Homework Assignment #4 — Solutions

Textbook problems: Ch. 9: 9.8, 9.9, 9.16, 9.17

9.8 *a)* Show that a classical oscillating electric dipole $\vec{p}$ with fields given by (9.18) radiates electromagnetic angular momentum to infinity at the rate

$$\frac{d\vec{L}}{dt} = \frac{k^3}{12\pi\epsilon_0}\Im[\vec{p}^* \times \vec{p}]$$

To obtain the electromagnetic angular momentum, we begin with the linear momentum density

$$\vec{g} = \frac{1}{2c^2}\vec{E} \times \vec{H}^*$$

Denoting the angular momentum density by $\vec{m}$, we then have simply

$$\vec{m} = \vec{r} \times \vec{g} = \frac{1}{2c^2}\vec{r} \times (\vec{E} \times \vec{H}^*) = \frac{1}{2c^2}[\vec{E}(\vec{r} \cdot \vec{H}^*) - \vec{H}^*(\vec{r} \cdot \vec{E})]$$

Note that $\vec{r} \cdot \vec{H} = 0$ for an electric dipole field, while it is straightforward to show that

$$\vec{r} \cdot \vec{E} = r\hat{n} \cdot \vec{E} = \frac{1}{4\pi\epsilon_0}(2\hat{n} \cdot \vec{p})\left(\frac{1}{r^2} - \frac{ik}{r}\right)e^{ikr}$$

Hence

$$\vec{m} = -\frac{1}{2c^2}\vec{H}^*(\vec{r} \cdot \vec{E})$$

$$= -\frac{1}{2c^2}\frac{ck^2}{4\pi}(\hat{n} \times \vec{p}^*)\frac{e^{-ikr}}{r}\left(1 + \frac{1}{ikr}\right)\frac{1}{4\pi\epsilon_0}(2\hat{n} \cdot \vec{p})\left(\frac{1}{r^2} - \frac{ik}{r}\right)e^{ikr}$$

$$= \frac{ik^3}{16\pi^2\epsilon_0 cr^2}\left(1 + \frac{1}{(kr)^2}\right)(\hat{n} \cdot \vec{p})(\hat{n} \times \vec{p}^*)$$

In principle, if we integrate this over all space, we will end up with the total angular momentum contained in the electromagnetic field. However, this is not what we want. Instead, we are interested in the amount of angular momentum radiated to infinity. We can calculate this by considering the amount of angular momentum that passes through a spherical shell of (large) radius r in a given unit of time. Essentially, $d\vec{L} = \vec{m}\,da\,dr = \vec{m}\,r^2\,dr\,d\Omega$, so that $d\vec{L}/dt = \vec{m}\,r^2(dr/dt)d\Omega$. Noting that the radiation travels outward at the speed of light gives

$$\frac{d\vec{L}}{dt} = r^2 c \int \vec{m}\,d\Omega = \frac{ik^3}{16\pi^2\epsilon_0}\left(1 + \frac{1}{(kr)^2}\right)\int(\hat{n} \cdot \vec{p})(\hat{n} \times \vec{p}^*)d\Omega$$

It is straightforward to evaluate the angular integral. By appealing to symmetry considerations, we have

$$\int \hat{n}_i \hat{n}_j \, d\Omega = \delta_{ij} \int \hat{n}_1 \hat{n}_1 \, d\Omega = \frac{1}{3}\delta_{ij} \int (\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2) d\Omega = \frac{1}{3}\delta_{ij} \int d\Omega = \frac{4\pi}{3}\delta_{ij}$$

Thus when we take the limit $r \to \infty$ we obtain simply

$$\frac{d\vec{L}}{dt} = \frac{ik^3}{16\pi^2\epsilon_0}\frac{4\pi}{3}\vec{p} \times \vec{p}^{\,*} = -\frac{ik^3}{12\pi\epsilon_0}\vec{p}^{\,*} \times \vec{p}$$

The time averaged value is the real part of the above expression. Noting the $-i$ factor, this gives

$$\frac{d\vec{L}}{dt} = \frac{k^3}{12\pi\epsilon_0}\Im[\vec{p}^{\,*} \times \vec{p}] \tag{1}$$

b) What is the ratio of angular momentum radiated to energy radiated? Interpret.

