**Physics 506** **Winter 2006**

## Homework Assignment #3 — Solutions

Textbook problems: Ch. 8: 8.18, 8.19
Ch. 9: 9.3, 9.6

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8.18 *a*) From the use of Green's theorem in two dimensions show that the TM and TE modes in a waveguide defined by the boundary-value problems (8.34) and (8.36) are orthogonal in the sense that

$$\int_A E_{z\,\lambda} E_{z\,\mu} da = 0 \qquad \text{for } \lambda \neq \mu$$

for TM modes, and a corresponding relation for $H_z$ for TE modes.

Orthogonality is a general property of the eigenfunctions of the wave equation. The general two-dimensional equation is given by

$$[\nabla_t^2 + \gamma_\lambda^2]\psi_\lambda = 0$$

where either

$$\psi_\lambda|_S = 0 \qquad \text{TM modes}$$

or

$$\left.\frac{\partial \psi_\lambda}{\partial n}\right|_S = 0 \qquad \text{TE modes}$$

To prove orthogonality, note that $\psi_\lambda$ and $\psi_\mu$ satisfy the equations

$$[\nabla_t^2 + \gamma_\lambda^2]\psi_\lambda = 0, \qquad [\nabla_t^2 + \gamma_\mu^2]\psi_\mu = 0$$

Multiplying the first by $\psi_\mu$ and the second by $\psi_\lambda$ and subtracting gives

$$(\gamma_\mu^2 - \gamma_\lambda^2)\psi_\mu\psi_\lambda = \psi_\mu \nabla_t^2 \psi_\lambda - \psi_\lambda \nabla_t^2 \psi_\mu$$

Integrating this over the cross-sectional area, and using Green's theorem yields

$$\begin{aligned}(\gamma_\mu^2 - \gamma_\lambda^2)\int_A \psi_\mu\psi_\lambda\, da &= \int_A [\psi_\mu \nabla_t^2 \psi_\lambda - \psi_\lambda \nabla_t^2 \psi_\mu]\, da \\ &= -\oint_C \left[\psi_\mu \frac{\partial \psi_\lambda}{\partial n} - \psi_\lambda \frac{\partial \psi_\mu}{\partial n}\right] dl\end{aligned}$$

where we have used an *inward pointing* normal direction. We now note that the right hand side vanishes for either TM or TE boundary conditions. Thus, provided $\gamma_\mu^2 \neq \gamma_\lambda^2$, we end up with

$$\int_A \psi_\mu \psi_\lambda\, da = 0 \qquad (\gamma_\mu^2 \neq \gamma_\lambda^2)$$

For non-degenerate eigenvalues, we conclude that

$$\int_A \psi_\mu \psi_\lambda \, da = 0 \qquad \text{for } \mu \neq \lambda$$

For degenerate eigenvalues, we note that linearity of the wave equation guarantees that we may find an orthogonal basis using, e.g., a Gram-Schmidt orthogonalization process.

*b*) Prove that the relations (8.131)–(8.134) form a consistent set of normalization conditions for the fields, including the circumstances when $\lambda$ is a TM mode and $\mu$ is a TE mode.

We start with relation (8.131), which states

$$\int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} \, da = \delta_{\lambda,\mu}$$

where $\vec{E}_{t,\lambda}$ may be either a TM or a TE mode. To handle this expression, we note that the transverse fields for TM and TE modes are given by

$$\begin{array}{llll} \text{TM:} & \vec{E}_t = \dfrac{ik}{\gamma^2} \vec{\nabla}_t E_z, & \vec{H}_t = \dfrac{1}{Z} \hat{z} \times \vec{E}_t & Z = \dfrac{k}{\epsilon\omega} \\ \text{TE:} & \vec{E}_t = -\dfrac{i\mu\omega}{\gamma^2} \hat{z} \times \vec{\nabla}_t H_z, & \vec{H}_t = \dfrac{1}{Z} \hat{z} \times \vec{E}_t & Z = \dfrac{\mu\omega}{k} \end{array} \tag{1}$$

Hence for two TM modes, we end up with

$$\begin{aligned} \int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} \, da &= -\frac{k^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \vec{\nabla}_t E_{z,\lambda} \cdot \vec{\nabla}_t E_{z,\mu} \, da \\ &= -\frac{k^2}{\gamma_\mu^2 \gamma_\lambda^2} \left[ -\oint_S E_{z,\lambda} \frac{\partial E_{z,\mu}}{\partial n} dl - \int_A E_{z,\lambda} \nabla_t^2 E_{z,\mu} \, da \right] \end{aligned}$$

