Physics 506 **Winter 2006**

Homework Assignment #6 — Solutions

Textbook problems: Ch. 10: 10.2, 10.3, 10.7, 10.10

10.2 Electromagnetic radiation with elliptic polarization, described (in the notation of Section 7.2) by the polarization vector,

$$\vec{\epsilon} = \frac{1}{\sqrt{1+r^2}}(\vec{\epsilon}_+ + re^{i\alpha}\vec{\epsilon}_-)$$

is scattered by a perfectly conducting sphere of radius a. Generalize the amplitude in the scattering cross section (10.71), which applies for $r = 0$ or $r = \infty$, and calculate the cross section for scattering in the long-wavelength limit. Show that

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right]$$

Compare with Problem 10.1.

Using a spherical wave expansion with the above polarization vector, we may write the incident plane wave as

$$\vec{E} = \sum_l i^l\sqrt{\frac{2\pi(2l+1)}{1+r^2}}\big[j_l(kr)\vec{X}_{l,1} + \frac{1}{k}\vec{\nabla}\times j_l(kr)\vec{X}_{l,1} + re^{i\alpha}(j_l(kr)\vec{X}_{l,-1} - \frac{1}{k}\vec{\nabla}\times j_l(kr)\vec{X}_{l,-1})\big]$$

(and a similar expression for $\vec{H}$). The scattered wave then takes the normalized form

$$\vec{E}_{\text{sc}} = \tfrac{1}{2}\sum_l i^l\sqrt{\frac{2\pi(2l+1)}{1+r^2}}\big[\alpha_+(l)h_l^{(1)}(kr)\vec{X}_{l,1} + \frac{\beta_+(l)}{k}\vec{\nabla}\times h_l^{(1)}(kr)\vec{X}_{l,1} + re^{i\alpha}(\alpha_-(l)h_l^{(1)}(kr)\vec{X}_{l,-1} - \frac{\beta_-(l)}{k}\vec{\nabla}\times h_l^{(1)}(kr)\vec{X}_{l,-1})\big]$$

Note that, since the incident wave has both $\vec{\epsilon}_+$ and $\vec{\epsilon}_-$ (coherently), the scattered electric field (essentially the scattering amplitude) has a coherent sum of positive and negative helicities. The scattering cross section may be written as

$$\frac{d\sigma_{\text{sc}}}{d\Omega} = \frac{\pi}{2k^2(1+r^2)}\Big|\sum_l\sqrt{2l+1}\big[\alpha_+(l)\vec{X}_{l,1} + i\beta_+(l)\hat{n}\times\vec{X}_{l,1} + re^{i\alpha}(\alpha_-(l)\vec{X}_{l,-1} - i\beta_-(l)\hat{n}\times\vec{X}_{l,-1})\big]\Big|^2$$

This is the generalization of (10.63), and leads to a total scattering cross section

$$\sigma_{\rm sc} = \frac{\pi}{2k^2(1+r^2)}\sum_l (2l+1)[|\alpha_+(l)|^2 + |\beta_+(l)|^2 + r^2(|\alpha_-(l)|^2 + |\beta_-(l)|^2)]$$

In the long wavelength limit, we only need to worry about the $l = 1$ terms in the above. The partial wave coefficients $\alpha_\pm(l)$ and $\beta_\pm(l)$ are those for a perfectly conducting sphere, and are unchanged by the elliptical polarization. For $l = 1$, we use the Jackson result

$$\alpha_\pm(1) = -\frac{1}{2}\beta_\pm(1) \approx -\frac{2i}{3}(ka)^3$$

As a result, we obtain

$$\frac{d\sigma_{\rm sc}}{d\Omega} \approx \frac{2\pi}{3k^2(1+r^2)}(ka)^6\big|\vec{X}_{1,1} - 2i\hat{n}\times\vec{X}_{1,1} + re^{i\alpha}(\vec{X}_{1,-1} + 2i\hat{n}\times\vec{X}_{1,-1})\big|^2 \quad (1)$$

We now work out the explicit functional forms of $\vec{X}_{1,1}$ and $\vec{X}_{1,-1}$. This is most straightforward in spherical coordinates where

$$\vec{X}_{1,\pm1} = \frac{1}{\sqrt{2}}\vec{L}Y_{1,\pm1}$$

Using

$$\vec{L} = \frac{r}{i}\hat{r}\times\vec{\nabla} = i\left(\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)$$

as well as $Y_{1,\pm1} = \mp\sqrt{3/8\pi}\sin\theta e^{\pm i\phi}$, we find

$$\vec{X}_{1,\pm1} = \mp i\sqrt{\frac{3}{16\pi}}\left(\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)\sin\theta e^{\pm i\phi} = \sqrt{\frac{3}{16\pi}}(\hat{\theta} \pm i\hat{\phi}\cos\theta)e^{\pm i\phi}$$