The energy radiated by the dipole is simply the radiated power. This was shown to be

$$\frac{dU}{dt} \equiv P - \frac{c^2 Z_0 k^4}{12\pi}|\vec{p}|^2$$

Hence

$$\frac{d\vec{L}/dt}{dU/dt} = \frac{1}{c^2 Z_0 \epsilon_0 k}\frac{\Im[\vec{p}^{\,*} \times \vec{p}]}{\vec{p}^{\,*} \cdot \vec{p}} = \frac{1}{\omega}\frac{\Im[\vec{p}^{\,*} \times \vec{p}]}{\vec{p}^{\,*} \cdot \vec{p}}$$

If we break up the dipole moment into real and imaginary parts, $\vec{p} = \vec{p}_1 + i\vec{p}_2$, we obtain

$$\frac{d\vec{L}/dt}{dU/dt} = \frac{1}{\omega}\frac{2\vec{p}_1 \times \vec{p}_2}{p_1^2 + p_2^2} = \frac{1}{\omega}\frac{2\hat{a}p_1 p_2 \sin\alpha}{p_1^2 + p_2^2}$$

where α represents the angle between $\vec{p}_1$ and $\vec{p}_2$. A simple application of the triangle inequality then demonstrates that

$$\frac{d\vec{L}}{dt} \leq \frac{1}{\omega}\frac{dU}{dt}$$

If we identify $U = \hbar\omega$ and $|\vec{L}| = \hbar$ (angular momentum 1 for a dipole field), then coherent radiation saturates this bound, while in general the radiated angular momentum is smaller. We may also interpret this inequality as an upper limit on the angular momentum carried away by a $l = 1$ dipole field.

c) For a charge e rotating in the x-y plane at radius a and angular speed ω, show that there is only a z component of radiated angular momentum with magnitude $dL_z/dt = e^2 k^3 a^2/6\pi\epsilon_0$. What about a charge oscillating along the z axis?

A moving charge corresponds to a time-dependent charge density

$$\rho = e\delta(x - a\cos\omega t)\delta(y - a\sin\omega t)\delta(z)$$

The dipole moment is thus the obvious time dependent expression

$$\vec{p}(\vec{x}, t) = \int \vec{x} \rho \, d^3x = ea(\hat{x} \cos \omega t + \hat{y} \sin \omega t)$$

which may be written as the real part of a complex quantity

$$\vec{p}(\vec{x}, t) = \Re[ea(\hat{x} + i\hat{y})e^{-i\omega t}]$$

Hiding the harmonic behavior gives a complex dipole moment

$$\vec{p} = ea(\hat{x} + i\hat{y})$$

This is the expression that ought to be substituted into (1). The result is

$$\frac{d\vec{L}}{dt} = \frac{k^3 e^2 a^2}{6\pi\epsilon_0} \hat{z}$$

Note that it is important that the charge is rotating in a circle. This gives a phase difference between the real and imaginary parts of $\vec{p}$. In fact, this 90° phase shift and equal magnitudes in $\hat{x}$ and $\hat{y}$ (perfect circle) saturates the radiated angular momentum bound.

For linear motion along the z axis, on the other hand, we have $\vec{p} = p\hat{z}$ (up to an unimportant overall phase). In this case we have simply $d\vec{L}/dt = 0$. It ought to be at least intuitively plausible that linear motion does not gives rise to angular momentum in the radiated fields.

d) What are the results corresponding to parts a) and b) for magnetic dipole radiation?

The simple treatment for magnetic dipole radiation is to make the substitution $\vec{E} \to Z_0 \vec{H}$, $Z_0 \vec{H} \to -\vec{E}$ and $\vec{p} \to \vec{m}/c$. In fact, all we need is the latter replacement in (1). This gives

$$\frac{d\vec{L}}{dt} = \frac{k^3}{12\pi\epsilon_0 c^2} \Im[\vec{m}^* \times \vec{m}] = \frac{\mu_0 k^3}{12\pi} \Im[\vec{m}^* \times \vec{m}]$$

Similarly, the power is

$$P = \frac{Z_0 k^4}{12\pi} |\vec{m}|^2$$

Hence the results of part b) are unchanged.

9.9 a) From the electric dipole fields with general time dependence of Problem 9.6, show that the total power and the total rate of radiation of angular momentum through a sphere at large radius r and time t are

$$P(t) = \frac{1}{6\pi\epsilon_0 c^3} \left(\frac{\partial^2 \vec{p}_{\text{ret}}}{\partial t^2} \right)^2$$

$$\frac{d\vec{L}_{\text{em}}}{dt} = \frac{1}{6\pi\epsilon_0 c^3} \left(\frac{\partial \vec{p}_{\text{ret}}}{\partial t} \times \frac{\partial^2 \vec{p}_{\text{ret}}}{\partial t^2} \right)$$

where the dipole moment $\vec{p}$ is evaluated at the retarded time $t' = t - r/c$.