The surface term vanishes, while $\nabla_t^2 E_{z,\mu} = -\gamma_\mu^2 E_{z,\mu}$. Hence we arrive at

$$\int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} \, da = -\frac{k^2}{\gamma_\lambda^2} \int_A E_{z,\lambda} E_{z,\mu} \, da = 0 \quad \text{for } \lambda \neq \mu \tag{2}$$

When properly normalized for $\lambda = \mu$, this gives (8.131) for two TM modes. The case of two TE modes is similar. We have

$$\begin{aligned} \int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} \, da &= -\frac{\mu^2 \omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A (\hat{z} \times \vec{\nabla}_t H_{z,\lambda}) \cdot (\hat{z} \times \vec{\nabla}_t H_{z,\mu}) \, da \\ &= -\frac{\mu^2 \omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \left[ \vec{\nabla}_t H_{z,\lambda} \cdot \vec{\nabla}_t H_{z,\mu} - (\hat{z} \cdot \vec{\nabla}_t H_{z,\lambda})(\hat{z} \cdot \vec{\nabla}_t H_{z,\mu}) \right] da \\ &= -\frac{\mu^2 \omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \vec{\nabla}_t H_{z,\lambda} \cdot \vec{\nabla}_t H_{z,\mu} \, da \end{aligned} \tag{3}$$

we we have noted that $\hat{z}\cdot\vec{\nabla}_t = 0$ identically (since the transverse gradient is orthogonal to $\hat{z}$). The proof of orthogonality of two TE modes then follows using the same integration method that was used above for the TM modes (but with $E_z$ replaced by $H_z$, and with $\partial H_z/\partial n$ vanishing on the boundary). Finally, for one TE mode and one TM mode, we have

$$\begin{aligned}\int_A \vec{E}_{t,\lambda}\cdot\vec{E}_{t,\mu}\,da &= \frac{\mu\omega k}{\gamma_\mu^2\gamma_\lambda^2}\int_A \vec{\nabla}_t E_{z,\lambda}\cdot(\hat{z}\times\vec{\nabla}_t H_{z,\mu})\,da\\ &= -\frac{\mu\omega k}{\gamma_\mu^2\gamma_\lambda^2}\int_A [\vec{\nabla}_t E_{z,\lambda}\times\vec{\nabla}_t H_{z,\mu}]\cdot\hat{z}\,da\\ &= -\frac{\mu\omega k}{\gamma_\mu^2\gamma_\lambda^2}\int_A \vec{\nabla}_t\times(E_{z,\lambda}\vec{\nabla}_t H_{z,\mu})\cdot\hat{z}\,da\\ &= -\frac{\mu\omega k}{\gamma_\mu^2\gamma_\lambda^2}\oint_S E_{z,\lambda}\vec{\nabla}_t H_{z,\mu}\cdot d\vec{l} = 0\end{aligned}$$

This integral vanishes because $E_{z,\lambda}$ vanishes on the boundary. As a result, all TE modes are orthogonal to all TM modes. Proper normalization then results in (8.131).

We now turn to relation (8.132), which states

$$\int_A \vec{H}_{t,\lambda}\cdot\vec{H}_{t,\mu}\,da = \frac{1}{Z_\lambda^2}\delta_{\lambda,\mu}$$

The best way to prove this is to note from (1) that

$$\vec{H}_{t,\lambda} = \frac{1}{Z_\lambda}\hat{z}\times\vec{E}_{t,\lambda}$$

for either TM or TE modes, provided $Z_\lambda$ is chosen accordingly. In this case

$$\begin{aligned}\int_A \vec{H}_{t,\lambda}\cdot\vec{H}_{t,\mu}\,da &= \frac{1}{Z_\mu Z_\lambda}\int_A (\hat{z}\times\vec{E}_{t,\lambda})(\hat{z}\times\vec{E}_{t,\mu})\,da\\ &= \frac{1}{Z_\mu Z_\lambda}\int_A \left[\vec{E}_{t,\lambda}\cdot\vec{E}_{t,\mu} - (\hat{z}\cdot\vec{E}_{t,\lambda})(\hat{z}\cdot\vec{E}_{t,\mu})\right]\,da\\ &= \frac{1}{Z_\mu Z_\lambda}\int_A \vec{E}_{t,\lambda}\cdot\vec{E}_{t,\mu}\,da = \frac{1}{Z_\mu Z_\lambda}\delta_{\lambda,\mu} = \frac{1}{Z_\lambda^2}\delta_{\lambda,\mu}\end{aligned}$$