This also yields

$$\hat{n}\times\vec{X}_{1,\pm1} = \hat{r}\times\vec{X}_{1,\pm1} = \sqrt{\frac{3}{16\pi}}(\hat{\phi} \mp i\hat{\theta}\cos\theta)e^{\pm i\phi}$$

Inserting these expressions into (1) gives

$$\begin{aligned}
\frac{d\sigma_{\rm sc}}{d\Omega} &= \frac{(ka)^6}{8k^2(1+r^2)}\big|\big[\hat{\theta}(1-2\cos\theta) + i\hat{\phi}(\cos\theta-2)\big]e^{i\phi} \\
&\qquad + re^{i(\alpha-\phi)}\big[\hat{\theta}(1-2\cos\theta) - i\hat{\phi}(\cos\theta-2)\big]\big|^2 \\
&= \frac{(ka)^6}{8k^2(1+r^2)}\big|\hat{\theta}(1-2\cos\theta)(1+re^{i(\alpha-2\phi)}) \\
&\qquad + i\hat{\phi}(\cos\theta-2)(1-re^{i(\alpha-2\phi)})\big|^2 \\
&= \frac{(ka)^6}{8k^2(1+r^2)}\big[(1-2\cos\theta)^2(1+r^2+2r\cos(\alpha-2\phi)) \\
&\qquad + (\cos\theta-2)^2(1+r^2-2r\cos(\alpha-2\phi))\big]
\end{aligned}$$

Multiplying this out and rearranging terms gives the final result

$$\frac{d\sigma_{\text{sc}}}{d\Omega} = k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right] \qquad (2)$$

Alternatively, we could take the result of 10.1a)

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{4} - |\hat\epsilon_0\cdot\hat n|^2 - \frac{1}{4}|\hat n\cdot\hat n_0\times\hat\epsilon_0|^2 - \hat n_0\cdot\hat n\right]$$

and substitute in the polarization

$$\hat\epsilon_0 = \frac{1}{\sqrt{1+r^2}}(\hat\epsilon_+ + re^{i\alpha}\hat\epsilon_-)$$

We take explicitly

$$\hat n_0 = \hat z, \qquad \hat\epsilon_\pm = \frac{1}{\sqrt{2}}(\hat x \pm i\hat y), \qquad \hat n_0\times\hat\epsilon_\pm = \mp i\hat\epsilon_\pm$$

as well as

$$\hat n = \sin\theta(\hat x\cos\phi + \hat y\sin\phi) + \cos\theta\hat z$$

so that

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= k^4a^6\left[\frac{5}{4} - \frac{\sin^2\theta}{2(1+r^2)}|e^{i\phi} + re^{i(\alpha-\phi)}|^2 - \frac{\sin^2\theta}{8(1+r^2)}|e^{i\phi} - re^{i(\alpha-\phi)}|^2 - \cos\theta\right]\\
&= k^4a^6\Bigg[\frac{5}{4} - \cos\theta - \frac{1}{2}\sin^2\theta\left(1+\frac{2r}{1+r^2}\cos(\alpha-2\phi)\right)\\
&\qquad\qquad - \frac{1}{8}\sin^2\theta\left(1-\frac{2r}{1+r^2}\cos(\alpha-2\phi)\right)\Bigg]\\
&= k^4a^6\left[\frac{5}{8}(2-\sin^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(\alpha-2\phi)\right]
\end{aligned}$$

This is identical to the $l=1$ partial wave result (2).

10.3 A solid uniform sphere of radius R and conductivity σ acts as a scatterer of a plane-wave beam of unpolarized radiation of frequency ω, with $\omega R/c \ll 1$. The conductivity is large enough that the skin depth δ is small compared to R.

a) Justify and use a magnetostatic scalar potential to determine the magnetic field around the sphere, assuming the conductivity is infinite. (Remember that $\omega \neq 0$.)