For real fields with explicit time dependence, the Poynting vector is simply $\vec{S} = \vec{E} \times \vec{H}$. Hence the angular power distribution is

$$\frac{dP}{d\Omega} = r^2 \hat{n} \cdot (\vec{E} \times \vec{H})$$

In the radiation zone, the result of Problem 9.6 may be given as

$$\vec{H} = -\frac{1}{4\pi cr} \hat{n} \times \frac{\partial^2 \vec{p}}{\partial t^2}, \qquad \vec{E} = -\frac{1}{\epsilon_0 c} \hat{n} \times \vec{H}$$

where all dipole expressions should be evaluated at the retarded time. This gives

$$\frac{dP}{d\Omega} = -\frac{r^2}{\epsilon_0 c} \hat{n} \cdot ((\hat{n} \times \vec{H}) \times \vec{H}) = \frac{r^2}{\epsilon_0 c} |\vec{H}|^2$$

where we have used the fact that $\hat{n} \cdot \vec{H} = 0$. Substituting in the expression for $\vec{H}$ and noting that (for any vector $\vec{V}$)

$$(\hat{n} \times \vec{V}) \cdot (\hat{n} \times \vec{V}) = |\vec{V}|^2 - |\hat{n} \cdot \vec{V}|^2 = V^2 - V^2 \cos^2 \alpha = V^2 \sin^2 \alpha$$

where θ is the angle between $\hat{n}$ and $\vec{V}$, we obtain

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2 \epsilon_0 c^3} \left(\frac{\partial^2 \vec{p}}{\partial t^2} \right)^2 \sin^2 \alpha$$

This expression is not entirely useful in itself, as the angle α may be some complicated function of the retarded time. However, this factor drops out after integrating over the entire solid angle. The standard result $\int \sin^2 \alpha \, d\Omega = 8\pi/3$ then gives us

$$P = \frac{1}{6\pi \epsilon_0 c^3} \left(\frac{\partial^2 \vec{p}}{\partial t^2} \right)^2 \tag{2}$$

(to be evaluated at the retarded time).

Working out the radiated angular momentum involves similar manipulations. We have

$$\frac{d\vec{L}}{dt} = cr^2 \int d\Omega \, \vec{r} \times (\frac{1}{c^2} \vec{E} \times \vec{H})$$

$$= \frac{r^3}{c} \int d\Omega [\vec{E}(\hat{n} \cdot \vec{H}) - \vec{H}(\hat{n} \cdot \vec{E})] = -\frac{r^3}{c} \vec{H}(\hat{n} \cdot \vec{E})$$

$$= -\frac{r^3}{c} \int d\Omega \left(-\frac{1}{4\pi r^2} \right) \left(1 + \frac{r}{c} \frac{\partial}{\partial t} \right) \hat{n} \times \frac{\partial \vec{p}}{\partial t} \left(\frac{1}{4\pi \epsilon_0} \left(1 + \frac{r}{c} \frac{\partial}{\partial t} \right) \frac{2\hat{n} \cdot \vec{p}}{r^3} \right)$$

$$= \frac{1}{8\pi^2 \epsilon_0 cr^2} \int d\Omega \left[\left(1 + \frac{r}{c} \frac{\partial}{\partial t} \right) \hat{n} \times \frac{\partial \vec{p}}{\partial t} \right] \left[\left(1 + \frac{r}{c} \frac{\partial}{\partial t} \right) \hat{n} \cdot \vec{p} \right]$$

Performing the angular integral as in the previous problem gives

$$\frac{d\vec{L}}{dt} = \frac{1}{6\pi\epsilon_0 cr^2} \left[\left(1 + \frac{r}{c}\frac{\partial}{\partial t} \right) \vec{p} \right] \times \left[\left(1 + \frac{r}{c}\frac{\partial}{\partial t} \right) \frac{\partial \vec{p}}{\partial t} \right]$$