Here we have made use of the fact that $\hat{z}\cdot\vec{E}_t$ vanishes because $\vec{E}_t$ is transverse to the $\hat{z}$ direction. The last line follows from applying (8.131), which we proved above.

The power flow relation (8.133)

$$\frac{1}{2}\int_A (\vec{E}_{t,\lambda}\times\vec{H}_{t,\mu})\cdot\hat{z}\,da = \frac{1}{2Z_\lambda}\delta_{\lambda,\mu}$$

follows similarly. Specifically, we have

$$\begin{aligned}\frac{1}{2}\int_A (\vec{E}_{t,\lambda}\times\vec{H}_{t,\mu})\cdot\hat{z}\,da &= \frac{1}{2Z_\mu}\int_A \hat{z}\cdot[\vec{E}_{t,\lambda}\times(\hat{z}\times\vec{E}_{t,\mu})]\,da\\ &= \frac{1}{2Z_\mu}\int_A \left[\vec{E}_{t,\lambda}\cdot\vec{E}_{t,\mu}-(\hat{z}\cdot\vec{E}_{t,\lambda})(\hat{z}\cdot\vec{E}_{t,\mu})\right]da\\ &= \frac{1}{2Z_\mu}\int_A \vec{E}_{t,\lambda}\cdot\vec{E}_{t,\mu}\,da = \frac{1}{2Z_\mu}\delta_{\lambda,\mu} = \frac{1}{2Z_\lambda}\delta_{\lambda,\mu}\end{aligned}$$

The relation (8.134) essentially normalizes the modes for the TM and TE case. Examination of (2) for TM modes and (3) for TE modes indicates that the proper normalization is

$$\begin{aligned}&\text{TM:} \qquad \int_A E_{z,\lambda}E_{z,\mu}\,da = -\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda,\mu}\\ &\text{TE:} \qquad \int_A E_{z,\lambda}E_{z,\mu}\,da = -\frac{\gamma_\lambda^2}{\mu^2\omega^2}\delta_{\lambda,\mu} = -\frac{\gamma_\lambda^2}{k_\lambda^2 Z_\lambda^2}\end{aligned} \tag{4}$$

8.19 The figure shows a cross-sectional view of an infinitely long rectangular waveguide with the center conductor of a coaxial line extending vertically a distance $h$ into its interior at $z = 0$. The current along the probe oscillates sinusoidally in time with frequency $\omega$, and its variation in space can be approximated as $I(y) = I_0 \sin[(\omega/c)(h - y)]$. The thickness of the probe can be neglected. The frequency is such that only the $\text{TE}_{10}$ mode can propagate in the guide.

*a*) Calculate the amplitudes for excitation of both TE and TM modes for all $(m, n)$ and show how the amplitudes depend on $m$ and $n$ for $m, n \gg 1$ for a fixed frequency $\omega$.

Before calculating the amplitudes, we work out the normalization for the rectangular waveguide normal modes. For the TM modes, we recall that

$$\begin{aligned}E_z &= E_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\ \vec{E}_t &= \frac{ik_{mn}}{\gamma_{mn}^2}\vec{\nabla}_t E_z \qquad\Rightarrow\qquad E_y = -\frac{ik_{mn}}{\gamma_{mn}^2}\frac{n\pi}{b}E_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\end{aligned} \tag{5}$$

where

$$k_{mn}^2 = \frac{\omega^2}{c^2} - \gamma_{mn}^2, \qquad \gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

We have explicitly written out the $y$ component above, since it will be needed to calculate the amplitude for excitation. Noting that $\sin^2$ averages to $1/2$, the TM normalization condition (4) gives

$$\frac{ab}{4}E_0^2 = -\frac{\gamma_{mn}^2}{k_{mn}^2} \qquad\Rightarrow\qquad E_0 = \frac{2i\gamma_{mn}}{k_{mn}\sqrt{ab}}$$

