Physics 506 **Winter 2006**

Homework Assignment #7 — Solutions

Textbook problems: Ch. 10: 10.14, 10.16, 11.4, 11.5

10.14 A rectangular opening with sides of length a and $b \geq a$ defined by $x = \pm(a/2)$, $y = \pm(b/2)$ exists in a flat, perfectly conducting plane sheet filling the x-y plane. A plane wave is normally incident with its polarization vector making an angle β with the long edges of the opening.

a) Calculate the diffracted fields and power per unit solid angle with the vector Smythe-Kirchhoff relation (10.109), assuming that the tangential electric field in the opening is the incident unperturbed field.

The vector Smythe-Kirchhoff relation states

$$\vec{E} = \frac{ie^{ikr}}{2\pi r}\vec{k} \times \int_{S_1} \hat{n}' \times \vec{E}(\vec{x}\,')e^{-i\vec{k}\cdot\vec{x}\,'}da'$$

where for a normally incident plane wave, the incident unperturbed field may be taken as

$$\vec{E}(\vec{x}\,') = E_0\hat{\epsilon}_0 e^{ikz'} = E_0(\hat{x}\sin\beta + \hat{y}\cos\beta)e^{ikz'}$$

For the rectangular screen, the surface S_1 is the rectangle at $z' = 0$ with $|x| \leq a/2$ and $|y| \leq b/2$ and surface normal $\hat{n}' = \hat{z}$. The resulting integral is then

$$\begin{aligned}\vec{E} &= \frac{iE_0e^{ikr}}{2\pi r}\vec{k} \times \int \hat{z} \times (\hat{x}\sin\beta + \hat{y}\cos\beta)e^{-i\vec{k}\cdot\vec{x}\,'}da' \\ &= \frac{iE_0e^{ikr}}{2\pi r}\vec{k} \times \int_{-a/2}^{a/2} dx' \int_{-b/2}^{b/2} dy'\, (\hat{y}\sin\beta - \hat{x}\cos\beta)e^{-i\vec{k}\cdot\vec{x}\,'} \\ &= \frac{iE_0e^{ikr}}{2\pi r}\vec{k} \times (\hat{y}\sin\beta - \hat{x}\cos\beta) \int_{-a/2}^{a/2} dx'\, e^{-ik_x x} \int_{-b/2}^{b/2} dy'\, e^{-ik_y y}\end{aligned}$$

The integrals are simple to perform, and yield

$$\vec{E} = \frac{2iE_0e^{ikr}}{\pi r}[-\hat{x}k_z\sin\beta - \hat{y}k_z\cos\beta + \hat{z}(k_x\sin\beta + k_y\cos\beta)]\frac{\sin(k_xa/2)\sin(k_yb/2)}{k_xk_y}$$

Because of the rectangular geometry, this expression is simplest when expressed in cartesian components. However, if we choose to write $\vec{k}$ in terms of spherical components, we may substitute in

$$k_x = k\sin\theta\cos\phi, \qquad k_y = k\sin\theta\sin\phi, \qquad k_z = k\cos\theta$$

to obtain

$$\vec{E} = \frac{2iE_0 e^{ikr}}{\pi kr}[-\hat{x}\cos\theta\sin\beta - \hat{y}\cos\theta\cos\beta + \hat{z}\sin\theta\sin(\phi+\beta)]$$
$$\times \frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\sin\theta\cos\phi}\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\sin\theta\sin\phi}$$

Note the standard $\sin\zeta/\zeta$ diffraction patterns for the x and y directions. The scattered power may be expressed as

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0}|\vec{E}|^2 = \frac{1}{2Z_0}\frac{4|E_0|^2}{\pi^2k^2}[\cos^2\theta + \sin^2\theta\sin^2(\phi+\beta)]$$
$$\times \frac{\sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{(\sin\theta\cos\phi)^2}\frac{\sin^2\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{(\sin\theta\sin\phi)^2}$$

In terms of the normally incident power on the aperture

$$P_i = \frac{|E_0|^2}{2Z_0}ab$$

the above becomes

$$\frac{dP}{d\Omega} = \frac{P_i}{\pi^2}[\cos^2\theta + \sin^2\theta\sin^2(\phi+\beta)]$$
$$\times \frac{\sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\frac{ka}{2}(\sin\theta\cos\phi)^2}\frac{\sin^2\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}(\sin\theta\sin\phi)^2} \tag{1}$$

Note that, for small openings, this reduces to

$$\frac{dP}{d\Omega} = \frac{P_i}{\pi^2}\frac{ka}{2}\frac{kb}{2}[\cos^2\theta + \sin^2\theta\sin^2(\phi+\beta)]$$

b) Calculate the corresponding result of the sclar Kirchhoff approximation.

