Homework Assignment #8 — Solutions

Textbook problems: Ch. 11: 11.13, 11.16, 11.18, 11.27

11.13 An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density q_0 in the inertial frame K'. The frame K' (and the wire) move with a velocity $\vec{v}$ parallel to the direction of the wire with respect to the laboratory frame K.

a) Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.

Starting in the K' frame, we take the wire to be oriented along the $\hat{z}$ direction. Since the charges are at rest in K', there is no magnetic field. The electric field is given by a simple application of Gauss' law. Thus (in cylindrical coordinates, and with Gaussian units)

$$\vec{E}' = \frac{2q_0}{\rho'}\hat{\rho}, \qquad \vec{B}' = 0$$

We now transform to the lab frame K using a boost along the $\hat{z}$ axis $\vec{\beta} = (v/c)\hat{z}$. In general, the electric and magnetic fields are related by

$$\vec{E} = \gamma(\vec{E}' - \vec{\beta} \times \vec{B}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{E}')$$

$$\vec{B} = \gamma(\vec{B}' + \vec{\beta} \times \vec{E}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{B}')$$

In this case, since $\vec{B}' = 0$ and $\vec{\beta} \cdot \vec{E}' = 0$ we have simply

$$\vec{E} = \gamma\vec{E}' = \frac{2\gamma q_0}{\rho'}\hat{\rho}$$

$$\vec{B} = \gamma\vec{\beta} \times \vec{E}' = \frac{2v\gamma q_0}{c\rho'}\hat{\phi}$$

Finally, noting that the $\hat{\rho}$ direction is transverse to the boost, we have the simple relation $\rho' = \rho$. Thus we find

$$\vec{E} = \frac{2\gamma q_0}{\rho}\hat{\rho}, \qquad \vec{B} = \frac{2v\gamma q_0}{c\rho}\hat{\phi} \tag{1}$$

b) What are the charge and current densities associated with the wire in its rest frame? In the laboratory?

In the rest frame, and in cylindrical coordinates, the charge density may be expressed as

$$\rho' = q_0 \frac{\delta(\rho')}{2\pi\rho'}$$

Since the 3-current vanishes, the 4-current density may be written as

$$J'^\mu = \left(cq_0 \frac{\delta(\rho')}{2\pi\rho'}, 0 \right)$$

Boosting back to the lab frame K gives

$$J^\mu = (\gamma J'^0, \vec{\beta}\gamma J'0) = \left(\gamma cq_0 \frac{\delta(\rho)}{2\pi\rho}, v\gamma q_0 \hat{z} \frac{\delta(\rho)}{2\pi\rho} \right)$$

where we once again used the fact that $\rho = \rho'$. The explicit charge and 3-current densities are

$$\rho = \gamma q_0 \frac{\delta(\rho)}{2\pi\rho}, \qquad \vec{J} = v\gamma q_0 \hat{z} \frac{\delta(\rho)}{2\pi\rho} \tag{2}$$

c) From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part a).

From (2), the wire in the lab frame has linear charge density γq_0 and carries a current $I = v\gamma q_0$ in the $\hat{z}$ direction. Gauss' law and Ampère's law then gives

$$\vec{E} = \frac{2\gamma q_0}{\rho} \hat{\rho}, \qquad \vec{B} = \frac{2v\gamma q_0}{c\rho} \hat{\phi}$$

in agreement with the result (1) of a).

11.16 In the rest frame of a conducting medium the current density satisfies Ohm's law, $\vec{J}' = \sigma\vec{E}'$, where σ is the conductivity and primes denote quantities in the rest frame.

a) Taking into account the possibility of convection current as well as conduction current, show that the covariant generalization of Ohm's law is

$$J^\alpha - \frac{1}{c^2}(U_\beta J^\beta)U^\alpha = \frac{\sigma}{c}F^{\alpha\beta}U_\beta$$

where U^α is the 4-velocity of the medium.