Similarly, the TE modes are

$$
\begin{aligned}
H_z &= H_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\
\vec{E}_t &= -\frac{i\mu_0\omega}{\gamma_{mn}^2}\hat{z}\times\vec{\nabla}_t H_z \qquad \Rightarrow \qquad E_y = \frac{i\mu_0\omega}{\gamma_{mn}^2}\frac{m\pi}{a}H_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}
\end{aligned}
\tag{6}
$$

with normalization

$$
\frac{ab}{4}H_0^2 = -\frac{\gamma_{mn}^2}{\mu_0^2\omega^2} \qquad \Rightarrow \qquad H_0 = \frac{2i\gamma_{mn}}{\mu_0\omega\sqrt{ab}}
$$

except that if $m = 0$ we should take $a \to 2a$ and if $n = 0$ we should take $b \to 2b$ in the square root, since the constant mode averages to 1 instead of 1/2. (This is not an issue for TM modes, since in that case the $m = 0$ or $n = 0$ possibilities are disallowed.) Before proceeding, we note here that the functional behavior of the $E_y$ components are identical for TM and TE modes; they only differ by constant factors.

Now that we have written out the normal modes, we turn to the excitation amplitude computation. The amplitudes are

$$
\mathcal{A}_{mn}^{(\pm)} = -\frac{Z_{mn}}{2}\int_V \vec{J}\cdot\vec{E}_{mn}^{(\mp)}\, d^3x
$$

where $Z_{mn} = k_{mn}/\epsilon_0\omega$ for TM modes and $Z_{mn} = \mu_0\omega/k_{mn}$ for TE modes. The source current density may be written as

$$
\vec{J} = \hat{y}I_0\sin[\tfrac{\omega}{c}(h-y)]\delta(x-X)\delta(z)\Theta(h-y)
$$

Hence

$$
\mathcal{A}_{mn}^{(\pm)} = -\frac{Z_{mn}}{2}I_0\int_0^h \sin[\tfrac{\omega}{c}(h-y)]E_{y,mn}^{(\mp)}(X,y,0)\,dy
$$

where $E_{y,mn}$ is given by (5) for TM modes and (6) for TE modes. Since we only need the electric field at $z = 0$, we see that this expression is independent of whether we choose a left-moving or a right-moving mode. As a result, the $(+)$ and $(-)$ amplitudes will be equally excited. For the TM mode of (5), we evaluate

$$
\begin{aligned}
\mathcal{A}_{mn}^{(\pm)} &= -\frac{Z_{mn}}{2}I_0\frac{ik_{mn}}{\gamma_{mn}^2}\frac{n\pi}{b}E_0\sin\frac{m\pi X}{a}\int_0^h \sin[\tfrac{\omega}{c}(h-y)]\cos\frac{n\pi y}{b}\,dy \\
&= \frac{k_{mn}}{\epsilon_0\omega\gamma_{mn}\sqrt{ab}}\frac{n\pi}{b}I_0\sin\frac{m\pi X}{a}\int_0^h \sin[\tfrac{\omega}{c}(h-y)]\cos\frac{n\pi y}{b}\,dy
\end{aligned}
$$

The integral may be performed by use of the trig identity

$$
\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]
$$

The result is

$$\int_0^h \sin[\tfrac{\omega}{c}(h-y)] \cos\frac{n\pi y}{b}\,dy = \frac{\omega}{c}\left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{-1}\left(\cos\frac{n\pi h}{b} - \cos\frac{\omega h}{c}\right)$$

which gives

$$\mathcal{A}_{mn}^{(\pm)} = \frac{k_{mn}}{\epsilon_0 c\gamma_{mn}\sqrt{ab}}\frac{n\pi}{b} I_0 \sin\frac{m\pi X}{a}\left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{-1}\left(\cos\frac{n\pi h}{b} - \cos\frac{\omega h}{c}\right)$$

