Physics 506 **Winter 2006**

Homework Assignment #5 — Solutions

Textbook problems: Ch. 9: 9.19, 9.22 *a*) and *b*), 9.23
Ch. 10: 10.1

9.19 Consider the excitation of a waveguide in Problem 8.19 from the point of view of multipole moments of the source.

a) For the linear probe antenna calculate the multipole moment components of $\vec{p}$, $\vec{m}$, $Q_{\alpha\beta}$, $Q^M_{\alpha\beta}$ that enter (9.69).

Note that the multipole moments depend on the choice of origin of the coordinates used to describe the source. Perhaps the most natural situation is to place the origin at the bottom of the probe. In this case, the source current density can be expressed as

$$\vec{J} = \hat{y} I_0 \sin[\tfrac{\omega}{c}(h-y)]\delta(x)\delta(z)\Theta(h-y)$$

The electric dipole moment can be computed directly from the current density

$$\begin{aligned}\vec{p} &= \frac{i}{\omega}\int \vec{J}\,d^3x = \frac{iI_0}{\omega}\hat{y}\int_0^h \sin[\tfrac{\omega}{c}(h-y)]\,dy \\ &= \frac{icI_0}{\omega^2}\left(1-\cos\frac{\omega h}{c}\right)\hat{y} = \frac{2icI_0}{\omega^2}\sin^2\left(\frac{\omega h}{2c}\right)\hat{y}\end{aligned}$$

For the magnetic dipole moment, we first compute the magnetization

$$\begin{aligned}\vec{\mathcal{M}} &= \tfrac{1}{2}\vec{x}\times\vec{J} = \tfrac{1}{2}(x,y,z)\times\hat{y}I_0\sin[\tfrac{\omega}{c}(h-y)]\delta(x)\delta(z)\Theta(h-y) \\ &= \tfrac{1}{2}I_0(-z,0,x)\sin[\tfrac{\omega}{c}(h-y)]\delta(x)\delta(z)\Theta(h-y) \\ &= 0\end{aligned}$$

since $x = z = 0$ is enforced by the delta functions. This indicates that the magnetic dipole moment $\vec{m} = \int \vec{\mathcal{M}}\,d^3x$ vanishes for the probe antenna. Furthermore, since the magnetic quadrupole moment $Q^M_{\alpha\beta}$ may be computed using the effective magnetic charge density $\rho^M = -\vec{\nabla}\cdot\vec{\mathcal{M}}$, it also vanishes. So we are left with the electric quadrupole moment

$$Q_{\alpha\beta} = \int (3x_\alpha x_\beta - \delta_{\alpha\beta}r^2)\rho\,d^3x$$

where

$$\rho = -\frac{i}{\omega}\vec{\nabla}\cdot\vec{J} = \frac{iI_0}{c}\cos[\tfrac{\omega}{c}(h-y)]\delta(x)\delta(z)\Theta(h-y)$$

Because of the delta functions, the only non-vanishing quadrupole moments are

$$Q_{xx} = Q_{zz} = -\tfrac{1}{2}Q_{yy} = -\int y^2 \rho\, d^3x = -\frac{iI_0}{c}\int_0^h y^2 \cos[\tfrac{\omega}{c}(h-y)]\, dy$$
$$= -\frac{2iI_0 c}{\omega^2}\left(h - \frac{c}{\omega}\sin\frac{\omega h}{c}\right)$$

The resulting multipole moments are

$$\begin{aligned}
\vec{p} &= \frac{2icI_0}{\omega^2}\sin^2\left(\frac{\omega h}{2c}\right)\hat{y} \\
\vec{m} &= 0 \\
Q_{xx} = Q_{zz} = -\tfrac{1}{2}Q_{yy} &= -\frac{2iI_0 c}{\omega^2}\left(h - \frac{c}{\omega}\sin\frac{\omega h}{c}\right) \\
Q^M_{\alpha\beta} &= 0
\end{aligned} \tag{1}$$

b) Calculate the amplitudes for excitation of the $\mathrm{TE}_{1,0}$ mode and evaluate the power flow. Compare the multipole expansion result with the answer given in Problem 8.19*b*). Discuss the reasons for agreement or disagreement. What about the comparison for excitation of other modes?

