**Physics 506** **Winter 2006**

## Homework Assignment #9 — Solutions

Textbook problems: Ch. 12: 12.2, 12.9, 12.13, 12.14

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12.2 *a*) Show from Hamilton's principle that Lagrangians that differ only by a total time derivative of some function of the coordinates and time are equivalent in the sense that they yield the same Euler-Lagrange equations of motion.

Suppose Lagrangians $L_1$ and $L_2$ differ by a total time derivative of the form

$$L_2 = L_1 + \frac{d}{dt} f(q_i(t), t)$$

Writing out the time derivative explicitly gives

$$L_2 = L_1 + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$$

The Euler-Lagrange equations for $L_2$ are derived from

$$\begin{aligned}\frac{\partial L_2}{\partial q_i} &= \frac{\partial L_1}{\partial q_i} + \frac{\partial^2 f}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 f}{\partial q_i \partial t} \\ \frac{\partial L_2}{\partial \dot{q}_i} &= \frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i}\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial L_2}{\partial q_i} - \frac{d}{dt}\frac{\partial L_2}{\partial \dot{q}_i} &= \frac{\partial L_1}{\partial q_i} - \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial^2 f}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 f}{\partial q_i \partial t} - \frac{d}{dt}\frac{\partial f}{\partial q_i} \\ &= \frac{\partial L_1}{\partial q_i} - \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial}{\partial q_i}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial q_i} \\ &= \frac{\partial L_1}{\partial q_i} - \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}_i}\end{aligned}$$

As a result, both $L_1$ and $L_2$ yield the same Euler-Lagrange equations.

Note that it is perhaps more straightforward to consider the change in the action

$$S_2 = \int_{t_1}^{t_2} L_2\, dt = \int_{t_1}^{t_2} (L_1 + \frac{df}{dt})\, dt = \int_{t_1}^{t_2} L_1\, dt + f(q_i(t_2), t_2) - f(q_i(t_1), t_1)$$

In other words, the additional of a total time derivative only changes the action by a surface term. So long as we do not vary the path at its endpoints ($\delta q_i(t_1) = \delta q_i(t_2) = 0$) we end up with the same equations of motion.

*b*) Show explicitly that the gauge transformation $A^\alpha \to A^\alpha + \partial^\alpha \Lambda$ of the potentials in the charged-particle Lagrangian (12.12) merely generates another equivalent Lagrangian.

We start with the Lagrangian

$$L = -mc^2\sqrt{1 - u^2/c^2} + \frac{e}{c}\vec{u}\cdot\vec{A} - e\Phi$$

In components, the gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ reads

$$\Phi \to \Phi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda, \qquad \vec{A} \to \vec{A} - \vec{\nabla}\Lambda$$

In this case, the Lagrangian changes by the term

$$\delta L = -\frac{e}{c}\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\Lambda$$

However, for $\Lambda = \Lambda(\vec{x}(t), t)$, the above is just the total time derivative

$$\delta L = -\frac{e}{c}\frac{d\Lambda}{dt}$$

As a result the Lagrangian changes by a total time derivative. Thus the gauge transformed Lagrangian is equivalent to the original one in the sense of part *a*).

12.9 The magnetic field of the earth can be represented approximately by a magnetic dipole of magnetic moment $M = 8.1 \times 10^{25}$ gauss-cm$^3$. Consider the motion of energetic electrons in the neighborhood of the earth under the action of this dipole field (Van Allen electron belts). [Note that $\vec{M}$ points south.]

*a*) Show that the equation for a line of magnetic force is $r = r_0 \sin^2\theta$, where $\theta$ is the usual polar angle (colatitude) measured from the axis of the dipole, and find an expression for the magnitude of $B$ along any line of force as a function of $\theta$.

Taking a spherical coordinate system with the $\hat{z}$ axis pointing north, the magnetic dipole moment of the earth can be represented by $\vec{m} = -M\hat{z}$. This gives rise to a magnetic field

$$\vec{B} = \frac{3\hat{r}(\hat{r}\cdot\vec{m}) - \vec{m}}{r^3} = \frac{M}{r^3}(\hat{z} - 3\cos\theta\hat{r})$$

To obtain the equation for a line of magnetic force, we first resolve the above into spherical coordinate components using $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$. This gives

$$\vec{B} = -\frac{M}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \tag{1}$$