We first note that for harmonic fields ($\omega \neq 0$) both the magnetic field and electric field must vanish inside a perfect conductor. Furthermore, there are no source currents outside the solid sphere. As a result of $\vec J = 0$, and since we are in the long wavelength limit $kR \ll 1$ (so we may work with a quasi-static magnetic field with $\vec{\nabla}\cdot\vec{B}\approx 0$), we may use a magnetostatic scalar potential $\vec{B}=-\vec{\nabla}\Phi_M$, at least in the vicinity (but always outside) of the sphere. Immediately outside the sphere, we may take a Legendre expansion

$$\begin{aligned}\Phi_M &= -B_0 z + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)\\ &= -B_0 r P_1(\cos\theta) + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)\end{aligned}$$

Note that we have taken the incident magnetic field to point along the z direction. (Since electromagnetic waves are transverse, this means the incident wave is actually traveling in the x-y plane.) We now use the fact that the perpendicular magnetic field must vanish at the surface $r=R$ of the conducting sphere. This gives

$$0 = B_r\big|_{r=R} = -\left.\frac{\partial\Phi_M}{\partial r}\right|_{r=R} = B_0 P_1(\cos\theta) + \sum_l \frac{(l+1)\alpha_l}{R^{l+2}} P_l(\cos\theta)$$

Since the Legendre polynomials form an orthogonal set, this indicates that all α_l must vanish for $l\neq 1$, while

$$\alpha_1 = -\tfrac{1}{2}B_0R^3$$

This gives

$$\Phi_M = -B_0\left(r+\frac{R^3}{2r^2}\right)P_1(\cos\theta) = -B_0 z\left(1+\frac{R^3}{2r^3}\right)$$

The resulting magnetic field is

$$\vec{B} = -\vec{\nabla}\Phi_M = B_0\left[\hat{z} - \frac{R^3}{2}\frac{3\hat{r}(\hat{r}\cdot\hat{z})-\hat{z}}{r^3}\right] \qquad (3)$$

The second term is clearly that of a magnetic dipole of strength

$$\vec{m} = -\frac{2\pi R^3}{\mu_0}\vec{B}_0 \qquad (\vec{B}_0 = B_0\hat{z})$$

This agrees with the conducting sphere result of (10.13). When combined with the electric dipole term, this gives the long wavelength scattering cross section of (10.14).

b) Use the technique of Section 8.1 to determine the absorption cross section of the sphere. Show that it varies as $(\omega)^{1/2}$ provided σ is independent of frequency.

We start with the power loss calculation

$$P_{\text{loss}} = \frac{1}{2\sigma\delta}\int |\hat{n}\times\vec{H}|^2 da$$

where

$$\hat{n} \times \vec{H} = \frac{1}{\mu_0}\hat{r} \times \vec{B} = \frac{B_0}{\mu_0}\left[1 + \frac{R^3}{2r^3}\right]_{r=R} \hat{r} \times \hat{z} = -\frac{3B_0}{2\mu_0}\sin\theta\hat{\phi}$$

We have used $\vec{B}$ given in (3), and evaluated the field at the surface of the conductor. Integrating this over the sphere gives

$$P_{\text{loss}} = \frac{1}{2\sigma\delta}\frac{9|B_0|^2}{4\mu_0^2}\int \sin^2\theta\, R^2 d\cos\theta\, d\phi = \frac{3\pi|B_0|^2R^2}{\sigma\delta\mu_0^2}$$

For normalization, note that the incident flux is

$$I_0 = \frac{1}{2Z_0}|\vec{E}_0|^2 = \frac{c^2}{2\sqrt{\mu_0/\epsilon_0}}|\vec{B}_0|^2 = \frac{Z_0}{2\mu_0^2}|B_0|^2$$

This gives an absorption cross section

$$\sigma_{\text{abs}} = \frac{P_{\text{loss}}}{I_0} = \frac{6\pi R^2}{\sigma\delta Z_0}$$

Using $\delta = \sqrt{2/\mu_0\sigma\omega}$ gives

$$\sigma_{\text{abs}} = 6\pi R^2\sqrt{\frac{\epsilon_0\omega}{2\sigma}}$$

which is clearly proportional to $(\omega)^{1/2}$ provide σ is independent of frequency.

10.7 Discuss the scattering of a plane wave of electromagnetic radiation by a nonpermeable, dielectric sphere of radius a and dielectric constant ϵ_r.

a) By finding the fields inside the sphere and matching to the incident plus scattered wave ouside the sphere, determine without any restriction on ka the multipole coefficients in the scattered wave. Define suitable phase shifts for the problem.