We consider the normalized TE_{mn} mode specified by the field

$$H_z = H_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

where

$$H_0 = \frac{2i\gamma_{mn}}{\mu_0\omega\sqrt{ab}}, \qquad (a \to 2a \text{ if } m = 0 \text{ and } b \to 2b \text{ if } n = 0)$$

and

$$\gamma_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Note that this choice of coordinates places $x = y = 0$ at the lower left of the rectangular waveguide. Since the magnetic moments vanish, we only need the explicit expressions for the electric field. Using

$$\vec{E}_t = -\frac{i\mu_0\omega}{\gamma_{mn}^2}\hat{z}\times\vec{\nabla}_t H_z$$

we obtain

$$\begin{aligned}
E_x &= -\frac{i\mu_0\omega}{\gamma_{mn}^2}\frac{n\pi}{b}H_0\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\
E_y &= \frac{i\mu_0\omega}{\gamma_{mn}^2}\frac{m\pi}{a}H_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\
E_z &= 0
\end{aligned} \tag{2}$$

The excitation amplitude for the TE_{mn} mode is given by an expansion in moments

$$\mathcal{A}_{mn}^{(\pm)} = \frac{i\omega Z_{mn}}{2}\Big[\vec{p}\cdot\vec{E}_{mn}^{(\mp)}(X,0,0) - \vec{m}\cdot\vec{B}_{mn}^{(\mp)}(X,0,0) \\ + \tfrac{1}{6}Q_{\alpha\beta}\partial_\beta E_{mn\,\alpha}^{(\mp)}(X,0,0) - \tfrac{1}{6}Q_{\alpha\beta}^{M}\partial_\beta B_{mn\,\alpha}^{(\mp)}(X,0,0) + \cdots\Big]$$

Using $Z_{mn} = \mu_0\omega/k_{mn}$ for a TE mode, and specializing to the non-vanishing components of (1), we have

$$\mathcal{A}^{(\pm)} = \frac{i\mu_0\omega^2}{2k_{mn}}\left[p_y E_y + \tfrac{1}{6}Q_{xx}(\partial_x E_x - 2\partial_y E_y + \partial_z E_z) + \cdots\right]$$

where the electric field components are evaluated at $(X,0,0)$. In particular

$$E_y = \frac{i\mu_0\omega}{\gamma_{mn}^2}\frac{m\pi}{a}H_0\sin\frac{m\pi X}{a}$$

while $\partial_x E_x$ and $\partial_y E_y$ vanish because they are evaluated at $y=0$. This results in an excitation amplitude

$$\begin{aligned}\mathcal{A}_{mn}^{(\pm)} &= \frac{i\mu_0\omega^2}{2k_{mn}}\left[\frac{2icI_0}{\omega^2}\sin^2\left(\frac{\omega h}{2c}\right)\frac{i\mu_0\omega}{\gamma_{mn}^2}\frac{m\pi}{a}H_0\sin\frac{m\pi X}{a}\right] \\ &= \frac{2\mu_0 c I_0}{k_{mn}\gamma_{mn}\sqrt{ab}}\frac{m\pi}{a}\sin\frac{m\pi X}{a}\sin^2\left(\frac{\omega h}{2c}\right)\end{aligned}$$

where we have substituted in the value of H_0. This may be compared with the exact TE result from Problem 8.19, rewritten as

$$\begin{aligned}\mathcal{A}_{mn\,\mathrm{exact}}^{(\pm)} &= \frac{2\mu_0 c I_0}{k_{mn}\gamma_{mn}\sqrt{ab}}\frac{m\pi}{a}\sin\frac{m\pi X}{a}\sin^2\left(\frac{\omega h}{2c}\right) \\ &\quad\times\left(1-\frac{\sin^2(n\pi h/2b)}{\sin^2(\omega h/2c)}\right)\left(1-\left(\frac{n\pi c}{b\omega}\right)^2\right)^{-1}\end{aligned}$$

This demonstrates that $\mathcal{A}_{m,0}^{(\pm)}$ calculated using the electric dipole moment is in fact exact. In particular

$$\mathcal{A}_{1,0}^{(\pm)} = \frac{2\mu_0 c I_0}{k_{10}\sqrt{2ab}}\sin\frac{\pi X}{a}\sin^2\left(\frac{\omega h}{2c}\right)$$

(where we fixed the normalization for the $n=0$ case), resulting in the exact power expression

$$P_{1,0} = \frac{\mu_0 c^2}{\omega k_{10} ab}|I_0|^2\sin^2\left(\frac{\pi X}{a}\right)\sin^4\left(\frac{\omega h}{2c}\right)$$