There are several methods for obtaining the covariant generalization of Ohm's law. We start by taking the rest frame result $\vec{J}' = \sigma\vec{E}'$ and converting 3-vectors into 4-vectors. To do so, we note that the rest frame may be defined as the frame where the 4-velocity has the form

$$U'^\mu = (c, 0)$$

This allows us to realize the electric field as a contraction of the Maxwell field strength tensor with the 4-velocity

$$F'^{\mu\nu}U'_\nu = (0, c\vec{E}')$$

Thus the right hand side of Ohm's law can be written as

$$(0, \sigma\vec{E}') = \frac{\sigma}{c}F'^{\mu\nu}U'_\nu \tag{3}$$

The left hand side of Ohm's law ought to be the 3-vector current density. It is natural to take $\vec{J}' \rightarrow J'^\mu$. However, this contains more than the 3-current, since it includes the charge density ρ' as well. Because ρ' is not involved in Ohm's law, we need to subtract it out. This can be done by noting that $J'^\nu U'_\nu = c^2\rho'$. Hence

$$(0, \vec{J}') = J'^\mu - \frac{1}{c^2}(J'^\nu U'_\nu)U'^\mu \tag{4}$$

Combining (3) and (4) gives the covariant form of Ohm's law

$$J^\mu - \frac{1}{c^2}(J^\nu U_\nu)U^\mu = \frac{\sigma}{c}F^{\mu\nu}U_\nu \tag{5}$$

where we have now dropped all primes, since the equation is in covariant form (ie frame independent). Note that this can equivalently be written as

$$J^\mu = \frac{\sigma}{c}F^{\mu\nu}U_\nu + \frac{1}{c^2}(J^\nu U_\nu)U^\mu$$

where the first term on the right corresponds to the conduction current and the second to the convection current (charge times velocity, where charge is given by the Lorentz invariant $J^\nu U_\nu$ and the velocity is given by U^μ).

b) Show that if the medium has a velocity $\vec{v} = c\vec{\beta}$ with respect to some inertial frame that the 3-vector current in that frame is

$$\vec{J} = \gamma\sigma[\vec{E} + \vec{\beta} \times \vec{B} - \vec{\beta}(\vec{\beta} \cdot \vec{E})] + \rho\vec{v}$$

where ρ is the charge density observed in that frame.

Working in the (inertial) lab frame, we take

$$U^\mu = (\gamma c, \gamma\vec{v}), \qquad J^\mu = (c\rho, \vec{J})$$

Substiting this into (5) and noting that

$$\frac{1}{c^2}J^\nu U_\nu = \gamma(\rho - \frac{1}{c^2}\vec{v} \cdot \vec{J})$$

we obtain the time component equation

$$-v^2\gamma^2\rho + \gamma^2\vec{v}\cdot\vec{J} = \sigma\gamma\vec{v}\cdot\vec{E} \tag{6}$$

as well as the space components equation

$$\vec{J} - \gamma^2\rho\vec{v} + \frac{\gamma^2}{c^2}\vec{v}(\vec{v}\cdot\vec{J}) = \sigma\gamma(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}) \tag{7}$$

Solving (6) for $\vec{v}\cdot\vec{J}$ and substituting this into (7) gives

$$\vec{J} = \sigma\gamma[\vec{E} + \vec{\beta}\times\vec{B} - \vec{\beta}(\vec{\beta}\cdot\vec{E})] + \rho\vec{v} \tag{8}$$

This gives an explicit realization of the conduction and convection currents (the latter being the $\rho\vec{v}$ term in the above).

c) If the medium is uncharged in its rest frame ($\rho' = 0$), what is the charge density and the expression for $\vec{J}$ in the frame of part b)? This is the relativistic generalization of the equation $\vec{J} = \sigma(\vec{E} + \vec{v}\times\vec{B})$ (see p. 320).