We now note that the equation for a line of magnetic force can be written parametrically as $r = r(\lambda)$, $\theta = \theta(\lambda)$. In this case, the tangent to the curve is given by

$$\frac{\partial}{\partial\lambda} = \frac{dr}{d\lambda}\hat{r} + r\frac{d\theta}{d\lambda}\hat{\theta} \tag{2}$$

Since this tangent vector must point in the same direction as $\vec{B}$, we may take ratios of $\hat{r}$ and $\hat{\theta}$ components of (1) and (2) to obtain

$$\frac{2\cos\theta}{\sin\theta} = \frac{dr/d\lambda}{rd\theta/d\lambda} = \frac{1}{r}\frac{dr}{d\theta}$$

This gives rise to the separable equation $dr/r = 2\cot\theta\, d\theta$ which may be integrated to yield

$$r(\theta) = r_0 \sin^2\theta$$

From (1), the magnitude of the magnetic field is

$$B = \frac{M\sqrt{1+3\cos^2\theta}}{r^3}$$

Along the line $R = r_0 \sin^2\theta$, this becomes

$$B(\theta) = \frac{M\sqrt{1+3\cos^2\theta}}{r_0^3 \sin^6\theta} \tag{3}$$

*b*) A positively charged particle circles around a line of force in the equatorial plane with a gyration radius $a$ and a mean radius $R$ ($a \ll R$). Show that the particle's azimuthal position (east longitude) changes approximately linearly in time according to

$$\phi(t) = \phi_0 - \frac{3}{2}\left(\frac{a}{R}\right)^2 \omega_B (t - t_0)$$

where $\omega_B$ is the frequency of gyration at radius $R$.

Since the magnetic field is non-uniform, the gyrating particle will pick up a drift velocity. We may use the general expression for the drift velocity

$$\vec{v}_D = \frac{1}{\omega_B R_c}(v_\parallel^2 + \tfrac{1}{2}v_\perp^2)(\hat{R}_c \times \hat{B})$$

where $\vec{R}_c$ is the 'radius of curvature vector'. So long as the particle mainly circles around a line of force, we assume the particle's velocity is almost entirely perpendicular to the lines of force. Thus

$$\vec{v}_D \approx \frac{1}{2\omega_B R_c} v_\perp^2 (\hat{r} \times (-\hat{\theta})) = -\frac{\omega_B}{2R_c} a^2 \hat{\phi} \tag{4}$$

where we have used the fact that the magnetic field in the equitorial plane points in the $-\hat{\theta}$ ($\hat{z}$) direction and that the radius of curvature vector points along $\hat{r}$. The radius of curvature can now be obtained from the equation for the magnetic force lines $r = r_0 \sin^2\theta$. However, we take a shortcut given by the relation right after (12.60) in Jackson

$$\frac{\vec{\nabla}_\perp B}{B} = -\frac{\hat{R}_c}{R_c}$$

for a curl-free field $\vec{B}$. In the equitorial plane, the magnitude of the magnetic field is $B = M/cr^3$ and the perpendicular gradient direction is the $\hat{r}$ direction. The above relation then gives

$$R_c = \frac{R}{3}$$

where $R$ is the distance from the center of the earth (taken as the mean radius). Substituting this into (4) gives

$$\vec{v}_D = -\frac{3a^2}{2R}\omega_B \hat{\phi}$$

This drift velocity (in the equitorial plane) is given by $\vec{v} = R\dot{\phi}\hat{\phi}$. As a result, a simple integration gives

$$\phi - \phi_0 = -\frac{3}{2}\left(\frac{a}{R}\right)^2 \omega_B (t - t_0) \tag{5}$$

*c*) If, in addition to its circular motion of part *b*), the particle has a small component of velocity parallel to the lines of force, show that it undergoes small oscillations in $\theta$ around $\theta = \pi/2$ with a frequency $\Omega = (3/\sqrt{2})(a/R)\omega_B$. Find the change in longitude per cycle of oscillation in lattitude.