The reason the electric dipole contribution is exact for $n = 0$ modes is that these modes are independent of y (zero modes in y). This is easily seen from (2), where the substitution $n = 0$ gives

$$
\begin{aligned}
E_x &= 0 \\
E_y &= \frac{i\mu_0\omega}{\gamma_{m0}^2}\frac{m\pi}{a}H_0 \sin\frac{m\pi x}{a} \\
E_z &= 0
\end{aligned}
$$

This constant electric field in the $\hat{y}$ direction couples only to the electric dipole moment, and not to any higher moments which we have ignored. The $n \neq 0$ modes are sensitive to higher multipole moments, however, and this is why the lowest multipole contributions to their excitation coefficients are no longer exact.

9.22 A spherical hole of radius a in a conducting medium can serve as an electromagnetic resonant cavity.

a) Assuming infinite conductivity, determine the transcendental equations for the characteristic frequencies ω_{lm} of the cavity for TE and TM modes.

We may treat this spherical cavity as imposing boundary conditions at $r = a$ on spherical waves. Starting with waves in free space, we recall that the vector spherical wave expansion may be written as

$$
\begin{aligned}
\vec{H} &= \sum_{lm}\left[a_E(l,m)j_l(kr)\vec{X}_{lm} - \frac{i}{k}a_M(l,m)\vec{\nabla}\times j_l(kr)\vec{X}_{lm}\right] \\
\vec{E} &= Z_0\sum_{lm}\left[\frac{i}{k}a_E(l,m)\vec{\nabla}\times j_l(kr)\vec{X}_{lm} + a_M(l,m)j_l(kr)\vec{X}_{lm}\right]
\end{aligned}
$$

where we used the spherical Bessel functions $j_l(kr)$ which are regular at $r = 0$. Noting that

$$
\begin{aligned}
\hat{r}\cdot\vec{H} &= -\frac{i}{k}\sum_{lm}a_M(l,m)\hat{r}\cdot\vec{\nabla}\times j_l(kr)\vec{X}_{lm} \\
&= -\frac{i}{k}\sum_{lm}a_M(l,m)\hat{r}\times\vec{\nabla}\cdot j_l(kr)\vec{X}_{lm} \\
&= \sum_{lm}\frac{1}{kr}a_M(l,m)j_l(kr)\vec{L}\cdot\vec{X}_{lm} \\
&= \sum_{lm}\frac{\sqrt{l(l+1)}}{kr}a_M(l,m)j_l(kr)Y_{lm}
\end{aligned}
$$

and similarly

$$
\hat{r}\cdot\vec{E} = -Z_0\sum_{lm}\frac{\sqrt{l(l+1)}}{kr}a_E(l,m)j_l(kr)Y_{lm}
$$

we see that the modes parametrized by $a_M(l,m)$ are TE modes, while those parametrized by $a_E(l,m)$ are TM modes.

In particular, the TE modes may be given by

$$\vec{H} = -\frac{i}{k}\vec{\nabla} \times j_l(kr)\vec{X}_{lm}, \qquad \vec{E} = Z_0 j_l(kr)\vec{X}_{lm} \tag{3}$$

We now impose the boundary conditions $H_\perp = 0$ and $E_\parallel = 0$, or more precisely

$$\hat{r}\cdot\vec{H}\Big|_{r=a} = 0, \qquad \hat{r}\times\vec{E}\Big|_{r=a} = 0$$

These are equivalent to the condition $j_l(ka) = 0$, and leads to the quantization $k_{lmn} = x_{ln}/a$ where x_{ln} is the n-th zero of the spherical Bessel function j_l. The TE_{lmn} frequencies are thus

$$\omega_{lmn} = \frac{x_{ln}c}{a}, \qquad j_l(x_{ln}) = 0, \qquad l \geq 1, \quad |m| \leq l$$

Each frequency specified by l and n is $(2l+1)$-fold degenerate, with azimuthal quantum number labeled by m.