Note that if the medium is uncharged in its rest frame, the 4-current must satisfy the relation $J^\nu U_\nu = 0$. In this case, the covariant Ohm's law (5) reduces to

$$J^\mu = \frac{\sigma}{c}F^{\mu\nu}U_\nu$$

Taking $J^\mu = (c\rho, \vec{J})$ and $U^\mu = (c\gamma, \vec{v}\gamma)$ in the above then gives directly

$$\rho = \frac{\sigma\gamma}{c}(\vec{\beta}\cdot\vec{E}), \qquad \vec{J} = \sigma\gamma(\vec{E} + \vec{\beta}\times\vec{B})$$

It is straightforward to check that this is consistent with (8) obtained above.

11.18 The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$, given by (11.152), become more and more concentrated as $\beta \to 1$, as is indicated in Fig. 11.9. Choose axes so that the charge moves along the z axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define the transverse vector $\vec{r}_\perp$, with components x and y. Consider the fields and the source in the limit of $\beta = 1$.

a) Show that the fields can be written as

$$\vec{E} = 2q\frac{\vec{r}_\perp}{r_\perp^2}\delta(ct - z); \qquad \vec{B} = 2q\frac{\hat{v}\times\vec{r}_\perp}{r_\perp^2}\delta(ct - z)$$

where $\hat{v}$ is a unit vector in the direction of the particle's velocity.

We take the particle motion to be along the $\hat{z}$ direction, so that $\vec{\beta} = (v/c)\hat{z}$. In this case, the fields in the rest frame are given by

$$\vec{E}' = \frac{q\vec{r}'}{r'^3}, \qquad \vec{B}' = 0$$

Boosting this back to the lab frame gives

$$\vec{E} = \gamma\vec{E}' - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}'), \qquad \vec{B} = \vec{\beta}\times\vec{E} \tag{9}$$

In particular, the latter expression indicates that once we compute the electric field $\vec{E}$, the magnetic field follows from a simple cross product with velocity. To proceed, we rewrite $\vec{r}'$ in terms of unprimed (lab) quantities. This may be done by considering the explicit boost transformation

$$x' = x, \qquad y' = y, \qquad z' = \gamma(z - vt)$$

This gives

$$\vec{r}' = \vec{r}_\perp + \gamma(z - vt)\hat{z}$$

so that

$$\vec{E}' = \frac{q(\vec{r}_\perp + \gamma(z - vt)\hat{z})}{(r_\perp^2 + \gamma^2(z - vt)^2))^{3/2}}$$

Boosting back using (9) gives the lab frame electric field

$$\vec{E} = \frac{\gamma q(\vec{r}_\perp + (z - vt)\hat{z})}{(r_\perp^2 + \gamma^2(z - vt)^2))^{3/2}}$$

which is an exact expression, valid for any velocity β. We are interested, however, in the limit $\beta \to 1$. This is equivalent to taking the limit $\gamma \to \infty$. Here we may use the fact that the denominator gets highly peaked, and approaches a delta function in this limit. More precisely,

$$\lim_{\gamma\to\infty}\frac{\gamma}{(A^2 + B^2\gamma^2)^{3/2}} = \frac{1}{A^2}\lim_{\xi\to\infty}\frac{\xi}{(1 + \xi^2 B^2)^{3/2}} = \frac{2}{A^2}\delta(B)$$

for any B and non-zero A. Taking $A = r_\perp$ and $B = z - vt$ then gives the result

$$\vec{E} = \frac{2q(\vec{r}_\perp + (z - vt)\hat{z})}{r_\perp^2}\delta(z - vt) = \frac{2q\vec{r}_\perp}{r_\perp^2}\delta(z - vt)$$

Since we have taken the limit $v \to c$, we finally arrive at

$$\vec{E} = 2q\frac{\vec{r}_\perp}{r_\perp^2}\delta(z - ct), \qquad \vec{B} = \vec{\beta}\times\vec{E} = 2q\frac{\hat{v}\times\vec{r}_\perp}{r_\perp^2}\delta(z - ct)$$

b) Show by substitution into the Maxwell equations that these fields are consistent with a 4-vector source density

$$J^\alpha = qcv^\alpha \delta^{(2)}(\vec{r}_\perp)\delta(ct - z)$$

where the 4-vector $v^\alpha = (1, \hat{v})$.

