Homework Assignment #9 — Due Thursday, March 23

Textbook problems: Ch. 12: 12.2, 12.9, 12.13, 12.14

12.2   a) Show from Hamilton's principle that Lagrangians that differ only by a total time derivative of some function of the coordinates and time are equivalent in the sense that they yield the same Euler-Lagrange equations of motion.

   b) Show explicitly that the gauge transformation $A^\alpha \to A^\alpha + \partial^\alpha \Lambda$ of the potentials in the charged-particle Lagrangian (12.12) merely generates another equivalent Lagrangian.

12.9 The magnetic field of the earth can be represented approximately by a magnetic dipole of magnetic moment $M = 8.1 \times 10^{25}$ gauss-cm^3. Consider the motion of energetic electrons in the neighborhood of the earth under the action of this dipole field (Van Allen electron belts). [Note that $\vec{M}$ points south.]

   a) Show that the equation for a line of magnetic force is $r = r_0 \sin^2 \theta$, where θ is the usual polar angle (colatitude) measured from the axis of the dipole, and find an expression for the magnitude of B along any line of force as a function of θ.

   b) A positively charged particle circles around a line of force in the equatorial plane with a gyration radius a and a mean radius R ($a \ll R$). Show that the particle's azimuthal position (east longitude) changes approximately linearly in time according to

$$\phi(t) = \phi_0 - \frac{3}{2}\left(\frac{a}{R}\right)^2 \omega_B(t - t_0)$$

   where ω_B is the frequency of gyration at radius R.

   c) If, in addition to its circular motion of part b), the particle has a small component of velocity parallel to the lines of force, show that it undergoes small oscillations in θ around $\theta = \pi/2$ with a frequency $\Omega = (3/\sqrt{2})(a/R)\omega_B$. Find the change in longitude per cycle of oscillation in lattitude.

   d) For an electron of 10 MeV kinetic energy at a mean radius $R = 3 \times 10^7$ m, find $|omega_b$ and a, and so determine how long it takes to drift once around the earth and how long it takes to execute one cycle of oscillation in latitude. Calculate the same quantities for an electron of 10 keV at the same radius.

12.13   a) Specialize the Darwin Lagrangian (12.82) to the interaction of two charged particles (m_1, q_1) and (m_2, q_2). Introduce reduced particle coordinates, $\vec{r} = \vec{x}_1 - \vec{x}_2$, $\vec{v} = \vec{v}_1 - \vec{v}_2$ and also center of mass coordinates. Write out the Lagrangian in the reference frame in which the velocity of the center of mass vanishes and evaluate the canonical momentum components, $p_x = \partial L / \partial v_x$, etc.

b) Calculate the Hamiltonian to first order in $1/c^2$ and show that it is

$$H = \frac{p^2}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) + \frac{q_1 q_2}{r} - \frac{p^4}{8c^2} \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) + \frac{q_1 q_2}{2m_1 m_2 c^2} \left(\frac{p^2 + (\vec{p} \cdot \hat{r})^2}{r} \right)$$

[You may disregard the comparison with Bethe and Salpeter.]

12.14 An alternative Lagrangian density for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{8\pi} \partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha$$

a) Derive the Euler-Lagrange equations of motion. Are they the Maxwell equations? Under what assumptions?

b) Show explicitly, and with what assumptions, that this Lagrangian density differs from (12.85) by a 4-divergence. Does this added 4-divergence affect the action or the equations of motion?