For the spherical wave analysis, we start with the outside solution, which is a combination of the incident and scattered wave

$$\begin{aligned}\vec{E} &= \sum_l i^l\sqrt{4\pi(2l+1)}[(j_l(kr) + \tfrac{1}{2}\alpha_\pm(l)h_l^{(1)}(kr))\vec{X}_{l,\pm1} \\ &\qquad \pm\frac{1}{k}\vec{\nabla}\times(j_l(kr) + \tfrac{1}{2}\beta_\pm(l)h_l^{(2)}(kr))\vec{X}_{l,\pm1}] \\ \vec{H} &= \frac{1}{Z_0}\sum_l i^l\sqrt{4\pi(2l+1)}[-\frac{i}{k}\vec{\nabla}\times(j_l(kr) + \tfrac{1}{2}\alpha_\pm(l)h_l^{(1)}(kr))\vec{X}_{l,\pm1} \\ &\qquad \mp i(j_l(kr) + \tfrac{1}{2}\beta_\pm(l)h_l^{(1)}(kr))\vec{X}_{l,\pm1}]\end{aligned} \tag{4}$$

Inside the dielectric sphere, we have no sources, and only a modified dielectric constant ϵ_r. As a result, the waves inside the sphere must be ordinary spherical waves, however with modified wave number

$$k' = \omega\sqrt{\mu_0\epsilon} = (\omega\sqrt{\mu_0\epsilon_0})\sqrt{\epsilon_r} = k\sqrt{\epsilon_r}$$

Defining also

$$Z = \sqrt{\frac{\mu_0}{\epsilon}} = \frac{Z_0}{\sqrt{\epsilon_r}}$$

the spherical waves inside the dielectric sphere may be parametrized by

$$\begin{aligned}
\vec{E} &= \sum_l i^l \sqrt{4\pi(2l+1)}[a_{M,\pm}(l) j_l(k'r)\vec{X}_{l,\pm1} \pm \frac{1}{k'} a_{E,\pm}(l)\vec{\nabla} \times j_l(k'r)\vec{X}_{l,\pm1}] \\
\vec{H} &= \frac{1}{Z} \sum_l i^l \sqrt{4\pi(2l+1)}[-\frac{i}{k'} a_{M,\pm}(l)\vec{\nabla} \times j_l(k'r)\vec{X}_{l,\pm1} \mp i a_{E,\pm}(l) j_l(k'r)\vec{X}_{l,\pm1}]
\end{aligned} \tag{5}$$

Note that the choice of constants was made to simplify the comparison with (4). Since we have a dielectric boundary, we now perform matching between the inside and outside fields. Note that this is the spherical generalization of matching plane waves incident on a flat dielectric boundary. There are four matching conditions, namely continuity of $B_\perp$, $E_\parallel$, $D_\perp$ and $H_\parallel$. For the perpendicular fields, note that

$$\hat{r} \cdot \vec{X}_{lm} = 0$$

while

$$\begin{aligned}
\hat{r} \cdot \vec{\nabla} \times f_l(kr)\vec{X}_{lm} = \hat{r} \times \vec{\nabla} \cdot f_l(kr)\vec{X}_{lm} &= i\vec{L} \cdot f_l(kr)\vec{X}_{lm} = i f_l(kr)\vec{L} \cdot \vec{X}_{lm} \\
&= i\sqrt{l(l+1)} f_l(kr) Y_{lm}
\end{aligned}$$

This indicates that only the curl terms in $\vec{E}$ and $\vec{H}$ survive in the perpendicular direction. For the parallel fields, on the other hand, both terms contribute. In particular

$$\hat{r} \times \vec{X}_{lm} \neq 0$$

and

$$\begin{aligned}
\hat{r} \times (\vec{\nabla} \times f_l(kr)\vec{X}_{lm}) &= \vec{\nabla}(f_l(kr)\hat{r} \cdot \vec{X}_{lm}) - \frac{1}{r} f_l(kr)(\vec{X}_{lm} - \hat{r}(\hat{r} \cdot \vec{X}_{lm})) \\
&\qquad - (\hat{r} \cdot \vec{\nabla}) f_l(kr)\vec{X}_{lm} \\
&= -\frac{1}{r}\frac{d}{dr}(r f_l(kr))\vec{X}_{lm}
\end{aligned}$$