According to the specifications of this problem, all the TM modes are cutoff modes. For large $m, n \gg 1$ the wavenumber is imaginary, $k_{mn} \approx i\gamma_{mn}$. Hence for fixed $\omega$, we see that

$$\mathcal{A}_{mn}^{(\pm)} \sim \frac{1}{n} \qquad \text{(TM)}$$

provided the trig functions are $\mathcal{O}(1)$ and do not vanish. For the TE mode of (6), we note that the amplitude calculation is identical, except for a different constant factor. The TE result is

$$\mathcal{A}_{mn}^{(\pm)} = \frac{\mu_0\omega^2}{ck_{mn}\gamma_{mn}\sqrt{ab}}\frac{m\pi}{a} I_0 \sin\frac{m\pi X}{a}\left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{-1}\left(\cos\frac{n\pi h}{b} - \cos\frac{\omega h}{c}\right)$$

for $m, n \gg 1$, all modes are cut off, and we see that

$$\mathcal{A}_{mn}^{(\pm)} \sim \frac{m}{n^2}\frac{1}{(m/a)^2 + (n/b)^2} \sim \frac{1}{N^3}$$

where $m \sim n \sim N \gg 1$. On the other hand, the propagating $\mathrm{TE}_{10}$ mode has amplitude

$$\mathcal{A}_{10}^{(\pm)} = \frac{\mu_0 c}{k_{10}\sqrt{2ab}} I_0 \sin\frac{\pi X}{a}\left(1 - \cos\frac{\omega h}{c}\right) = \frac{\sqrt{2}\mu_0 c}{k_{10}\sqrt{ab}} I_0 \sin\frac{\pi X}{a}\sin^2\frac{\omega h}{2c} \qquad (7)$$

where we have made sure to use the normalization appropriate for an $n = 0$ mode.

*b*) For the propagating mode show that the power radiated in the positive $z$ direction is

$$P = \frac{\mu c^2 I_0^2}{\omega k a b}\sin^2\left(\frac{\pi X}{a}\right)\sin^4\left(\frac{\omega h}{2c}\right)$$

with an equal amount in the opposite direction. Here $k$ is the wave number for the $\mathrm{TE}_{10}$ mode.

For an expansion in properly normalized normal modes, the radiated power in the $+z$ direction is given by

$$P_{mn} = \frac{1}{2Z_{mn}}\left|\mathcal{A}_{mn}^{(+)}\right|^2$$

Using the amplitude coefficient (7) for the $TE_{10}$ mode, we find

$$P = \frac{k_{10}}{2\mu_0\omega}\left|\mathcal{A}_{10}^{(+)}\right|^2 = \frac{\mu_0 c^2}{\omega k_{10} ab} I_0^2 \sin^2\frac{\pi X}{a}\sin^4\frac{\omega h}{2c} \tag{8}$$

*c*) Discuss the modifications that occur if the guide, instead of running off to infinity in both directions, is terminated with a perfectly conducting surface at $z = L$. What values of $L$ with maximize the power flow for a fixed current $I_0$? What is the radiation resistance of the probe (defined as the ratio of power flow to one-half the square of the current at the base of the probe) at maximum?

If we place a perfectly conducting surface at $z = L$ (we take $L$ positive), then the right-moving wave will be perfectly reflected at this surface. As a result, the wave flowing out of the left end ($z < 0$) of the waveguide will be the linear superposition of two components: the left-moving wave generated by the probe, and the right-moving wave reflected off of the conducting surface at $z = L$. We further note that, since $E_\parallel$ must vanish at the surface of a perfect conductor, the reflected wave must come back 180° out of phase. We start by writing the left-moving wave as

$$\vec{E}^{(-)} = \mathcal{A}_{10}^{(-)}\vec{E}_{t,10}e^{-ikz}$$

The right-moving wave leaves the source as

$$\vec{E}^{(+)} = \mathcal{A}_{10}^{(+)}\vec{E}_{t,10}e^{ikz}$$

and it is easy to see that the reflected wave must be

$$\vec{E}^{(\mathrm{refl})} = -\mathcal{A}_{10}^{(+)}\vec{E}_{t,10}e^{ik(2L-z)}$$

so that it satisfies the conductor boundary condition

$$\vec{E}^{(+)}(z = L) + \vec{E}^{(\mathrm{refl})}(z = L) = 0$$

Therefore, for $z < 0$, the total left-moving wave is given by

$$\vec{E} = \vec{E}^{(-)} + \vec{E}^{(\mathrm{refl})} = (\mathcal{A}_{10}^{(-)} - \mathcal{A}_{10}^{(+)}e^{2ikL})\vec{E}_{t,10}e^{-ikz} = \mathcal{A}_{10}(1 - e^{2ikL})\vec{E}_{t,10}e^{-ikz}$$