So long as $v_\parallel \ll v_\perp$, we can ignore its effect on the drift velocity. On the other hand, by conservation of energy and of flux through orbits of the particle, we have

$$v_0^2 = v_\parallel^2 + v_{\perp,0}^2 \frac{B(z)}{B_0} \approx v_\parallel^2 + \omega_B^2 a^2 \frac{B(z)}{B_0}$$

where $z$ is the coordinate parallel to the field line. Using the geometrical relation $\theta \approx \frac{\pi}{2} - (z/R)$ valid near the equitorial plane, and substituting it into (3), we obtain

$$\begin{aligned} B(z) &\approx \frac{M\sqrt{1 + 3\sin^2(z/R)}}{R^3 \cos^6(z/R)} \\ &\approx \frac{M}{R^3}\left(1 + 3\left(\frac{z}{R}\right)^2 - \cdots\right)^{1/2}\left(1 - \frac{1}{2}\left(\frac{z}{R}\right)^2 + \cdots\right)^{-6} \\ &\approx \frac{M}{R^3}\left(1 + \frac{9}{2}\left(\frac{z}{R}\right)^2\right) \end{aligned}$$

Hence $B(z)/B_0 \approx 1 + \frac{9}{2}(z/R)^2$, and

$$v_0^2 = v_\parallel^2 + (\omega_B a)^2 + \frac{9}{2}\left(\frac{\omega_B a}{R}\right)^2 z^2$$

This can be written in terms of an effective conservation of energy equation $E = \frac{1}{2}mv_\|^2 + V(z)$ where

$$V(z) = \frac{1}{2}m\left(\frac{3\omega_B a}{\sqrt{2}R}\right)^2 z^2$$

This is an effective harmonic oscillator potential $V = \frac{1}{2}m\Omega^2 z^2$ with

$$\Omega = \frac{3}{\sqrt{2}}\left(\frac{a}{R}\right)\omega_B$$

A complete period takes a time of

$$T_\Omega = \frac{2\pi}{\Omega} = \frac{2\sqrt{2}\pi}{3\omega_B}\left(\frac{R}{a}\right) \tag{6}$$

Inserting this into (5) gives a change of longitude of

$$\Delta\phi = -\frac{3}{2}\left(\frac{a}{R}\right)^2 \omega_B T_\Omega = -\sqrt{2}\pi\left(\frac{a}{R}\right)$$

over one complete oscillation in lattitude.

*d*) For an electron of 10 MeV kinetic energy at a mean radius $R = 3\times 10^7$ m, find $\omega_B$ and $a$, and so determine how long it takes to drift once around the earth and how long it takes to execute one cycle of oscillation in latitude. Calculate the same quantities for an electron of 10 keV at the same radius.

Note that $\omega_B = eB/\gamma mc$, while to a good approximation the strength of the magnetic field is $B_0 = M/R^3$. As a result

$$\omega_B = \frac{eM}{\gamma mcR^3} = \frac{ecM}{ER^3}$$

where $E$ is the relativistic energy of the electron, $E = \gamma mc^2 = mc^2 + KE$. Here $KE = (\gamma - 1)mc^2$ is the kinetic energy of the electron. Solving this for velocity gives

$$v = c\frac{\sqrt{2KE/mc^2 + (KE/mc^2)^2}}{1 + KE/mc^2}$$

which may be substituted into the relation $v \approx v_\perp = \omega_B a$ to get

$$a = \frac{mc^2R^3\sqrt{2KE/mc^2 + (KE/mc^2)^2}}{eM}$$

The time for an electron to drift once around the earth ($\Delta\phi = 2\pi$) can be obtained from (5). The result is

$$T_{2\pi} = \frac{4\pi}{3}\left(\frac{R}{a}\right)^2 \frac{1}{\omega_B}$$

In addition, the period for a lattitude oscillation is given by (6). Using $M = 8.1\times10^{25}$ gauss-cm$^3$, $R = 3\times10^9$ cm, $e = 4.8\times10^{-10}$ statcoul, $c = 3\times10^{10}$ cm/s and $mc^2 = 511\,\mathrm{keV}$, $1\,\mathrm{eV} = 1.6\times10^{-12}$ erg, as well as $KE = 10\,\mathrm{MeV}$ gives

$$\begin{aligned} 10\,\mathrm{MeV}: \qquad & \omega_B = 2.6\times10^3\,\mathrm{s}^{-1}, \qquad a = 1.2\times10^7\,\mathrm{cm} \\ & T_{2\pi} = 110\,\mathrm{s}, \qquad T_\Omega = 0.30\,\mathrm{s} \end{aligned}$$

Although the radius of gyration is rather large (120 km), it is less than half a percent of the distance from the center of the earth. As a result, the gradient of the magnetic field is still quite small over the orbit of the electron, and hence the approximations we have used are still valid. On the other hand, for $KE = 10\,\mathrm{keV}$, we find instead

$$\begin{aligned} 10\,\mathrm{MeV}: \qquad & \omega_B = 5.2\times10^4\,\mathrm{s}^{-1}, \qquad a = 1.1\times10^5\,\mathrm{cm} \\ & T_{2\pi} = 5.7\times10^4\,\mathrm{s}, \qquad T_\Omega = 1.50\,\mathrm{s} \end{aligned}$$