where we have used $\hat{r} \cdot \vec{X}_{lm} = 0$. Matching linearly independent terms in the inside (5) and outside (4) solutions gives

$$\begin{aligned}
B_\perp: &\qquad a_{M,\pm}(l) j_l(x') = j_l(x) + \tfrac{1}{2}\alpha_\pm(l) h_l^{(1)}(x) \\
H_\parallel: &\qquad \sqrt{\epsilon_r} a_{E,\pm}(l) j_l(x') = j_l(x) + \tfrac{1}{2}\beta_\pm(l) h_l^{(1)}(x) \\
&\qquad a_{M,\pm}(l) \frac{d}{dx'} x' j_l(x') = \frac{d}{dx} x(j_l(x) + \tfrac{1}{2}\alpha_\pm(l) h_l^{(1)}(x)) \\
D_\perp: &\qquad \sqrt{\epsilon_r} a_{E,\pm}(l) j_l(x') = j_l(x) + \tfrac{1}{2}\beta_\pm(l) h_l^{(1)}(x) \\
E_\parallel: &\qquad a_{M,\pm}(l) j_l(x') = j_l(x) + \tfrac{1}{2}\alpha_\pm(l) h_l^{(1)}(x) \\
&\qquad a_{E,\pm}(l) \frac{d}{dx'} x' j_l(x') = \sqrt{\epsilon_r}\frac{d}{dx} x(j_l(x) + \tfrac{1}{2}\beta_\pm(l) h_l^{(1)}(x))
\end{aligned}$$

where we have defined

$$x = ka, \qquad x' = k'a = x\sqrt{\epsilon_r}$$

We note that two of the six equations are redundant (this also happened in the case of plane waves reflecting and refracting off of a plane dielectric boundary). This allows us to solve four equations for four unknowns α, β, a_E and a_M. Since we are only directly interested in the multipole coefficients α and β, we eliminate a_E and a_M from the above to obtain the solution

$$\begin{aligned}
\alpha_\pm(l) + 1 &= -\frac{h_l^{(2)}(x)\frac{d}{dx'}x'j_l(x') - j_l(x')\frac{d}{dx}xh_l^{(2)}(x)}{h_l^{(1)}(x)\frac{d}{dx'}x'j_l(x') - j_l(x')\frac{d}{dx}xh_l^{(1)}(x)} \\
\beta_\pm(l) + 1 &= -\frac{h_l^{(2)}(x)\frac{d}{dx'}x'j_l(x') - \epsilon_r j_l(x')\frac{d}{dx}xh_l^{(2)}(x)}{h_l^{(1)}(x)\frac{d}{dx'}x'j_l(x') - \epsilon_r j_l(x')\frac{d}{dx}xh_l^{(1)}(x)}
\end{aligned}$$

We now note that (at least for real ϵ_r) the above expressions are of the form of a ratio of a complex quantity divided by its complex conjugate. This indicates that the fractions have unit magnitude, and can be written in terms of real phase shifts

$$\alpha_\pm(l) = 1 = e^{2i\delta_l}, \qquad \beta_\pm(l) + 1 = e^{2i\delta_l'}$$

Noting that

$$e^{2i\delta_l} = -\frac{a - ib}{a + ib} \qquad \leftrightarrow \qquad \tan\delta_l = \frac{a}{b}$$

gives

$$\begin{aligned}
\tan\delta_l &= \frac{j_l(x)\frac{d}{dx'}x'j_l(x') - j_l(x')\frac{d}{dx}xj_l(x)}{n_l(x)\frac{d}{dx'}x'j_l(x') - j_l(x')\frac{d}{dx}xn_l(x)} \\
\tan\delta_l' &= \frac{j_l(x)\frac{d}{dx'}x'j_l(x') - \epsilon_r j_l(x')\frac{d}{dx}xj_l(x)}{n_l(x)\frac{d}{dx'}x'j_l(x') - \epsilon_r j_l(x')\frac{d}{dx}xn_l(x)}
\end{aligned} \tag{6}$$

With a bit of simplification, these can be rewritten in the form

$$\begin{aligned}
\tan\delta_l &= \frac{xj_l'(x) - B_l j_l(x)}{xn_l'(x) - B_l n_l(x)} \\
\tan\delta_l' &= \frac{xj_l'(x) - B_l' j_l(x)}{xn_l'(x) - B_l' n_l(x)}
\end{aligned} \tag{7}$$

where the coefficients B_l and B_l' are

$$B_l = x'\frac{j_l'(x')}{j_l(x')}, \qquad B_l' = \frac{1}{\epsilon_r}\left(x'\frac{j_l'(x')}{j_l(x')} + 1 - \epsilon_r\right) \tag{8}$$

and may be thought of as parametrizing the matching conditions at the boundary of the dielectric sphere. Note that the expressions for $\tan\delta_l$ and B_l are identical to that from the quantum mechanical scattering problem. The presence of the primed quantities is the result of vector waves as opposed to scalar waves.

b) Consider the long-wavelength limit ($ka \ll 1$) and determine explicitly the differential and total scattering cross sections. Compare your results with those of Section 10.1.B.