The maximum amplitude case occurs when there is constructive interference

$$kL = (n + \tfrac{1}{2})\pi$$

Since the amplitude doubles, the power is increased by a factor of four compared with (8). For this maximum power case, the radiation resistance is given by

$$P = \frac{4\mu_0 c^2}{\omega k_{10} ab} I_0^2 \sin^2\frac{\pi X}{a}\sin^4\frac{\omega h}{2c} = \frac{1}{2}I_0^2 R_{\mathrm{rad}}$$

or

$$R_{\rm rad} = \frac{8\mu_0 c^2}{\omega k_{10} ab} \sin^2 \frac{\pi X}{a} \sin^4 \frac{\omega h}{2c}$$

9.3 Two halves of a spherical metallic shell of radius $R$ and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V \cos\omega t$. In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

In the long wavelength limit, we may appeal to the multipole expansion of the source. In this case, the source is essentially a harmonically ($e^{-i\omega t}$) varying version of the electrostatic problem with hemispheres at opposite potential. The long wavelength limit is also equivalent to the low frequency limit. In this case, it is valid to think of the source as a quasi-static object. Using azimuthal symmetry, the potential then admits an expansion in Legendre polynomials

$$\Phi(r,\theta) = \sum_l \alpha_l \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)$$

where

$$\alpha_l = \frac{2l+1}{2} \int_{-1}^{1} \Phi(R,\cos\theta) P_l(\cos\theta) d\cos\theta$$

For hemispheres at opposite potential $\pm V$ (times $e^{-i\omega t}$, which is to be understood), the expansion coefficients are

$$\alpha_l = (2l+1)V \int_0^1 P_l(x)\, dx \qquad \text{odd } l \text{ only}$$

The dipole is dominant, with $\alpha_1 = \frac{3}{2}V$. This gives rise to a dipole potential of the form

$$\Phi = \frac{3}{2}V \left(\frac{R}{r}\right)^2 P_1(\cos\theta) = \frac{3}{2} V R^2 \frac{z}{r^3}$$

This makes it straightforward to read off an electric dipole moment

$$\vec{p} = 4\pi\epsilon_0(\tfrac{3}{2}VR^2\hat{z}) = 6\pi\epsilon_0 VR^2\hat{z}$$

Working in the radiation zone, this electric dipole gives

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{r}\times\vec{p})\frac{e^{ikr}}{r} = -\frac{ck^2}{4\pi} 6\pi\epsilon_0 VR^2 \frac{e^{ikr}}{r}\sin\theta\hat{\phi} = -\frac{3}{2}Z_0^{-1}V(kR)^2\frac{e^{ikr}}{r}\sin\theta\hat{\phi}$$

and

$$\vec{E} = -Z_0\hat{r}\times\vec{H} = -\frac{3}{2}V(kR)^2\frac{e^{ikr}}{r}\sin\theta\hat{\theta}$$

The angular distribution of dipole radiation gives

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\vec{p}\,|^2 \sin^2\theta = \frac{c^2 Z_0}{32\pi^2} k^4 36\pi^2 \epsilon_0^2 V^2 R^4 \sin^2\theta = \frac{9}{8} Z_0^{-1} V^2 (kR)^4 \sin^2\theta$$

and the total radiated power is

$$P = 3\pi Z_0^{-1} V^2 (kR)^4$$

9.6 *a)* Starting from the general expression (9.2) for $\vec{A}$ and the corresponding expression for $\Phi$, expand both $R = |\vec{x} - \vec{x}\,'|$ and $t' = t - R/c$ to first order in $|\vec{x}\,'|/r$ to obtain the electric dipole potentials for arbitrary time variation

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^2}\vec{n}\cdot\vec{p}_{\rm ret} + \frac{1}{cr}\vec{n}\cdot\frac{\partial\vec{p}_{\rm ret}}{\partial t}\right]$$
$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi r}\frac{\partial\vec{p}_{\rm ret}}{\partial t}$$

where $\vec{p}_{\rm ret} = \vec{p}(t' = t - r/c)$ is the dipole moment evaluated at the retarded time measured from the origin.