For the scalar Kirchhoff approximation, we have

$$\psi = -\frac{e^{ikr}}{4\pi r}\int_{S_1}[\hat{n}'\cdot\vec{\nabla}'\psi + i\vec{k}\cdot\hat{n}'\psi]e^{-i\vec{k}\cdot\vec{x}\,'}da'$$

Here we take

$$\psi(\vec{x}\,') = \psi_0 e^{ikz'}, \qquad \hat{n}'\cdot\vec{\nabla}'\psi = \hat{z}\cdot\vec{\nabla}'\psi = \frac{\partial}{\partial z'}\psi = ik\psi_0 e^{ikz'}$$

Hence

$$\psi = -\frac{e^{ikr}}{4\pi r}\int(ik\psi_0 + ik_z\psi_0)e^{-i\vec{k}\cdot\vec{x}\,'}da'$$
$$= -\frac{i\psi_0 e^{ikr}}{4\pi r}(k+k_z)\int_{-a/2}^{a/2}dx'\,e^{-ik_x x}\int_{-b/2}^{b/2}dy'\,e^{-ik_y y}$$

The integrals are identical to the ones performed above. The result (using spherical components of $\vec{k}$) is

$$\psi = -\frac{i\psi_0 e^{ikr}}{\pi kr}(1+\cos\theta)\frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\sin\theta\cos\phi}\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\sin\theta\sin\phi}$$

Using $dP/d\Omega = r^2|\psi|^2$ and $P_i = |\psi|^2 ab$, the scalar expression for scattered power becomes

$$\frac{dP}{d\Omega} = \frac{P_i}{\pi^2}\left[\cos^4\frac{\theta}{2}\right]\frac{\sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\frac{ka}{2}(\sin\theta\cos\phi)^2}\frac{\sin^2\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}(\sin\theta\sin\phi)^2}$$

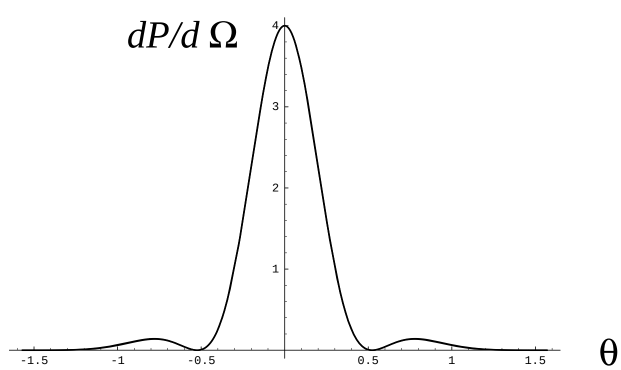
Comparing this scalar expression to the vector expression (1), we see that the only difference lies in the additional polarization factors enclosed in the square brackets.

c) For $b = a$, $\beta = 45°$, $ka = 4\pi$, compute the vector and scalar approximations to the diffracted power per unit solid angle as a function of the angle θ for $\phi = 0$. Plot a graph showing a comparison between the two results.

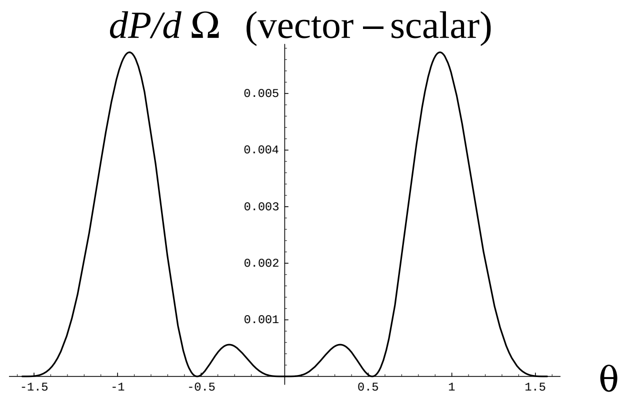
For the above parameters, the vector and scalar expressions reduce to

$$\left.\frac{dP}{d\Omega}\right|_{\text{vector}} = \frac{P_i}{\pi^2}[\tfrac{1}{2}(1+\cos^2\theta)]\frac{\sin^2(2\pi\sin\theta)}{\sin^2\theta}$$
$$\left.\frac{dP}{d\Omega}\right|_{\text{scalar}} = \frac{P_i}{\pi^2}[\cos^4(\theta/2)]\frac{\sin^2(2\pi\sin\theta)}{\sin^2\theta}$$