In the limit that $r \to \infty$, the only term that contributes is

$$\frac{d\vec{L}}{dt} = \frac{1}{6\pi\epsilon_0 c^3} \left(\frac{\partial \vec{p}}{\partial t} \times \frac{\partial^2 \vec{p}}{\partial t^2} \right) \tag{3}$$

b) The dipole moment is caused by a particle of mass m and charge e moving nonrelativistically in a fixed central potential $V(r)$. Show that the radiated power and angular momentum for such a particle can be written as

$$P(t) = \frac{\tau}{m} \left(\frac{dV}{dr} \right)^2$$

$$\frac{d\vec{L}_{\text{em}}}{dt} = \frac{\tau}{m} \left(\frac{dV}{rdr} \right) \vec{L}$$

where $\tau = e^2/6\pi\epsilon_0 mc^3$ ($= 2e^2/3mc^3$ in Gaussian units) is a characteristic time, $\vec{L}$ is the particle's angular momentum, and the right-hand sides are evaluated at the retarded time. Related these results to those from the Abraham-Lorentz equation for radiation damping [Section 16.2].

For a single particle of mass m and charge e, the dipole moment is $\vec{p} = e\vec{x}$. Hence

$$\frac{\partial \vec{p}}{\partial t} = e\dot{\vec{x}} = \frac{e}{m}(m\dot{\vec{x}})$$

$$\frac{\partial^2 \vec{p}}{\partial t^2} = e\ddot{\vec{x}} = \frac{e}{m}(m\ddot{\vec{x}}) = \frac{e}{m}\vec{F} = -\frac{e}{m}\vec{\nabla}V = -\frac{e}{m}\hat{n}\frac{dV}{dr}$$

where we have used $F = ma$ in the second line as well as the fact that $V(r)$ is a central potential. Substituting this into (2) gives

$$P = \frac{e^2}{6\pi\epsilon_0 m^2 c^3} \left(\frac{dV}{dr} \right)^2 = \frac{\tau}{m} \left(\frac{dV}{dr} \right)^2$$

where $\tau = e^2/6\pi\epsilon_0 mc^3$. Similarly, substituting into (3) yields

$$\frac{d\vec{L}}{dt} = -\frac{e^2}{6\pi\epsilon_0 m^2 c^3}(m\dot{\vec{x}}) \times \vec{x} \left(\frac{1}{r}\frac{dV}{dr} \right) = \frac{\tau}{m} \left(\frac{dV}{r\,dr} \right) \vec{L}$$

where $\vec{L} = \vec{x} \times (m\dot{\vec{x}})$.

c) Suppose the charged particle is an electron in a hydrogen atom. Show that the inverse time defined by the ratio of the rate of angular momentum radiated to the particle's angular momentum is of the order of $\alpha^4 c/a_0$, where $\alpha = e^2/4\pi\epsilon_0 \hbar c \approx 1/137$ is the fine structure constant and a_0 is the Bohr radius. How does this inverse time compare to the observed rate of radiation in hydrogen atoms?

We estimate

$$\Gamma = \frac{dL/dt}{L} \approx \frac{\tau}{m} \left| \frac{dV}{r\, dr} \right|$$

For the hydrogen atom, we use the Coulomb potential $V = e^2/4\pi\epsilon_0 r$. As a result

$$\Gamma \approx \frac{e^2}{6\pi^2\epsilon_0 m^2 c^3} \left| \frac{1}{r} \frac{d}{dr} \frac{e^2}{4\pi\epsilon_0 r} \right| \approx \frac{e^4}{24\pi^3\epsilon_0^2 m^2 c^3} \frac{1}{a_0^3}$$

where in the last step we assumed that $r \approx a_0$, the Bohr radius. Noting that $a_0 = \hbar/\alpha m c$ as well as $\alpha = e^2/4\pi\epsilon_0 \hbar c$, the above expression simplifies to

$$\Gamma \approx \frac{2}{3\pi} \frac{\alpha^4 c}{a_0} \approx 0.212 \frac{\alpha^4 c}{a_0}$$

This may be compared, for example, with the quantum mechanical calculation of the width of the $2p \to 1s$ E1 transition of the hydrogen atom

$$\Gamma_{2p\to 1s} = \left(\frac{2}{3} \right)^8 \frac{\alpha^4 c}{a_0} = 0.039 \frac{\alpha^4 c}{a_0}$$

While the general agreement is fairly reasonably, it does indicate that small (or large) numerical factors, which may superficially be estimated as $\mathcal{O}(1)$, can show up in the detailed calculation.

d) Relate the expressions in parts a) and b) to those for harmonic time dependence in Problem 9.8.