Maxwell's equations are $\partial_\mu F^{\mu\nu} = (4\pi/c)J^\nu$. In particular, the time component gives

$$\frac{4\pi}{c}J^0 = \vec{\nabla} \cdot \vec{E} = 2q\vec{\nabla}_\perp \cdot \left(\frac{\vec{r}_\perp}{r_\perp^2}\right)\delta(z - ct)$$

Using Gauss' in two dimensions, it is easy to see that

$$\vec{\nabla}_\perp \cdot \left(\frac{\vec{r}_\perp}{r_\perp^2}\right) = 2\pi\delta^{(2)}(\vec{r}_\perp)$$

Hence we obtain

$$J^0 = cq\delta^{(2)}(\vec{r}_\perp)\delta(z - ct)$$

In the space directions, we have

$$\frac{4\pi}{c}\vec{J} = -\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = 2q\frac{\vec{r}_\perp}{r_\perp^2}\delta'(z - ct) + 2q\vec{\nabla} \times \left[\hat{z} \times \left(\frac{\vec{r}_\perp}{r_\perp^2}\right)\delta(z - ct)\right]$$

$$= 2q\frac{\vec{r}_\perp}{r_\perp^2}\delta'(z - ct) + 2q\hat{z} \times \left[\hat{z} \times \left(\frac{\vec{r}_\perp}{r_\perp^2}\right)\right]\delta'(z - ct)$$

$$+ 2q\left[\hat{z}\vec{\nabla} \cdot \left(\frac{\vec{r}_\perp}{r_\perp^2}\right) - (\hat{z} \cdot \vec{\nabla})\left(\frac{\vec{r}_\perp}{r_\perp^2}\right)\right]\delta(z - ct)$$

$$= 4\pi q\hat{z}\delta^{(2)}(\vec{r}_\perp)\delta(z - ct)$$

(Note that $\hat{z} \cdot \vec{\nabla} = \partial_z$.) Noting that $\hat{v} = \hat{z}$, this is equivalent to

$$\vec{J} = cq\hat{v}\delta^{(2)}(\vec{r}_\perp)\delta(z - ct)$$

Putting together the time and space components gives

$$J^\mu = cq(1, \hat{v})\delta^{(2)}(\vec{r}_\perp)\delta(z - ct)$$

c) Show that the fields of part a) are derivable from either of the following 4-vector potentials

$$A^0 = A^z = -2q\delta(ct - z)\ln(\lambda r_\perp); \qquad \vec{A}_\perp = 0$$

or

$$A^0 = 0 = A^z; \qquad \vec{A}_\perp = -2q\Theta(ct - z)\vec{\nabla}_\perp \ln(\lambda r_\perp)$$

where λ is an irrelevant parameter setting the scale of the logarithm. Show that the two potentials differ by a gauge transformation and find the gauge function, χ.

For the first case ($A^0 = A^z$ non-vanishing) we have

$$\vec{E} = -\vec{\nabla}A^0 - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = 2q\vec{\nabla}[\delta(z-ct)\ln(\lambda r_\perp)] + \frac{2q}{c}\hat{z}\frac{\partial}{\partial t}[\delta(z-ct)\ln(\lambda r_\perp)]$$
$$= 2q\delta(z-ct)\vec{\nabla}_\perp \ln(\lambda r_\perp) + 2q\hat{z}\delta'(z-ct)\ln(\lambda r_\perp)$$
$$- 2q\hat{z}\delta'(z-ct)\ln(\lambda r_\perp)$$
$$= 2q\delta(z-ct)\frac{\hat{r}_\perp}{r_\perp^2}$$