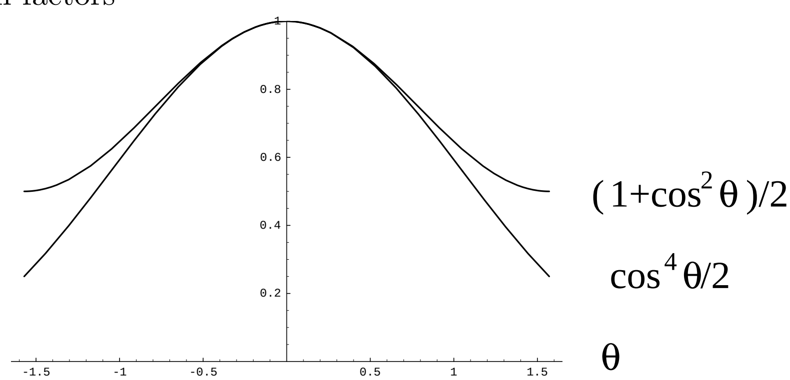
These two expressions (normalized to unit power) may be plotted on the same graph



In fact, they are virtually indistinguishable. To show that the vector and scalar expressions are actually not identical, we may plot the difference $dP/d\Omega|_{\text{vector}} - dP/d\Omega|_{\text{scalar}}$



(note the different scale on the vectical axis). This difference is entirely dependent on the polarization factors



These factors are nearly identical in the forward direction (at the diffraction peak). Although the difference gets large off axis, there is so little power there that this difference is essentially unimportant.

10.16 *a*) Show from (10.125) that the integral of the shadow scattering differential cross section, summed over outgoing polarizations, can be written in the short wavelength limit as

$$\sigma_{\rm sh} = \int d^2x_\perp \int d^2x'_\perp \cdot \frac{1}{4\pi^2} \int e^{i(\vec{x}_\perp - \vec{x}'_\perp)\cdot k_\perp} d^2k_\perp$$

and therefore is equal to the projected area of the scatterer, independent of its detailed shape.

With

$$\hat{\epsilon}^* \cdot \vec{f}_{\rm sh} \approx \frac{ik}{2\pi}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0) \int_{\rm sh} e^{-i\vec{k}_\perp \cdot \vec{x}_\perp} d^2x_\perp \qquad (2)$$

the differential cross section may be written as

$$\frac{d\sigma_{\rm sh}}{d\Omega} = |\hat{\epsilon}^* \cdot \vec{f}_{\rm sh}|^2 = \frac{k^2}{(2\pi)^2}|\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 \int_{\rm sh} e^{i\vec{k}_\perp \cdot \vec{x}_\perp} d^2x_\perp \int_{\rm sh} e^{-i\vec{k}_\perp \cdot \vec{x}'_\perp} d^2x'_\perp$$
$$= \frac{1}{(2\pi)^2} \int_{\rm sh} d^2x_\perp \int_{\rm sh} d^2x'_\perp e^{i\vec{k}_\perp \cdot (\vec{x}_\perp - \vec{x}'_\perp)} |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 k^2$$

Note that in the short wavelength limit ($ka \gg 1$) this expression is highly peaked in the forward direction. This means when we integrate over solid angle to obtain σ_{sh}, the integral is dominated by an extremely narrow solid angle around the forward direction. The general integrated expression is

$$\begin{aligned}\sigma_{\text{sh}} &= \frac{1}{(2\pi)^2}\int_{\text{sh}} d^2x_\perp \int_{\text{sh}} d^2x'_\perp \int e^{i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{x}'_\perp)}|\hat{\epsilon}^*\cdot\hat{\epsilon}_0|^2 k^2 d\Omega \\ &= \frac{1}{(2\pi)^2}\int_{\text{sh}} d^2x_\perp \int_{\text{sh}} d^2x'_\perp \int e^{i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{x}'_\perp)}\frac{|\hat{\epsilon}^*\cdot\hat{\epsilon}_0|^2}{\cos\theta} d^2k_\perp\end{aligned}$$

where we made use of the fact that the projected solid angle involves a cosine of the angle to the surface normal ($d^2k_\perp = k^2\cos\theta\, d\Omega$). Since we are working in the short wavelength limit, is is now consistent to replace $|\hat{\epsilon}^*\cdot\hat{\epsilon}_0| \approx 1$ when summed over outgoing polarizations (since polarization is preserved) and to replace $\cos\theta \approx 1$ (since the $d^2x_\perp$ integrals are only important in the forward direction). The result is then

$$\sigma_{\text{sh}} = \frac{1}{(2\pi)^2}\int_{\text{sh}} d^2x_\perp \int_{\text{sh}} d^2x'_\perp \int e^{i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{x}'_\perp)} d^2k_\perp \qquad (3)$$