The TM modes are similar, although the boundary conditions are somewhat more involved. The modes themselves are given by

$$\vec{H} = j_l(kr)\vec{X}_{lm}, \qquad \vec{E} = Z_0\frac{i}{k}\vec{\nabla}\times j_l(kr)\vec{X}_{lm} \tag{4}$$

This time, the $H_\perp = 0$ boundary condition is automatic, while the $E_\parallel = 0$ condition gives

$$\vec{r}\times(\vec{\nabla}\times j_l(kr)\vec{X}_{lm})\Big|_{r=a} = 0$$

This vector quantity may be simplified using

$$\vec{r}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{r}\cdot\vec{V}) - \vec{V} - (\vec{r}\cdot\vec{\nabla})\vec{V} = \vec{\nabla}(\vec{r}\cdot\vec{V}) - (1+r\partial_r)\vec{V} = \vec{\nabla}(\vec{r}\cdot\vec{V}) - \partial_r r\vec{V}$$

Using $\vec{V} = j_l(kr)\vec{X}_{lm}$ with $\vec{r}\cdot\vec{X}_{lm} = 0$ gives

$$\vec{r}\times(\nabla\times j_l(kr)\vec{X}_{lm}) = -\partial_r(rj_l(kr))\vec{X}_{lm} \tag{5}$$

Hence the $E_\parallel = 0$ boundary condition leads to the TM_{lmn} frequencies

$$\omega_{lmn} = \frac{y_{ln}c}{a}, \qquad \frac{d}{dx}[xj_l(x)]\Big|_{z=y_{ln}} = 0, \qquad l \geq 1, \quad |m| \leq 1$$

The y_{ln} correspond to zeros of $[xj_l(x)]'$ or equivalently $j_l(x) + xj_l'(x)$.

b) Calculate numerical values for the wavelength λ_{lm} in units of the radius a for the four lowest modes for TE and TM waves.

The numerical values for the wavelengths are obtained from the zeros x_{ln} and y_{ln}. For TE modes, the first four zeros of $j_l(x)$ are

$$x_{11} = 4.4934, \qquad x_{21} = 5.7635, \qquad x_{31} = 6.9879, \qquad x_{12} = 7.7253$$

Since $k_{lmn} = x_{ln}/a$ and $\lambda_{lmn} = 2\pi/k_{lmn}$, we end up with $\lambda_{lmn}/a = 2\pi/x_{ln}$ or

$$\frac{\lambda_{1m1}}{a} = 1.398, \qquad \frac{\lambda_{2m1}}{a} = 1.090, \qquad \frac{\lambda_{3m1}}{a} = 0.899, \qquad \frac{\lambda_{1m2}}{a} = 0.813$$

All these modes are $(2l+1)$-fold degenerate. For TM modes, the first four zeros of $[xj_l(x)]'$ are

$$y_{11} = 2.7437, \qquad y_{21} = 3.8702, \qquad y_{31} = 4.9734, \qquad y_{41} = 6.0619$$

with corresponding wavelengths

$$\frac{\lambda_{1m1}}{a} = 2.290, \qquad \frac{\lambda_{2m1}}{a} = 1.623, \qquad \frac{\lambda_{3m1}}{a} = 1.263, \qquad \frac{\lambda_{4m1}}{a} = 1.036$$

Note that the next mode, given by $y_{12} = 6.1168$ is nearly degenerate with y_{41}.

9.23 The spherical resonant cavity of Problem 9.22 has nonpermeable walls of large, but finite, conductivity. In the approximation that the skin depth δ is small compared to the cavity radius a, show that the Q of the cavity, defined by equation (8.86), is given by

$$Q = \frac{a}{\delta} \qquad \text{for all TE modes}$$
$$Q = \frac{a}{\delta}\left(1 - \frac{l(l+1)}{x_{lm}^2}\right) \qquad \text{for TM modes}$$

where $x_{lm} = (a/c)\omega_{lm}$ for TM modes.

In order to calculate the Q factor, we need to obtain both the stored energy and the power loss at the walls. We start with the simpler case of TE modes, given by (3). The energy density for harmonic fields is

$$u = \frac{\epsilon_0}{4}|\vec{E}|^2 + \frac{\mu_0}{4}|\vec{H}|^2$$

However, the energy is equally distributed between $\vec{E}$ and $\vec{H}$. Thus for TE modes we may immediately write down

$$u = \frac{\epsilon_0}{2}|\vec{E}|^2 = \frac{\mu_0}{2} j_l(kr)^2 |\vec{X}_{lm}|^2$$

The stored energy is given by integrating this over the volume of the sphere

$$U = \frac{\mu_0}{2} \int j_l(kr)^2 |\vec{X}_{lm}|^2 \, r^2 dr d\Omega = \frac{\mu_0}{2} \int_0^a j_l(kr)^2 r^2 dr$$