Consider the mapping of Problem 9.8

$$\vec{p}_{\text{ret}} \to \vec{p}\, e^{i(kr-\omega t)}, \qquad \frac{\partial}{\partial t} \to -i\omega$$

As a result

$$\frac{\partial \vec{p}}{\partial t} \to -i\omega \vec{p}\, e^{i(kr-\omega t)} \qquad \text{and} \qquad \frac{\partial^2 \vec{p}}{\partial t^2} \to -\omega^2 \vec{p}\, e^{i(kr-\omega t)}$$

In addition, for harmonic fields, we would like to the the complex conjugate of one of the harmonic factors in either (2) or (3) and divide by two to compute the time averaged quantity. Thus

$$P = \frac{1}{6\pi\epsilon_0 c^3} \left(\frac{\partial^2 \vec{p}}{\partial t^2} \right)^2 \to \frac{1}{6\pi\epsilon_0 c^3} \omega^4 \left(\tfrac{1}{2} \vec{p} \cdot \vec{p}* \right) = \frac{ck^4}{12\pi\epsilon_0} |\vec{p}|^2$$

and

$$\frac{d\vec{L}}{dt} = \frac{1}{6\pi\epsilon_0 c^3}\left(\frac{\partial\vec{p}}{\partial t}\times\frac{\partial^2\vec{p}}{\partial t^2}\right) \to \frac{1}{6\pi\epsilon_0 c^3}i\omega^3(\tfrac{1}{2}\vec{p}\times\vec{p}*) = \frac{ik^3}{12\pi\epsilon_0}\vec{p}\times\vec{p}^*$$

We should, of course, take the real part of this expression. The result is the expected one

$$\frac{d\vec{L}}{dt} = \frac{k^3}{12\pi\epsilon_0}\Im[\vec{p}^*\times\vec{p}]$$

9.16 A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in the figure.

a) Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

Note that the current flows in opposite directions in the top and bottom half of this antenna. As a result, we may write the source current density as

$$\vec{J}(z) = \hat{z}I\sin(kz)\delta(x)\delta(y)\Theta(d/2 - |z|) \tag{4}$$

where

$$k = \frac{2\pi}{d}$$

In the radiation zone, the vector potential is given by

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')e^{-ik\hat{n}\cdot\vec{x}'}d^3x'$$

$$= \hat{z}\frac{\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_{-d/2}^{d/2}\sin(kz)e^{-ikz\cos\theta}dz$$

Since the source current is odd under $z \to -z$, this integral may be written as

$$\vec{A} = -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_0^{d/2}2\sin(kz)\sin(kz\cos\theta)\,dz$$

$$= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_0^{d/2}[\cos((1-\cos\theta)kz) - \cos((1+\cos\theta)kz)]\,dz$$

$$= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{kr}\left[\frac{1}{1-\cos\theta}\sin((1-\cos\theta)kz) - \frac{1}{1+\cos\theta}\sin((1+\cos\theta)kz)\right]_0^{d/2}$$

$$= -\hat{z}\frac{i\mu_0 I}{2\pi}e^{ikr}kr\frac{\sin(\pi\cos\theta)}{\sin^2\theta}$$

In the radiation zone, the magnetic field is

$$\vec{H} = \frac{ik}{\mu_0}\hat{n}\times\vec{A} = -\hat{\phi}\frac{I}{2\pi}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin\theta}$$

where we have used $\hat{n} \times \hat{z} \equiv \hat{r} \times \hat{z} = -\hat{\phi} \sin \theta$. This gives rise to a radiated power

$$\frac{dP}{d\Omega} = \frac{Z^2 r^2}{2}|\vec{H}|^2 = \frac{Z_0|I|^2}{8\pi^2}\frac{\sin^2(\pi \cos \theta)}{\sin^2 \theta} \tag{5}$$

This looks almost (but not quite) like a quadrupole pattern.
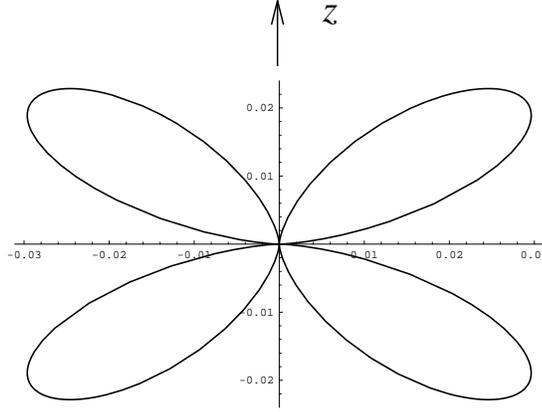
b) Determine the total power radiated and find a numerical value for the radiation resistance.