We may now integrate $d^2k_\perp$ to obtain a two-dimensional delta function

$$\int e^{i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{x}'_\perp)} d^2k_\perp = (2\pi)^2\delta^2(\vec{x}_\perp-\vec{x}'_\perp)$$

Substituting this into (3) gives

$$\sigma_{\text{sh}} = \int_{\text{sh}} d^2x_\perp = A_{\text{sh}}^\perp$$

where $A_{\text{sh}}^\perp$ denotes the projected area of the shadow region.

b) Apply the optical theorem to the "shadow" amplitude (10.125) to obtain the total cross section under the assumption that in the forward direction the contribution from the illuminated side of the scatterer is negligible in comparison.

The obtical theorem states

$$\sigma_t = \frac{4\pi}{k}\Im(\hat{\epsilon}_0^*\cdot\vec{f}(\vec{k}=\vec{k}_0))$$

Assuming the forward scattering is dominated by the shadow contribution, this reduces to

$$\sigma_t \approx \frac{4\pi}{k}\Im(\hat{\epsilon}_0^*\cdot f_{\text{sh}}(\vec{k}_\perp=0))$$

However, from (2), we obtain

$$\hat{\epsilon}_0^*\cdot f_{\text{sh}}(\vec{k}_\perp=0) \approx \frac{ik}{2\pi}(\hat{\epsilon}_0^*\cdot\hat{\epsilon})\int_{\text{sh}} d^2x_\perp = \frac{ik}{2\pi}A_{\text{sh}}^\perp$$

where we assume the polarization $\hat{\epsilon}_0$ is properly normalized (as it needs to be). Inserting this in the above then gives simply

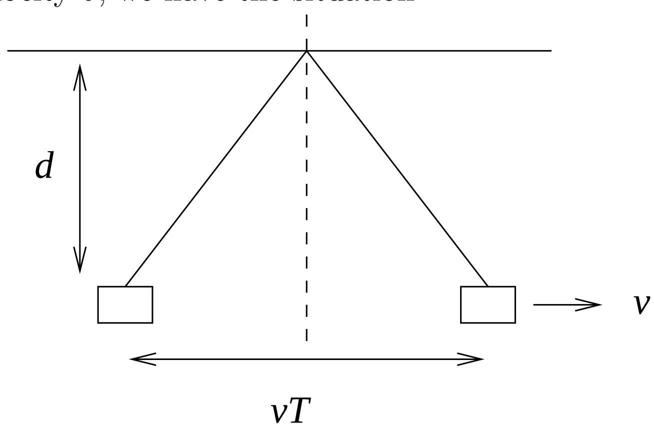
$$\sigma_t \approx 2A_{\rm sh}^{\perp}$$

which reproduces the phenomenon that the wave nature of light results in a total cross section twice that of the geometrical cross section.

11.4 A possible clock is shown in the figure. It consists of a flashtube F and a photocell P shielded so that each views only the mirror M, locate a distance d away, and mounted rigidly with respect to the flashtube-photocell assembly. The electronic innards of the box are such that when the photocell responds to a light flash from the mirror, the flashtube is triggered with a negligible delay and emits a short flash toward the mirror. The clock thus "ticks" once every $(2d/c)$ seconds when at rest.

a) Suppose that the clock moves with a uniform velocity v, perpendicular to the line from PF to M, relative to an observer. Using the second postulate of relativity, show by explicit geometrical or algebraic construction that the observer sees the relativistic time dilatation as the clock moves by.