12.13 *a*) Specialize the Darwin Lagrangian (12.82) to the interaction of two charged particles $(m_1, q_1)$ and $(m_2, q_2)$. Introduce reduced particle coordinates, $\vec r = \vec x_1 - \vec x_2$, $\vec v = \vec v_1 - \vec v_2$ and also center of mass coordinates. Write out the Lagrangian in the reference frame in which the velocity of the center of mass vanishes and evaluate the canonical momentum components, $p_x = \partial L/\partial v_x$, etc.

The two particle Darwin Lagrangian reads

$$L = \frac12 m_1 v_1^2 + \frac12 m_2 v_2^2 + \frac{1}{8c^2}(m_1 v_1^4 + m_2 v_2^4) - \frac{q_1}{q_2} r_{12} + \frac{q_1 q_2}{2 r_{12} c^2}[\vec v_1\cdot\vec v_2 + (\vec v_1\cdot\hat r)(\vec v_2\cdot\hat r)] \quad (7)$$

We take a standard (non-relativistic) transformation to center of mass coordinates

$$\vec r = \vec x_1 - \vec x_2, \qquad \vec R = \frac{m_1\vec x_1 + m_2\vec x_2}{M}$$

where $M = m_1 + m_2$. Inverting this gives

$$\vec x_1 = \vec R + \frac{m_2}{M}\vec r, \qquad \vec x_2 = \vec R - \frac{m_1}{M}\vec r$$

As a result, the individual terms in the Lagrangian (7) become

$$\begin{aligned} \frac12 m_1 v_1^2 + \frac12 m_2 v_2^2 &= \frac12 M V^2 + \frac12 \mu v^2 \\ \frac{(m_1 v_1^4 + m_2 v_2^4)}{8c^2} &= \frac{1}{8c^2}\left(M V^4 + 6\mu V^2 v^2 + 4\mu\frac{m_2 - m_1}{M}(\vec V\cdot\vec v)v^2 + \mu\frac{m_1^3 + m_2^3}{M^3}v^4\right) \\ \vec v_1\cdot\vec v_2 &= V^2 + \frac{m_2 - m_1}{M}\vec V\cdot\vec v - \frac{\mu}{M}v^2 \\ (\vec v_1\cdot\hat r)(\vec v_2\cdot\hat r) &= (\vec V\cdot\hat r)^2 + \frac{m_2 - m_1}{M}(\vec V\cdot\hat r)(\vec v\cdot\hat r) - \frac{\mu}{M}(\vec v\cdot\hat r)^2 \end{aligned}$$

where $\mu = m_1 m_2/M$ is the reduced mass. For vanishing center of mass velocity ($\vec{V} = 0$) the Lagrangian becomes

$$L = \frac{1}{2}\mu v^2 + \frac{1}{8c^2}\mu\frac{m_1^3 + m_2^3}{M^3}v^4 - \frac{q_1 q_2}{r} - \frac{\mu q_1 q_2}{2Mrc^2}[v^2 + (\vec{v}\cdot\hat{r})^2] \tag{8}$$

The canonical momentum is

$$\vec{p} = \vec{\nabla}_v L = \mu\vec{v} + \frac{1}{2c^2}\mu\frac{m_1^3 + m_2^3}{M^3}v^2\vec{v} - \frac{\mu q_1 q_2}{2Mrc^2}[\vec{v} + (\vec{v}\cdot\hat{r})\hat{r}] \tag{9}$$

*b*) Calculate the Hamiltonian to first order in $1/c^2$ and show that it is

$$H = \frac{p^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \frac{q_1 q_2}{r} - \frac{p^4}{8c^2}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right) + \frac{q_1 q_2}{2m_1 m_2 c^2}\left(\frac{p^2 + (\vec{p}\cdot\hat{r})^2}{r}\right)$$

[You may disregard the comparison with Bethe and Salpeter.]