and

$$\vec{B} = \vec{\nabla}\times\vec{A} = -2q\vec{\nabla}\times[\hat{z}\delta(z-ct)\ln(\lambda r_\perp)]$$
$$= -2q\vec{\nabla}_\perp \times[\delta(z-ct)\ln(\lambda r_\perp)\hat{z}]$$
$$= -2q\frac{\vec{r}_\perp}{r_\perp^2}\times\hat{z}\delta(z-ct) = 2q\frac{\hat{v}\times\vec{r}_\perp}{r_\perp^2}\delta(z-ct)$$

For the second case ($\vec{A}_\perp$ non-vanishing) we have

$$\vec{A}_\perp = -2q\Theta(ct-z)\vec{\nabla}_\perp \ln(\lambda r_\perp) = -2q\Theta(ct-z)\frac{\vec{r}_\perp}{r_\perp^2}$$

This gives

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = 2q\delta(ct-z)\frac{\vec{r}_\perp}{r_\perp^2}$$

and

$$\vec{B} = \vec{\nabla}\times\vec{A} = -2q\vec{\nabla}\times\left[\Theta(ct-z)\frac{\vec{r}_\perp}{r_\perp^2}\right]$$
$$= 2q\delta(ct-z)\hat{z}\times\frac{\vec{r}_\perp}{r_\perp^2} - 2q\Theta(ct-z)\vec{\nabla}\times\left(\frac{\vec{r}_\perp}{r_\perp^2}\right)$$
$$= 2q\frac{\hat{v}\times\vec{r}_\perp}{r_\perp^2}\delta(ct-z)$$

(since the function $\vec{r}_\perp/r_\perp^2 = \vec{\nabla}_\perp \ln(\lambda r_\perp)$ is curl-free).

The above two vector potentials are necessarily related by a gauge transformation χ through

$$A^\mu_{(1)} - A^\mu_{(2)} = \partial^\mu \chi$$

To find the explicit gauge transformation, we write

$$\partial^\mu \chi = A^\mu_{(1)} - A^\mu_{(2)}$$
$$= 2q\big(-\delta(ct-z)\ln(\lambda r_\perp), \Theta(ct-z)\vec{\nabla}_\perp \ln(\lambda r_\perp) - \delta(ct-z)\ln(\lambda r_\perp)\hat{z}\big)$$
$$= 2q\left(-\frac{\partial}{\partial ct}\Theta(ct-z)\ln(\lambda r_\perp), \Theta(ct-z)\vec{\nabla}_\perp \ln(\lambda r_\perp) + \hat{z}\frac{\partial}{\partial z}\Theta(ct-z)\ln(\lambda r_\perp)\right)$$
$$= -2q\left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right)\Theta(ct-z)\ln(\lambda r_\perp)$$

We thus conclude that the gauge function is

$$\chi = -2q\Theta(ct-z)\ln(\lambda r_\perp)$$

11.27 a) A charge density ρ' of zero total charge, but with a dipole moment $\vec{p}$, exists in reference frame K'. There is no current density in K'. The frame K' moves with a velocity $\vec{v} = \vec{\beta}c$ in the frame K. Find the charge and current densities ρ and $\vec{J}$ in the frame K and show that there is a magnetic dipole moment, $\vec{m} = (\vec{p} \times \vec{\beta})/2$, correct to first order in β. What is the electric dipole moment in K to the same order in β?