For $ka \ll 1$ only the lowest ($l = 1$) phase shift is important. In this case, we may approximate the spherical Bessel functions

$$j_1(x) = \frac{x}{3}(1 - \frac{x^2}{10} + \cdots), \qquad n_1(x) = -\frac{1}{x^2}(1 + \frac{x^2}{2} + \cdots)$$

Substituting this directly into (6), and keeping only the lowest non-trivial terms gives

$$\begin{aligned} \tan\delta_1 &= \frac{1}{45}x^3(x'^2 - x^2) = \frac{1}{45}(\epsilon_r - 1)(ka)^5 \\ \tan\delta_1' &= \frac{2}{3}\frac{\epsilon_r - 1}{\epsilon_r + 2}x^3 = \frac{2}{3}\frac{\epsilon_r - 1}{\epsilon_r + 2}(ka)^3 \end{aligned}$$

The multipole expansions are then approximated by

$$\begin{aligned} \alpha_\pm(1) &= e^{2i\delta_1} - 1 \approx 2i\delta_1 = \frac{2i}{45}(\epsilon_r - 1)(ka)^5 \\ \beta_\pm(1) &= e^{2i\delta_1'} - 1 \approx 2i\delta_1' = \frac{4i}{3}\frac{\epsilon_r - 1}{\epsilon_r + 2}(ka)^3 \end{aligned}$$

We see that only the β_1 (electric dipole) coefficient dominates at low energies. The scattering cross section is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\pi}{2k^2}\left|\sum_l \sqrt{2l+1}[\alpha_\pm(l)\vec{X}_{l,\pm1} \pm i\beta_\pm(l)\hat{n}\times\vec{X}_{l,\pm1}]\right|^2 \\ &\approx \frac{\pi}{2k^2}\left|\sqrt{3}\frac{4i}{3}\frac{\epsilon_r - 1}{\epsilon_r + 2}(ka)^3\hat{n}\times\vec{X}_{1,\pm1}\right|^2 \\ &= \frac{16\pi}{6k^2}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2(ka)^6|\vec{X}_{1,\pm1}|^2 \\ &= \tfrac{1}{2}k^4a^6\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2(1 + \cos^2\theta) \end{aligned}$$

This agrees perfectly with the dipole approximation.

c) In the limit $\epsilon_r \to \infty$ compare your results to those for the perfectly conducting sphere.

When we take $\epsilon_r \to \infty$ we are taking $x' = \sqrt{\epsilon_r}x \to \infty$. In this case, we use a large argument approximation to the spherical Bessel function $j_l(x')$

$$j_l(x') \sim \frac{1}{x'}\sin(x' - \frac{l\pi}{2})$$

This results in the asymptotic forms of the coefficients (8)

$$B_l \sim x' \cot(x' - \frac{l\pi}{2}) - 1 \to \infty$$
$$B_l' \sim \frac{1}{\sqrt{\epsilon_r}} x \cot(x' - \frac{l\pi}{2}) - 1 \to -1$$

Substituting this into (7) gives

$$\tan\delta_l = \frac{j_l(x)}{n_l(x)}, \qquad \tan\delta_l' = \frac{x j_l'(x) + j_l(x)}{x n_l'(x) + n_l(x)} = \frac{\frac{d}{dx} x j_l(x)}{\frac{d}{dx} x n_l(x)}$$

which reproduce exactly the perfectly conducting sphere phase shifts.