We now use the normalization integral for spherical Bessel functions

$$\int_0^a j_l(x_{lm}\rho/a) j_l(x_{ln}\rho/a)\rho^2 d\rho = \tfrac{1}{2} a^3 [j_l'(x_{ln})]^2 \delta_{mn}$$

to obtain

$$U_{lmn} = \frac{\mu_0 a^3}{4} j_l'(x_{ln})^2 \tag{6}$$

The power loss is given in terms of the tangential magnetic field at the conducting surface

$$P = \frac{1}{2\sigma\delta} \int |\hat{r} \times \vec{H}|^2 da$$

Using $\vec{h} = -(i/k)\vec{\nabla} \times j_l(kr)\vec{X}_{lm}$ as well as the vector identity (5) gives

$$\begin{aligned} P_{lmn} &= \frac{1}{2\sigma\delta} \int_{r=a} \left(\frac{1}{kr} \frac{d}{dr} r j_l(kr) \right)^2 |\vec{X}_{lm}|^2 r^2 d\Omega \\ &= \frac{1}{2\sigma\delta k^2} ([r j_l(kr)]')^2 \bigg|_{r=a} \\ &= \frac{1}{2\sigma\delta k^2} (j_l(ka) + ka j_l'(ka))^2 = \frac{a^2}{2\sigma\delta} j_l'(x_{ln})^2 \end{aligned} \tag{7}$$

where in the last line we made use of the fact that $ka = x_{ln}$ and that $j_l(x_{ln}) = 0$. Combining (6) and (7) then gives the Q factor for TE modes

$$Q_{lmn} = \omega \frac{U_{lmn}}{P_{lmn}} = \frac{\mu_0 \sigma \omega \delta a}{2} = \frac{a}{\delta}$$

where we made use of the definition of the skin depth $\delta = \sqrt{2/\mu_0\sigma\omega}$. The calculation for TM modes is similar. However, the appropriate spherical Bessel function normalization integral needs to be modified for integrating to zeros of $[x j_l(x)]'$. Here we simply state that the appropriate normalization integral may be written as

$$\int_0^a j_l(\alpha_m \rho/a) j_l(\alpha_n \rho/a)\rho^2 d\rho = \tfrac{1}{2} a^3 \left(1 + \frac{p(p-1) - l(l+1)}{\alpha_n^2} \right) [j_l(\alpha_n)]^2 \delta_{mn}$$

where α_n is the n-th positive zero of

$$[x^p j_l(x)]' = 0$$

Setting $p = 1$ for TM modes, and using the notation y_{ln} to denote the n-th zero of $[xj_l(x)]' = 0$, the expression for the stored energy becomes

$$U_{lmn} = \frac{\mu_0}{2}\int_0^a j_l(kr)^2 r^2 dr = \frac{\mu_0 a^3}{4}\left(1 - \frac{l(l+1)}{y_{ln}^2}\right) j_l(y_{ln})^2$$

The power loss is

$$P_{lmn} = \frac{1}{2\sigma\delta}\int |\hat{r}\times\vec{H}|^2 da = \frac{1}{2\sigma\delta}\int_{r=a} j_l(kr)^2|\hat{r}\times\vec{X}_{lm}^2|r^2 d\Omega = \frac{a^2}{2\sigma\delta} j_l(y_{mn})^2$$

As a result, the Q factor for a TM_{lmn} mode is

$$Q_{lmn} = \omega\frac{U_{lmn}}{P_{lmn}} = \frac{\mu_0\sigma\omega\delta a}{2}\left(1 - \frac{l(l+1)}{y_{ln}^2}\right) = \frac{a}{\delta}\left(1 - \frac{l(l+1)}{y_{ln}^2}\right)$$

10.1 *a*) Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius a, summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega}(\vec{\epsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6\left[\frac{5}{4} - |\vec{\epsilon}_0\cdot\hat{n}|^2 - \frac{1}{4}|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 - \hat{n}_0\cdot\hat{n}\right]$$

where $\hat{n}_0$ and $\hat{n}$ are the directions of the incident and scattered radiations, respectively, while $\vec{\epsilon}_0$ is the (perhaps complex) unit polarization vector of the incident radiation ($\vec{\epsilon}_0{}^*\cdot\vec{\epsilon}_0 = 1$; $\hat{n}_0\cdot\vec{\epsilon}_0 = 0$).