The perpendicular motion is fairly easy to handle. If the box moves to the right at a uniform velocity v, we have the situation



Denoting the round-trip time T, the box moves a horizontal distance of vT during one complete period. Using a bit of geometry, the distance D traveled by a beam of light from the box to the mirror and back is simply

$$D = 2\sqrt{d^2 + (vT/2)^2} = \sqrt{(2d)^2 + (vT)^2}$$

For a constant speed of light, this gives

$$T = D/c \qquad \Rightarrow \qquad cT = \sqrt{(2d)^2 + (vT)^2}$$

Solving this for T gives the familiar time dilatation expression

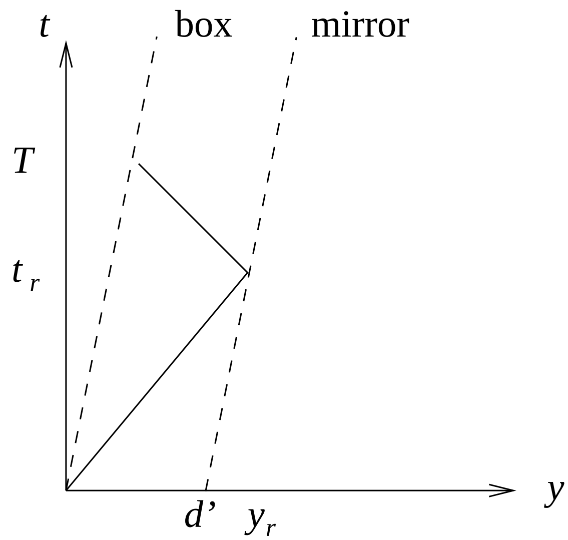
$$T = \gamma T_0$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \text{and} \qquad T_0 = \frac{2d}{c}$$

b) Suppose that the clock moves with a velocity v parallel to the line from PF to M. Verify that here, too, the clock is observed to tick more slowly, by the same time dilatation factor.

Here, consider a spacetime diagram



Here the box (and mirror) is moving in its parallel direction. We ought to know that this gives rise to a length contraction $d \to d/\gamma$. However, for now, we simply suppose that the box–mirror contraption appears to have length d'. The light beam reflects at position y_r at time t_r, and is recaptured by the box at time T. We first work out t_r algebraically. From the figure, the mirror's position is given by $y_r = d' + vt_r$, while the light ray travels according to $y_r = ct_r$. Solving this set of equations gives

$$y_r = \frac{d'}{1 - v/c}, \qquad t_r = \frac{d'/c}{1 - v/c}$$

On the return, the light ray is captured by the box at time T and position $y = vT$. Noting that the return path of the light ray is given by

$$y = y_r - c(t - t_r) = 2y_r - ct = \frac{2d'}{1 - v/c} - ct$$

we equate this to vT to obtain

$$T = \frac{2d'/c}{1 - v^2/c^2} = \gamma^2 \frac{2d'}{c}$$

Here, we realize that if lengths are contracted, $d' = d/\gamma$, then

$$T = \gamma^2 \frac{2d'}{c} = \gamma \frac{2d}{c} = \gamma T_0$$

gives the same time dilatation factor as part *a*). Alternatively, by demanding that the time dilatation factor is universal, we may obtain the length contraction relation $d' = d/\gamma$ as a result of this computation.

11.5 A coordinate system K' moves with a velocity $\vec{v}$ relative to another system K. In K' a particle has a velocity $\vec{u}\,'$ and an acceleration $\vec{a}\,'$. Find the Lorentz transformation law for accelerations, and show that in the system K the components of acceleration parallel and perpendicular to $\vec{v}$ are

$$\begin{aligned} \vec{a}_\parallel &= \frac{(1-v^2/c^2)^{3/2}}{(1+\vec{v}\cdot\vec{u}\,'/c^2)^3}\vec{a}_\parallel{}' \\ \vec{a}_\perp &= \frac{(1-v^2/c^2)}{(1+\vec{v}\cdot\vec{u}\,'/c^2)^3}\left(\vec{a}_\perp{}' + \frac{\vec{v}}{c^2}\times(\vec{a}\,'\times\vec{u}\,')\right) \end{aligned}$$

Instead of working direction with perpendicular and parallel components, we may start with a particular boost in the x-t direction, and then generalize our results. We thus take a boost of the form

$$x^0 = \gamma(x^{0\prime} + \beta x'), \qquad x = \gamma(x' + \beta x^{0\prime}), \qquad y = y', \qquad z = z' \tag{4}$$

Note that $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v/c$ are *constants* specifying the Lorentz boost. In frame K, the path of a particle is specified by the vector function $\vec{x}(x^0)$, while in frame K' this is instead $\vec{x}'(x^{0\prime})$. Three-velocities and 3-accelerations are then defined in a frame dependent manner

$$\begin{aligned} &\text{frame } K: && \vec{u} = c\frac{\partial\vec{x}}{\partial x^0}, && \vec{a} = c\frac{\partial\vec{u}}{\partial x^0} \\ &\text{frame } K': && \vec{u}\,' = c\frac{\partial\vec{x}\,'}{\partial x^{0\prime}}, && \vec{a}\,' = c\frac{\partial\vec{u}}{\partial x^{0\prime}} \end{aligned}$$