10.10 The aperture or apertures in a perfectly conducting plane screen can be viewed as the location of effective sources that produce radiation (the diffracted fields). An aperture whose dimensions are small compared with a wavelength acts as a source of dipole radiation with the contributions of other multipoles being negligible.

a) Beginning with (10.101) show that the effective electric and magnetic dipole moments can be expressed in terms of integrals of the tangential electric field in the aperture as follows:

$$\vec{p} = \epsilon\hat{n} \int (\vec{x} \cdot \vec{E}_{\text{tan}})\, da$$
$$\vec{m} = \frac{2}{i\omega\mu} \int (\hat{n} \times \vec{E}_{\text{tan}})\, da$$

where $\vec{E}_{\text{tan}}$ is the *exact* tangential electric field in the aperture, $\hat{n}$ is the normal to the plane screen, directed into the region of interest, and the integration is over the area of the openings.

The diffraction result (10.101) states

$$\vec{E}(\vec{x}\,) = \frac{1}{2\pi}\vec{\nabla} \times \int_{\text{apertures}} (\hat{n}' \times \vec{E})\frac{e^{ikR}}{R}da' \tag{9}$$

In the radiation zone, we may take

$$\frac{e^{ikR}}{R} \approx \frac{e^{ikr}}{r} e^{-i\vec{k}\cdot\vec{x}'}$$

Furthermore, for a small aperture (long wavelength limit), we may expand the second exponential

$$\frac{e^{ikR}}{R} \approx \frac{e^{ikr}}{r}(1 - i\vec{k}\cdot\vec{x}')$$

Inserting this into (9) and noting that we may use the replacement $\vec{\nabla} \to i\vec{k}$ in the radiation zone, we obtain the expansion

$$\vec{E} = \frac{i}{2\pi}\frac{e^{ikr}}{r}\vec{k} \times \int (\hat{n}' \times \vec{E})(1 - i\vec{k}\cdot\vec{x}'\,)\,da' \tag{10}$$

We start with the first term in the expansion

$$\vec{E}_1 = \frac{i}{2\pi}\frac{e^{ikr}}{r}\vec{k} \times \int \hat{n}' \times \vec{E}\,da'$$

which may be compared with the electric field of magnetic dipole radiation (in the radiation zone)

$$\vec{E} = -\frac{Z_0}{4\pi}k^2\frac{e^{ikr}}{r}\hat{k} \times \vec{m}$$

This allows us to read off the effective magnetic dipole moment

$$\vec{m} = \frac{2}{ikZ_0}\int \hat{n}' \times \vec{E}\,da' = \frac{2}{i\omega\mu_0}\int \hat{n}' \times \vec{E}\,da' \tag{11}$$

The effective electric dipole moment is somewhat trickier to extract. It is related to the second term in (10), which we write as

$$\vec{E}_2 = \frac{1}{2\pi}\frac{e^{ikr}}{r}\vec{k} \times \int (\hat{n}' \times \vec{E})(\vec{k}\cdot\vec{x}\,')\,da' \tag{12}$$

Since we have a flat screen, the normal vector $\hat{n}'$ is constant. Furthermore, the outgoing momentum vector $\vec{k}$ is unrelated to the integration coordinates (which line on the screen). Thus these two vectors may be pulled out of the integral. This means, we need to evaluate the integral (given in components)

$$\int E_i x'_j\,da'$$

where the indices i and j only lie in the screen directions (ie $i, j = 1, 2$ if we take $\hat{n}' = \hat{z}$). We now show that

$$\int E_i x'_j\,da' = \tfrac{1}{2}\delta_{ij}\int \vec{E}\cdot\vec{x}\,'\,da' \tag{13}$$

where we reemphasize that i and j lie in the screen directions only. Perhaps the most direct way to prove this is to write $E_i x'_j$ in tensor form

$$\begin{aligned}\vec{E}\otimes\vec{x}\,' = \begin{pmatrix} E_1x'_1 & E_1x'_2 \\ E_2x'_1 & E_2x'_2 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} E_1x'_1 + E_2x'_2 & 0 \\ 0 & E_1x'_1 + E_2x'_2 \end{pmatrix} \\ &\quad + \frac{1}{2}\begin{pmatrix} E_1x'_1 - E_2x'_2 & 2E_1x'_2 \\ 2E_2x'_1 & -E_1x'_1 + E_2x'_2 \end{pmatrix} \\ &= \tfrac{1}{2}\delta_{ij}(\vec{E}\cdot\vec{x}\,') + \tfrac{1}{2}[E_i x'_j - (\hat{n}'\times\vec{x}\,')_i(\hat{n}'\times\vec{E})_j]\end{aligned} \tag{14}$$