If all polarizations are specified, the conducting sphere scattering cross section is given by

$$\frac{d\sigma}{d\Omega}(\hat{n}, \vec{\epsilon}; \hat{n}_0, \vec{\epsilon}_0) = k^4 a^6|\vec{\epsilon}^*\cdot\vec{\epsilon}_0 - \tfrac{1}{2}(\hat{n}\times\vec{\epsilon}^*)\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 \tag{8}$$

What we would like to do is to sum this over both orthogonal outgoing polarizations. One way to do this is to introduce a linear polarization basis transverse to the outgoing direction $\hat{n}$. To do so, we first assume the scattering is not in the forward direction. Then the incoming direction $\hat{n}_0$ may be used to define orthogonal polarizations

$$\vec{\epsilon}^{\,1} = \frac{\hat{n}\times\hat{n}_0}{\sin\theta}, \qquad \vec{\epsilon}^{\,2} = \hat{n}\times\vec{\epsilon}^{\,1} = \frac{\hat{n}(\hat{n}\cdot\hat{n}_0) - \hat{n}_0}{\sin\theta}$$

where θ is the angle between $\hat{n}$ and $\hat{n}_0$. In particular, we may write $\sin^2\theta = 1 - (\hat{n}\cdot\hat{n}_0)^2$. In this case, the cross section summed over outgoing polarizations becomes

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\hat n;\hat n_0,\vec\epsilon_0) = \frac{k^4a^6}{1-(\hat n\cdot\hat n_0)^2}\big[&|(\hat n\times\hat n_0)\cdot\vec\epsilon_0 - \tfrac12(\hat n\times(\hat n\times\hat n_0))\cdot(\hat n_0\times\vec\epsilon_0)|^2\\
&+|(\hat n(\hat n\cdot\hat n_0)-\hat n_0)\cdot\vec\epsilon_0 - \tfrac12(\hat n\times(\hat n(\hat n\cdot\hat n_0)-\hat n_0))\cdot(\hat n_0\times\vec\epsilon_0)|^2\big]\\
= \frac{k^4a^6}{1-(\hat n\cdot\hat n_0)^2}\big[&|(\hat n\times\hat n_0)\cdot\vec\epsilon_0 - \tfrac12(\hat n(\hat n\cdot\hat n_0)-\hat n_0)\cdot(\hat n_0\times\vec\epsilon_0)|^2\\
&+|(\hat n\cdot\hat n_0)(\hat n\cdot\vec\epsilon_0) - \tfrac12(\hat n_0\times\hat n)\cdot(\hat n_0\times\vec\epsilon_0)|^2\big]\\
= \frac{k^4a^6}{1-(\hat n\cdot\hat n_0)^2}\big[&|\hat n\cdot(\hat n_0\times\vec\epsilon_0) - \tfrac12(\hat n\cdot\hat n_0)\hat n\cdot(\hat n_0\times\vec\epsilon_0)|^2\\
&+|(\hat n\cdot\hat n_0)(\hat n\cdot\vec\epsilon_0) - \tfrac12(\hat n\cdot\vec\epsilon_0)|^2\big]\\
= \frac{k^4a^6}{1-(\hat n\cdot\hat n_0)^2}\big[&|\hat n\cdot(\hat n_0\times\vec\epsilon_0)|^2(1-\tfrac12(\hat n\cdot\hat n_0))^2\\
&+|\hat n\cdot\vec\epsilon_0|^2(\tfrac12-(\hat n\cdot\hat n_0))^2\big]
\end{aligned}$$

Note that we have used transversality of the initial polarization, $\hat n_0\cdot\vec\epsilon_0=0$. To proceed, we expand the squares and rewrite the above as

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\hat n;\hat n_0,\vec\epsilon_0) = \frac{k^4a^6}{1-(\hat n\cdot\hat n_0)^2}\big[&(\tfrac54-(\hat n\cdot\hat n_0))(|\hat n\cdot(\hat n_0\times\vec\epsilon_0)|^2+|\hat n\cdot\vec\epsilon_0|^2)\\
&-(1-(\hat n\cdot\hat n_0)^2)(\tfrac14|\hat n\cdot(\hat n_0\times\vec\epsilon_0)|^2+|\hat n\cdot\vec\epsilon_0|^2\big]
\end{aligned}\tag{9}$$

