\pi}\left[-\epsilon_{ijk}E^j B^k - \mu^2 A^0 A^i\right] = \frac{1}{4\pi}\left[(\vec{E}\times\vec{B})^i + \mu^2 A^0 A^i\right]
\end{aligned}$$

12.19 Source-free electromagnetic fields exist in a localized region of space. Consider the various conservation laws that are contained in the integral of $\partial_\alpha M^{\alpha\beta\gamma} = 0$ over all space, where $M^{\alpha\beta\gamma}$ is defined by (12.117).

a) Show that when β and γ are both space indices conservation of the total field angular momentum follows.

Note that

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\beta}x^\gamma - \Theta^{\alpha\gamma}x^\beta$$

Hence

$$M^{0ij} = \Theta^{0i}x^j - \Theta^{0j}x^i = c(g^i x^j - g^j x^i) = c\epsilon^{ijk}(\vec{g}\times\vec{x})^k = -c\epsilon^{ijk}(\vec{x}\times\vec{g})^k$$

where $\vec{g}$ is the linear momentum density of the electromagnetic field. Since $\vec{x}\times\vec{g}$ is the angular momentum density, integrating M^{0ij} over 3-space gives the field angular momentum

$$M^{ij} \equiv \int M^{0ij} d^3x = -c\epsilon^{ijk}\int(\vec{x}\times\vec{g})^k\, d^3x = -c\epsilon^{ijk}L^k$$

The conservation law $\partial_\mu M^{\mu ij} = 0$ then corresponds to the conservation of angular momentum in the electromagnetic field.

b) Show that when $\beta = 0$ the conservation law is

$$\frac{d\vec{X}}{dt} = \frac{c^2\vec{P}_{\text{em}}}{E_{\text{em}}}$$

where $\vec{X}$ is the coordinate of the center of mass of the electromagnetic fields, defined by

$$\vec{X}\int u\, d^3x = \int \vec{x}u\, d^3x$$

where u is the electromagnetic energy density and E_{em} and $\vec{P}_{\text{em}}$ are the total energy and momentum of the fields.

In this case, we have

$$\begin{aligned} M^{0i} \equiv \int M^{00i} d^3x &= \int(\Theta^{00}x^i - \Theta^{0i}x^0)\, d^3x \\ &= \int(ux^i - cg^i x^0)\, d^3x = \int(ux^i - c^2 t g^i)\, d^3x \end{aligned}$$

Making use of the definition $\int ux^i d^3x = EX^i$ where $E = \int u\, d^3x$ is the total field energy, we have simply

$$M^{0i} = EX^i - c^2 t P^i$$

where $\vec{P} = \int \vec{g}\, d^3x$ is the (linear) field momentum. Since M^{0i} is a conserved charge, its time derivative must vanish. This gives

$$0 = \frac{d}{dt}(E\vec{X}) - c^2 \frac{d}{dt}(t\vec{P}) = E\frac{d\vec{X}}{dt} - c^2\vec{P}$$

(where we used the fact that energy and momentum are conserved, namely $dE/dt = 0$ and $d\vec{P}/dt = 0$). The result $d\vec{X}/dt = c^2\vec{P}/E$ then follows.

12.20 A uniform superconductor with London penetration depth λ_L fills the half-space $x > 0$. The vector potential is tangential and for $x < 0$ is given by

$$A_y = (ae^{ikx} + be^{-ikx})e^{-i\omega t}$$

Find the vector potential inside the superconductor. Determine expressions for the electric and magnetic fields at the surface. Evaluate the surface impedance Z_s (in Gaussian units, $4\pi/c$ times the ratio of tangential electric field to tangential magnetic field). Show that in the appropriate limit your result for Z_s reduces to that given in Section 12.9.

The behavior of the vector potential inside the superconductor may be described by the massive Proca equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \mu^2\right]\vec{A} = 0$$

Working with a harmonic time behavior $e^{-i\omega t}$, the Proca equation may be rewritten as

$$[\nabla^2 + (\omega^2/c^2 - \mu^2)]\vec{A} = 0$$

This has a generic solution of the form

$$\vec{A}(\vec{x}, t) = \vec{A}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

where

$$|\vec{k}| = \sqrt{\omega^2/c^2 - \mu^2} = i\sqrt{\mu^2 - \omega^2/c^2}$$

The second form of the square root is appropriate for sufficiently low frequencies. Since the vector potential outside the superconductor ($x < 0$) only points in the $\hat{y}$ direction, and since the wave is normally incident (ie only a function of x), it is natural to expect the solution inside the superconductor to be of the form

$$A_y = (\alpha e^{-\sqrt{\mu^2 - \omega^2/c^2}x} + \beta e^{\sqrt{\mu^2 - \omega^2/c^2}x})e^{-i\omega t}$$

for appropriate constants α and β. To avoid an exponentially growing behavior, we take $\beta = 0$. Then it is straightforward to see that matching at $x = 0$ gives

$$A_y(x, t) = \begin{cases} (ae^{ikx} + be^{-ikx})e^{-i\omega t} & x < 0 \\ (a + b)e^{-\sqrt{\mu^2 - \omega^2/c^2}x}e^{-i\omega t} & x > 0 \end{cases}$$

In the absence of a scalar potential, the electric and magnetic fields are

$$\vec{E}(x=0^+) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}\bigg|_{x=0^+} = \frac{i\omega}{c}(a+b)\hat{y}e^{-i\omega t}$$

and

$$\vec{B}(x=0^+) = \vec{\nabla}\times\vec{A}\bigg|_{x=0^+} = -\sqrt{\mu^2-\omega^2/c^2}(a+b)\hat{z}e^{-i\omega t}$$

The surface impedance is given by

$$Z_s = \frac{4\pi}{c}\frac{E_y}{B_z} = -\frac{4\pi i\omega}{c^2\sqrt{\mu^2-\omega^2/c^2}}$$

Setting $\mu = 1/\lambda_{\mathrm{L}}$ and $\omega = 2\pi c/\lambda$ finally yields

$$Z_s = -\frac{8\pi^2 i}{c}\frac{\lambda_{\mathrm{L}}}{\lambda}(1-(2\pi\lambda_{\mathrm{L}}/\lambda)^2)^{-1/2}$$

This reduces in the long wavelength limit ($\lambda \gg \lambda_{\mathrm{L}}$) to the expected result

$$Z_s = -\frac{8\pi^2 i}{c}\frac{\lambda_{\mathrm{L}}}{\lambda}$$