To transform between the two frames, we need not just the transformation of the 3-vectors, but also the transformation relating times x^0 and $x^{0\prime}$. Noting from (4) that a particle following a path $\vec{x}\,'(x^{0\prime})$ yields a time relation

$$x^0 = \gamma(x^{0\prime} + \beta x'(x^{0\prime}))$$

we may write

$$\frac{dx^0}{dx^{0\prime}} = \gamma(1+\beta u'_x/c)$$

The inverse relation is simply

$$\frac{dx^{0\prime}}{dx^0} = \frac{1}{\gamma(1+\beta u'_x/c)}$$

This useful expression is basically all we need. We start with velocities

$$u_x = c\frac{dx}{dx^0} = c\frac{dx^{0\prime}}{dx^0}\frac{dx}{dx^{0\prime}} = \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx^{0\prime}}\gamma(x'+\beta x^{0\prime}) = \frac{u'_x + c\beta}{1+\beta u'_x/c} \tag{5}$$

and

$$u_y = c\frac{dy}{dx^0} = c\frac{dx^{0\prime}}{dx^0}\frac{dy}{dx^{0\prime}} = \frac{c}{\gamma(1+\beta u'_x/c)}(u'_y/c) = \frac{u'_y}{\gamma(1+\beta u'_x/c)} \tag{6}$$

Writing $\beta u'_x = \vec{\beta}\cdot\vec{u}'$, it is easy to see that these velocity transformations may be written as

$$\vec{u}_\parallel = \frac{\vec{u}'_\parallel + c\vec{\beta}}{1+\vec{\beta}\cdot\vec{u}\,'/c}, \qquad \vec{u}_\perp = \frac{\vec{u}'_\perp}{\gamma(1+\vec{\beta}\cdot\vec{u}\,'/c)}$$

We now go on to accelerations. From (5), we have

$$\begin{aligned} a_x = c\frac{du_x}{dx^0} &= \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx^{0\prime}}\frac{u'_x+c\beta}{1+\beta u'_x/c} \\ &= \frac{c}{\gamma(1+\beta u'_x/c)}\frac{(1+\beta u'_x/c)(a'_x/c) - (u'_x+c\beta)(\beta a'_x/c^2)}{(1+\beta u'_x/c)^2} \\ &= \frac{(1-\beta^2)a'_x}{\gamma(1+\beta u'_x/c)^3} = \frac{a'_x}{\gamma^3(1+\beta u'_x/c)^3} \end{aligned}$$

And from (6) we have

$$\begin{aligned} a_y = c\frac{du_y}{dx^0} &= \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx^{0\prime}}\frac{u'_y}{\gamma(1+\beta u'_x/c)} \\ &= \frac{c}{\gamma^2(1+\beta u'_x/c)}\frac{(1+\beta u'_x/c)(a'_y/c) - u'_y(\beta a'_x/c^2)}{(1+\beta u'_x/c)^2} \\ &= \frac{a'_y + \beta(u'_x a'_y - u'_y a'_x)/c}{\gamma^2(1+\beta u'_x/c)^3} \end{aligned} \tag{7}$$

It is straightforward to convert the expression for a_x into one for $\vec{a}_\parallel$. The result is

$$\vec{a}_\parallel = \frac{\vec{a}'_\parallel}{\gamma^3(1+\vec{\beta}\cdot\vec{u}\,'/c)^3}$$

For the perpendicular direction, we have to be a bit more clever. Noting that x components in (7) are related to $\vec{\beta}\cdot(\ )$, while y components are directly related to the $\perp$ direction, we have

$$\vec{a}_\perp = \frac{\vec{a}'_\perp + \vec{a}\,'(\vec{\beta}\cdot\vec{u}\,') - \vec{u}'(\vec{\beta}\cdot\vec{a}\,')/c}{\gamma^2(1+\vec{\beta}\cdot\vec{u}\,'/c)^3}$$

Use of the BAC–CAB rule finally gives

$$\vec{a}_\perp = \frac{\vec{a}'_\perp + \vec{\beta}\times(\vec{a}\,'\times\vec{u}\,')/c}{\gamma^2(1+\vec{\beta}\cdot\vec{u}\,'/c)^3}$$