The second line cancels the denominator. However the first line needs a bit of work. We now use the fact that ϵ_0 is a unit polarization vector orthogonal to $\hat n_0$. As a result, the three vectors

$$\hat n_0,\qquad \vec\epsilon_0,\qquad \hat n_0\times\vec\epsilon_0\tag{10}$$

form a normalized right-handed coordinate basis spanning the three-dimensional space. (There is a slight subtlety if $\vec\epsilon_0$ is complex, although the end result is okay, provided we are careful with magnitude squares.) The components of $\hat n$ expanded in this basis are

$$\hat n\cdot\hat n_0,\qquad \hat n\cdot\vec\epsilon_0,\qquad \hat n\cdot(\hat n_0\times\vec\epsilon_0)$$

and since $\hat n$ is a unit vector, the sum of the squares of these components must be one. In other words

$$(\hat n\cdot\hat n_0)^2+|\hat n\cdot\vec\epsilon_0|^2+|\hat n\cdot(\hat n_0\times\vec\epsilon_0)|^2=1$$

where we have been careful about complex quantities. Using this result, we see that the denominator in (9) can be completely eliminated, resulting in

$$\frac{d\sigma}{d\Omega}(\hat n;\hat n_0,\vec\epsilon_0) = k^4a^6[\tfrac54-(\hat n\cdot\hat n_0)-\tfrac14|\hat n\cdot(\hat n_0\times\vec\epsilon_0)|^2-|\hat n\cdot\vec\epsilon_0|^2]\tag{11}$$

b) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega}(\vec{\epsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right]$$

where $\hat{n}\cdot\hat{n}_0 = \cos\theta$ and the azimuthal angle ϕ is measured from the direction of the linear polarization.

As stated, the scattering angle θ is given by $\hat{n}\cdot\hat{n}_0 = \cos\theta$. The azimuthal angle ϕ is the one between $\hat{n}$ and $\vec{\epsilon}_0$, measured in the plan perpendicular to $\hat{n}_0$. What this means is that, using the basis vectors (10) with $\vec{\epsilon}_0$ real, the components of $\hat{n}$ can be written as

$$\hat{n} = \hat{n}_0\cos\theta + \vec{\epsilon}_0\sin\theta\cos\phi + (\hat{n}_0\times\vec{\epsilon}_0)\sin\theta\sin\phi$$

or alternatively

$$\hat{n}\cdot\hat{n}_0 = \cos\theta, \qquad \hat{n}\cdot\vec{\epsilon}_0 = \sin\theta\cos\phi, \qquad \hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0) = \sin\theta\sin\phi$$

Substituting this into (11) gives

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\theta,\phi) &= k^4a^6[\tfrac{5}{4} - \cos\theta - \tfrac{1}{4}\sin^2\theta\sin^2\phi - \sin^2\theta\cos^2\phi] \\ &= k^4a^6[\tfrac{5}{4} - \cos\theta - \tfrac{1}{8}\sin^2\theta(1-\cos 2\phi) - \tfrac{1}{2}\sin^2\theta(1+\cos 2\phi)] \\ &= k^4a^6[\tfrac{5}{8}(1+\cos^2\theta) - \cos\theta - \tfrac{3}{8}\sin^2\theta\cos 2\phi]\end{aligned}$$

c) What is the ratio of scattered intensities at $\theta = \pi/2$, $\phi = 0$ and $\theta = \pi/2$, $\phi = \pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

At $\theta = \pi/2$, we have

$$\frac{d\sigma}{d}\Omega(\pi/2,\phi) = k^4a^6[\tfrac{5}{8} - \tfrac{3}{8}\cos 2\phi]$$

Hence

$$\frac{d\sigma}{d\Omega}(\pi/2, 0) = \tfrac{1}{4}k^4a^6, \qquad \frac{d\sigma}{d\Omega}(\pi/2,\pi/2) = k^4a^6$$

Scattering at 90° is fairly easy to understand physically. For $\phi = 0$, the scattered wave is lined up with the incident polarization ϵ_0. Since the polarization is given by the electric field vector, this indicates that the induced electric dipole of the sphere is lined up with the direction of the scattered wave. Since the radiation must be transverse, no dipole radiation can be emitted on axis, and in this case the scattering must be purely magnetic dipole in nature. On the other hand, for $\phi = \pi/2$, the scattered wave is lined up with the incident magnetic field, and hence the scattering must be purely electric dipole in nature. This demonstrates that the maximum strength of magnetic dipole scattering is a quarter that of electric dipole scattering. This is in fact evident by the factor of 1/2 in the magnetic dipole term in the cross section expression (8).