We assume the charge density ρ' is static (independent of time t') in the rest frame. Thus $\rho' = \rho'(\vec{x}')$ is only a function of $\vec{x}'$. Furthermore, we define

$$0 = q' \equiv \int \rho' d^3x', \qquad \vec{p}' \equiv \int \vec{x}' \rho' d^3x'$$

We use the prime notation (ie $\vec{p}'$) to denote the electric dipole in the rest frame. To boost to the lab frame K, we first construct the 4-vector current $J'^\mu = (c\rho', 0)$. The boosted current is then

$$J^\mu = (\gamma c\rho', \gamma\vec{\beta}c\rho')$$

while the coordinates are related by

$$x^0 = \gamma(x'^0 + \vec{\beta}\cdot\vec{x}')$$
$$\vec{x} = \vec{x}' + \frac{\gamma-1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{x}') + \gamma\vec{\beta}x'^0 \tag{10}$$

As a result, the lab frame charge and current densities are

$$\rho(x^0, \vec{x}) = \gamma\rho'(\vec{x}'), \qquad \vec{J}(x^0, \vec{x}) = \gamma\vec{v}\rho'(\vec{x}')$$

where $\vec{x}'$ may be given by the boost $\vec{x}' = \vec{x} + [(\gamma-1)/\beta^2]\vec{\beta}(\vec{\beta}\cdot\vec{x}) - \gamma\vec{\beta}x^0$, and we recall that ρ' is independent of x'^0.

In the lab frame, we define the electric and magnetic dipole moments as integrations over the source distrbutions at a fixed lab time x^0. For simplicity, we take $x^0 = 0$. In this case, the boost relations (10) may be solved to give $x'^0 = -\vec{\beta} \cdot \vec{x}'$ as well as

$$\vec{x} = \vec{x}' - \frac{\gamma}{\gamma + 1} \vec{\beta}(\vec{\beta} \cdot \vec{x}')$$

At fixed x^0 time, the 3-volume integration in frames K and K' are related by

$$\left[\int d^3 x = \frac{1}{\gamma} \int d^3 x'\right]_{x^0 = \text{const}}$$

We now have enough information to relate moments computed in frames K and K'. In particular, for the magnetic dipole moment, we have

$$\vec{m} = \frac{1}{2c} \int \vec{x} \times \vec{J} d^3 x = \frac{1}{2c} \int \left(\vec{x}' - \frac{\gamma}{\gamma + 1} \vec{\beta}(\vec{\beta} \cdot \vec{x}')\right) \times (\gamma \vec{\beta} c \rho') \frac{d^3 x'}{\gamma}$$

$$= \frac{1}{2} \int \vec{x}' \times \vec{\beta} \rho' d^3 x' = \left(\frac{1}{2} \int \vec{x}' \rho' d^3 x'\right) \times \vec{\beta}$$

$$= \tfrac{1}{2} \vec{p}' \times \vec{\beta}$$

In fact, this is exact, not just to first order, but to all orders in β. The electric dipole moment calculation is similar

$$\vec{p} = \int \vec{x} \rho \, d^3 x = \int \left(\vec{x}' - \frac{\gamma}{\gamma + 1} \vec{\beta}(\vec{\beta} \cdot \vec{x}')\right) \gamma \rho' \frac{d^3 x'}{\gamma}$$

$$= \int \left(\vec{x}' - \frac{\gamma}{\gamma + 1} \vec{\beta}(\vec{\beta} \cdot \vec{x}')\right) \rho' d^3 x'$$

$$= p' - \frac{\gamma}{\gamma + 1} \vec{\beta}(\vec{\beta} \cdot \vec{p}')$$

(If we had non-zero total charge, this expression would be corrected by the addition of a $q' \vec{v} t$ term; this was not apparent in the above, since we had only worked at fixed time $t = 0$.) This indicates that, the electric dipole moment remains uncorrected to first order in β, ie $\vec{p} = \vec{p}' + \mathcal{O}(\beta^2)$.

b) Instead of the charge density, but no current density, in K' consider no charge density, but a current density $\vec{J}'$ that has a magnetic dipole moment $\vec{m}$. Find the charge and current densities in K and show that to first order in β there is an electric dipole moment $\vec{p} = \vec{\beta} \times \vec{m}$ in addition to the magnetic dipole moment.