The Hamiltonian is obtained from the Lagrangian (8) by the transformation $H = \vec{p}\cdot\vec{v} - L$. Note, however, that we must invert the relation (9) to write the resulting $H$ as a function of $\vec{p}$ and $\vec{r}$. We start with

$$\begin{aligned} H &= \vec{p}\cdot\vec{v} - \frac{1}{2}\mu v^2 - \frac{1}{8c^2}\mu\frac{m_1^3 + m_2^3}{M^3}v^4 + \frac{q_1 q_2}{r} + \frac{\mu q_1 q_2}{2Mrc^2}[v^2 + (\vec{v}\cdot\hat{r})^2] \\ &= \frac{p^2}{2\mu} - \frac{1}{2\mu}(\vec{p} - \mu\vec{v})^2 - \frac{1}{8c^2}\mu\frac{m_1^3 + m_2^3}{M^3}v^4 + \frac{q_1 q_2}{r} + \frac{\mu q_1 q_2}{2Mrc^2}[v^2 + (\vec{v}\cdot\hat{r})^2] \end{aligned} \tag{10}$$

Since we only work to first order in $1/c^2$, we do not need to completely solve (9) for $\vec{v}$ in terms of $\vec{p}$. Instead, it is sufficient to note that

$$\vec{v} = \frac{1}{\mu}\vec{p} + \mathcal{O}\left(\frac{1}{c^2}\right)$$

Inserting this into (10) gives (to order $1/c^2$)

$$\begin{aligned} H &= \frac{p^2}{2\mu} - \frac{1}{8c^2}\frac{m_1^3 + m_2^3}{M^3\mu^3}p^4 + \frac{q_1 q_2}{r} + \frac{q_1 q_2}{2M\mu rc^2}[p^2 + (\vec{p}\cdot\hat{r})^2] \\ &= \frac{p^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - \frac{p^4}{8c^2}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right) + \frac{q_1 q_2}{r} + \frac{q_1 q_2}{2m_1 m_2 rc^2}[p^2 + (\vec{p}\cdot\hat{r})^2] \end{aligned}$$

12.14 An alternative Lagrangian density for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c}J_\alpha A^\alpha$$

*a*) Derive the Euler-Lagrange equations of motion. Are they the Maxwell equations? Under what assumptions?

The variations of the Lagrangian density are

$$\frac{\partial\mathcal{L}}{\partial\mathcal{A}_\mu} = -\frac{1}{c}J^\nu, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\nu\mathcal{A}_\mu)} = -\frac{1}{4\pi}\partial^\nu A^\mu$$

This leads to the Euler-Lagrange equations

$$-\frac{1}{c}J^\mu + \frac{1}{4\pi}\partial_\nu\partial^\nu A^\mu = 0$$

or

$$\partial^\nu\partial_\nu A_\mu = \frac{4\pi}{c}J_\nu \tag{11}$$

This can be recognized as the equation of motion for the vector potential $A_\mu$ in the Lorenz gauge $\partial^\mu A_\mu = 0$. To see this, recall that if we define $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ then the Maxwell equation $\partial^\mu F_{\mu\nu} = (4\pi/c)A_\nu$ becomes

$$\partial^\mu\partial_\mu A_\nu - \partial_\nu(\partial^\mu A_\mu) = \frac{4\pi}{c}J_\nu$$

which yields (11) provided $\partial^\mu A_\mu = 0$.

*b*) Show explicitly, and with what assumptions, that this Lagrangian density differs from (12.85) by a 4-divergence. Does this added 4-divergence affect the action or the equations of motion?

The difference between this alternative Lagrangian density and the standard Maxwell Lagrangian density

$$\mathcal{L}_0 = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}J^\mu A_m u$$

can be expressed as

$$\begin{aligned}
\Delta\mathcal{L} &= \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{8\pi}\partial_\mu A_\nu\partial^\mu A^\nu \\
&= -\frac{1}{8\pi}\partial_\mu A_\nu\partial^\nu A^\mu \\
&= -\frac{1}{8\pi}\partial_\mu(A_\nu\partial^\nu A^\mu - A^\mu\partial^\nu A_\nu) - \frac{1}{8\pi}(\partial^\mu A_\mu)^2
\end{aligned}$$

where the last line follows by simple manipulation of derivatives. This demonstrates that the alternative Lagrangian density differs from the 'correct' one by a 4-divergence so long as we are restricted to Lorenz gauge, $\partial^\mu A_\mu = 0$.

Finally, we recall that the action is the four-dimensional integral of the Lagrangian density. In this case, since the integral of a 4-divergence gives a surface term, the action is only affected by a possible surface term. Assuming the vector potential $A_\mu$ falls off sufficiently at infinity, this surface term will in fact vanish, so the action is actually unchanged. This then demonstrates that the equations of motion are unaffected by the addition of a 4-divergence to the Lagrangian density.