The second term vanishes when integrated over the openings. This is because we may use $\vec{\nabla}\times\vec{E}=0$ in a source-free region. Then

$$
\begin{aligned}
0 &= \int x_i' x_j' \hat{n}'\cdot(\vec{\nabla}'\times\vec{E})\,da' = \epsilon_{klm}\hat{n}_k' \int x_i' x_j' \partial_l E_m\,da' \\
&= -\epsilon_{klm}\hat{n}_k' \int \partial_l(x_i' x_j')E_m\,da' = \hat{n}_k' \int (\epsilon_{ikm}x_j' + \epsilon_{jkm}x_i')E_m\,da' \\
&= \int [x_i'(\hat{n}'\times\vec{E})_j + x_j'(\hat{n}'\times\vec{E})_i]\,da'
\end{aligned}
\tag{15}
$$

Note that the surface term arising from the integration by parts vanishes because it is proportional to $E_{\|}$, which must vanish on the boundaries of the openings. Substituting in explicit components $ij = 11$, 12, and 22 then proves that the integral of $E_2 x_1'$, $E_1 x_1' - E_2 x_2'$, and $E_1 x_2'$ vanish, as needed to remove the second term from (14). This can also be seen directly by taking a cross product of (15) with $\hat{n}'$ in the ith component to get

$$
\int [(\hat{n}'\times\vec{x}\,')_i(\hat{n}'\times\vec{E})_j - x_j' E_i]\,da' = 0
$$

In any case, the result is simply (13), which may be substituted into (12) to obtain

$$
\begin{aligned}
\vec{E}_2 &= \frac{1}{4\pi}\frac{e^{ikr}}{r}\vec{k}\times(\hat{n}'\times\vec{k})\int \vec{x}\,'\cdot\vec{E}\,da' \\
&= -\frac{1}{4\pi}\frac{e^{ikr}}{r}\vec{k}\times(\vec{k}\times\hat{n}')\int \vec{x}\,'\cdot\vec{E}\,da'
\end{aligned}
$$

Comparing this with the radiation patter for electric dipole radiation

$$
\vec{E} = -\frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\hat{k}\times(\hat{k}\times\vec{p})
$$

gives an effective electric dipole moment

$$
\vec{p} = \epsilon_0 \hat{n}' \int \vec{x}\,'\cdot\vec{E}\,da'
$$

Note the curious fact that the magnetic dipole term comes from the lowest order in the expansion of (9), while the electric dipole term comes from the next order. This is ‘backwards’ from what happens for a conventional source given by a specified current density.

b) Show that the expression for the magnetic moment can be transformed into

$$
\vec{m} = \frac{2}{\mu}\int \vec{x}(\hat{n}\cdot\vec{B})\,da
$$

Be careful about possible contributions from the edge of the aperture where some components of the fields are singular if the screen is infinitesimally thick.

To relate the electric field to the magnetic field, we may use Faraday's equation for harmonic fields $\vec{\nabla} \times \vec{E} - i\omega\vec{B} = 0$ to write

$$\hat{n}' \cdot (\vec{\nabla}' \times \vec{E}) = i\omega(\hat{n}' \cdot \vec{B})$$

Multiplying this by a vector $\vec{x}\,'$ and integrating gives

$$\begin{aligned} i\omega \int \vec{x}\,'(\hat{n}' \cdot \vec{B})\, da' &= \int \vec{x}\,'[\hat{n}' \cdot (\vec{\nabla}' \times \vec{E})]\, da' \\ &= \int \vec{x}\,'\epsilon_{ijk}\hat{n}_i'\partial_j E_k\, da' \\ &= -\int \partial_j(\vec{x}\,')\epsilon_{ijk}\hat{n}_i' E_k\, da' = \int \hat{n}' \times \vec{E}\, da' \end{aligned}$$

Note that for integration by parts, we use the fact that $\hat{n}'$ is a constant surface normal vector and that $E_{||}$ vanishes at the edges of the aperture. More precisely, the generalization of Stokes' theorem indicates that the surface term is of the form

$$\oint \vec{x}\,'(\vec{E} \cdot d\vec{l}\,)$$

so the electric field contribution indeed arises only from the parallel component to the edge of the aperture. Finally, substituting this integrated relation between $\vec{E}$ and $\vec{B}$ into (11) gives

$$\vec{m} = \frac{2}{i\omega\mu_0} \int \hat{n}' \times \vec{E}\, da' = \frac{2}{\mu_0} \int \vec{x}\,'(\hat{n}' \cdot \vec{B})\, da'$$