The total radiated power is given by integrating the angular distribution over the solid angle

$$P = \frac{Z_0|I|^2}{8\pi^2}2\pi \int_{-1}^{1} \frac{\sin^2(\pi \cos \theta)}{1 - \cos^2 \theta} d\cos \theta = \frac{Z_0|I|^2}{4\pi} \int_{-1}^{1} \frac{\sin^2(\pi x)}{1 - x^2} dx \approx \frac{Z_0|I|^2}{4\pi} \times 1.557$$

Comparing this with $P = \frac{1}{2}R_{\mathrm{rad}}|I|^2$ gives a radiation resistance

$$R_{\mathrm{rad}} = \frac{Z_0}{2\pi} \times 1.557 = 93.4 \,\Omega \tag{6}$$

9.17 Treat the linear antenna of Problem 9.16 by the multipole expansion method.

a) Calculate the multipole moments (electric dipole, magnetic dipole, and electric quadrupole) exactly and in the long-wavelength approximation.

Although the length of the antenna is equal to the wavelength (so the multipole expansion method is not particularly valid), we may still see what we get. Using the current density (4), we may obtain a charge density

$$\rho = \frac{1}{i\omega}\vec{\nabla} \cdot \vec{J} = \frac{1}{i\omega}\frac{d\vec{J}}{dz} = -\frac{iI}{c}\cos(kz)\delta(x)\delta(y)\Theta(d/2 - |z|)$$

where we used $\omega = ck$. The electric dipole moment is then

$$\vec{p} = \int \vec{x}\,\rho\,d^3x = -\hat{z}\frac{iI}{c}\int_{-d/2}^{d/2} z\cos(kz)\,dz = 0$$

Simple symmetry arguments under $z \to -z$ demonstrates that there is no electric dipole. The magnetic dipole moment also vanishes since

$$\vec{m} = \frac{1}{2} \int \vec{x} \times \vec{J} \, d^3x = \frac{I}{2} \int_{-d/2}^{d/2} \sin(kz)(z\hat{z}) \times \hat{z} \, dz = 0$$

We are left with an electric quadrupole moment

$$Q_{ij} = \int (3x_i x_j - r^2 \delta_{ij}) \rho \, d^3x = -\frac{iI}{c} \int_{-d/2}^{d/2} (3(z\delta_{i3})(z\delta_{j3}) - z^2 \delta_{ij}) \cos(kz) \, dz$$

The only non-vanishing moments are

$$Q_{33} = -2Q_{11} = -2Q_{22} = -\frac{2iI}{c} \int_{-d/2}^{d/2} z^2 \cos(kz) \, dz \qquad (7)$$

The integral is straightforward, and the result is

$$Q_{33} = -2Q_{11} = -2Q_{22} = \frac{8\pi i I}{ck^3}$$

So far, we have worked with the 'exact' multipole expressions. In the long wavelength limit $(kd \to 0)$, we really ought to modify the current density as appropriate. Nevertheless, by assuming the same symmetry as (4) under $z \to -z$, both dipole moments will of course vanish. For the quadrupole moment in the long wavelength limit, we take $\cos(kz) \to 1$ in (7), which corresponds to a uniform charge density. The result of the trivial integration is then

$$Q_{33} = -2Q_{11} = -2Q_{22} = -\frac{iId^3}{6c} = -\frac{8\pi i I}{ck^3} \frac{(kd)^3}{48\pi}$$

Since $kd \to 0$, this greatly underestimates the actual quadrupole moment given above. But of course we have to be extremely careful with this interpretation, as the current and charge densities (in particular, the current I) in the exact expressions do not directly carry over to the same counterparts in the long wavelength limit.

b) Compare the shape of the angular distribution of radiated power for the lowest nonvanishing multipole with the exact distribution of Problem 9.16.