In this case, the 4-current densities are

$$J'^\mu = (0, \vec{J}'), \qquad J^\mu = \left(\gamma \vec{\beta} \cdot \vec{J}', \vec{J}' + \frac{\gamma - 1}{\beta^2} \vec{\beta}(\vec{\beta} \cdot \vec{J}')\right)$$

For the lab electric dipole moment, we have

$$\vec{p} = \int \vec{x}\rho \, d^3x = \int \left(\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')\right)\frac{\gamma}{c}\vec{\beta}\cdot\vec{J}'\frac{d^3x'}{\gamma}$$

$$= \frac{1}{c}\int \left(\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')\right)(\vec{\beta}\cdot\vec{J}')\,d^3x' \tag{11}$$

We now recall the identity (written in 3-space, and without primes for simplicity, but valid in either K or K')

$$\int x_i J_j d^3x = \tfrac{1}{2}\int (x_i J_j + x_j J_i)d^3x + \tfrac{1}{2}\int (x_i J_j - x_j J_i)d^3x$$

$$= \tfrac{1}{2}\int J_k\partial_k(x_j x_j)\,d^3x + \tfrac{1}{2}\epsilon_{ijk}\int [\vec{x}\times\vec{J}]_k\,d^3x$$

$$= -\tfrac{1}{2}\int x_i x_j(\vec{\nabla}\cdot\vec{J})\,d^3x + \tfrac{1}{2}\epsilon_{ijk}\int [\vec{x}\times\vec{J}]_k\,d^3x$$

(where we have used integration by parts). In the K' frame, and with a static current density, $\vec{\nabla}'\cdot\vec{J}' = 0$. Moreover, the antisymmetric term corresponds to the magnetic dipole moment $\vec{m}'$. This indicates that

$$\int x_i' J_j' d^3x' = c\epsilon_{ijk}m_k'$$

Inserting this into (11) gives

$$p_i = \epsilon_{ijk}\beta_j m_k' - \frac{\gamma}{\gamma+1}\beta_i\beta_j\epsilon_{jln}\beta_l m_n' = \epsilon_{ijk}\beta_j m_k'$$

or, in vector notation

$$\vec{p} = \vec{\beta}\times\vec{m}\,'$$

The magnetic dipole moment picks up a correction

$$\vec{m} = \frac{1}{2c}\int \vec{x}\times\vec{J}d^3x$$

$$= \frac{1}{2c}\int \left(\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')\right)\times\left(\vec{J}' + \frac{\gamma-1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{J}')\right)\frac{d^3x'}{\gamma}$$

$$= \frac{1}{2c\gamma}\int \left(\vec{x}\times\vec{J}' - \frac{\gamma}{\gamma+1}(\vec{\beta}\times\vec{J}')(\vec{\beta}\cdot\vec{x}\,') - \frac{\gamma-1}{\beta^2}(\vec{\beta}\times\vec{x}\,')(\vec{\beta}\cdot\vec{J}')\right)d^3x'$$

$$= \frac{1}{\gamma}\vec{m}\,' + \frac{1}{2\gamma}\left(\frac{\gamma-1}{\beta^2} - \frac{\gamma}{\gamma+1}\right)(\beta^2\vec{m}\,' - \vec{\beta}(\vec{\beta}\cdot\vec{m}\,'))$$

$$= (1 - \tfrac{1}{2}\beta^2)\vec{m}\,' - \frac{\gamma-1}{2(\gamma+1)}\vec{\beta}(\vec{\beta}\cdot\vec{m}\,')$$

To first order in β, this is simply $\vec{m} = \vec{m}\,'+\mathcal{O}(\beta^2)$. Finally, we note that the results of a) and b) are not directly related by electric/magnetic duality. This is because physically there is actually a subtle difference between magnetic dipole moments generated by current loops versus ones generated by magnetic monopoles.