The lowest multipole is the electric quadrupole. In this case, the angular distribution is

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{512\pi^2} |Q_0|^2 \sin^2\theta \cos^2\theta = \frac{Z_0 |I|^2}{8} \sin^2\theta \cos^2\theta$$

where $Q_0 = Q_{33} = -2Q_{11} = -2Q_{22}$. This is plotted in black, and may be compared with the exact distribution (5), plotted in green in using the same scale
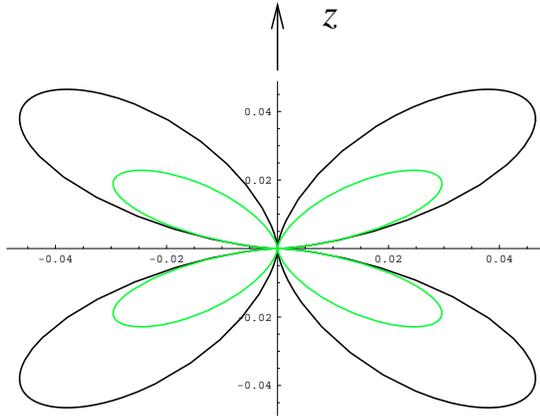
The comparison is more direct when the electric quadrupole radiation is normalized to the same total radiated power
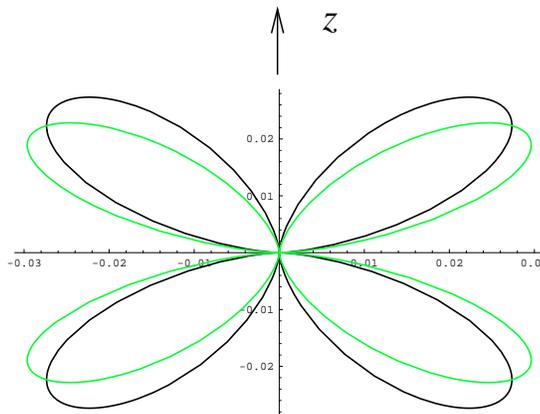
c) Determine the total power radiated for the lowest multipole and the corresponding radiation resistance using both multipole moments from part *a*). Compare with Problem 9.16*b*). Is there a paradox here?

The total power radiated using the exact electric quadrupole moment is

$$P = \frac{Z_0 |I|^2}{8} 2\pi \int_{-1}^{1} \sin^2 \theta \cos^2 \theta \, d\cos\theta = Z_0 |I|^2 \frac{\pi}{15}$$

This gives a radiation resistance of $R_{\text{rad}} = 2\pi Z_0 / 15 = 158\,\Omega$. Curiously, this (and the radiated power) is *larger* than the exact expression of (6). The reason this is not a paradox is that destructive interference from higher order terms in the source expansion will be able to bring this power down to the exact expression of (6). Here it is worth noting that this expansion in terms of *source multipole moments* is not the same as the the one for radiation multipoles. In the latter case, we would get a total radiated power

$$P = \frac{Z_0}{2k^2} \sum_{l,m} [|a_E(l,m)|^2 + |a_M(l,m)|^2]$$

which is a sum of squares without interference. (Of course, interference still shows up in the angular distribution.) If the quadrupole factor $a_E(2, m)$ was too large, then we would have a true paradox. However, this is not the case. In fact, it is easy to show (by reversing the parity argument) that for this antenna the radiation multipole coefficients are given by Jackson (9.184) with *even l*, instead of odd

$$a_E(l, 0) = \frac{I}{\pi d} \sqrt{\frac{4\pi(2l+1)}{l(l+1)}} \left(\frac{kd}{2}\right)^2 j_l\left(\frac{kd}{2}\right) \qquad l \text{ even}$$

Substituting in $kd = 2\pi$ gives

$$a_E(l, 0) = \frac{Ik}{2} \sqrt{\frac{4\pi(2l+1)}{l(l+1)}} j_l(\pi)$$

so in particular the $l = 2$ moment is

$$a_E(2, 0) = \frac{Ik}{2} \sqrt{\frac{30}{\pi^3}}$$

The angular power distribution for the quadrupole is

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} |a_E(2, 0)|^2 \frac{15}{8\pi} \sin^2\theta \cos^2\theta = \frac{Z_0 I^2}{8} \left(\frac{15}{2\pi^2}\right)^2 \sin^2\theta \cos^2\theta$$

and the total quadrupole power is

$$P = Z_0 |I|^2 \frac{15}{4\pi^3} \qquad \Rightarrow \qquad R_{\text{rad}} = 91.2\,\Omega$$

which is around 2% less than the exact result for the total power radiated in all modes, (6). This shows that here the quadrupole really is the dominant mode.