We start with the scalar potential, which is given by

$$\Phi(\vec{x}\,) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}\,', t - |\vec{x} - \vec{x}\,'|/c)}{|\vec{x} - \vec{x}\,'|} d^3x' \tag{9}$$

We now use the expansion

$$|\vec{x} - \vec{x}\,'| \approx r - \hat{n}\cdot\vec{x}\,'$$

as well as

$$t' = t - \frac{|\vec{x} - \vec{x}\,'|}{c} \approx t - \frac{r}{c} + \frac{\hat{n}\cdot\vec{x}\,'}{c} = t_{\rm ret} + \frac{\hat{n}\cdot\vec{x}\,'}{c}$$

where $t_{\rm ret} = t - r/c$. Since $\rho$ is a function of time $t'$, we make the expansion

$$\rho(\vec{x}\,', t') = \rho(\vec{x}\,', t_{\rm ret}) + \frac{\hat{n}\cdot\vec{x}\,'}{c}\frac{\partial\rho(\vec{x}\,', t_{\rm ret})}{\partial t} + \cdots$$

As a result, the expansion of (9) becomes

$$\begin{aligned}\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0 r}\int\left[\rho + \hat{n}\cdot\vec{x}\,'\left(\frac{1}{r}\rho + \frac{1}{c}\frac{\partial\rho}{\partial t}\right) + \cdots\right] d^3x' \\ &= \frac{1}{4\pi\epsilon_0 r}\left[Q + \hat{n}\cdot\left(\frac{1}{r}\vec{p} + \frac{1}{c}\frac{\partial\vec{p}}{\partial t}\right) + \cdots\right]\end{aligned}$$

where the retarded time dependence is to be understood, and where we have used the fact that

$$Q = \int \rho \, d^3x', \qquad \vec{p} = \int \vec{x}\,'\rho \, d^3x'$$

Dropping the static Coulomb potential (which does not radiate) then gives

$$\Phi(\vec{x}\,) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^2}\hat{n}\cdot\vec{p} + \frac{1}{cr}\hat{n}\cdot\frac{\partial\vec{p}}{\partial t}\right] \tag{10}$$

For the vector potential, the expansion is even simpler. We only need to keep the lowest order behavior

$$\vec{A}(\vec{x}\,) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}\,', t - |\vec{x} - \vec{x}\,'|/c)}{|\vec{x} - \vec{x}\,'|}d^3x' = \frac{\mu_0}{4\pi r}\int \left[\vec{J} + \cdots\right] d^3x'$$

Using integration by parts, we note that

$$\int J_i \, d^3x' = \int \frac{\partial x_i'}{\partial x_j'} J_j \, d^3x = -\int x_i'(\vec{\nabla}\cdot\vec{J}\,)\, d^3x' = \int x_i' \frac{\partial\rho}{\partial t} d^3x' = \frac{\partial p_i}{\partial t}$$

Hence

$$\vec{A}(\vec{x}\,') \approx \frac{\mu_0}{4\pi r}\frac{\partial\vec{p}}{\partial t} \tag{11}$$

*b*) Calculate the dipole electric and magnetic fields directly from these potentials and show that

$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi}\left[-\frac{1}{cr^2}\vec{n}\times\frac{\partial\vec{p}_{\text{ret}}}{\partial t} - \frac{1}{c^2 r}\vec{n}\times\frac{\partial^2\vec{p}_{\text{ret}}}{\partial t^2}\right]$$

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0}\left\{\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\left[\frac{3\vec{n}(\vec{n}\cdot\vec{p}_{\text{ret}}) - \vec{p}_{\text{ret}}}{r^3}\right] + \frac{1}{c^2 r}\vec{n}\times\left(\vec{n}\times\frac{\partial^2\vec{p}_{\text{ret}}}{\partial t^2}\right)\right\}$$

For the magnetic field, we use $\vec{B} = \vec{\nabla}\times\vec{A}$, where the vector potential is given by (11). It is important to note that the electric dipole $\vec{p}$ in (11) is actually a function of retarded time

$$\vec{p}_{\text{ret}} = \vec{p}(t - r/c)$$

Application of the chain rule then gives

$$\frac{\partial\vec{p}_{\text{ret}}}{\partial r} = -\frac{1}{c}\frac{\partial\vec{p}_{\text{ret}}}{\partial t}$$

Since $\vec{\nabla} r = \hat{n}$, the magnetic field turns out to be

$$\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\frac{1}{r}\frac{\partial\vec{p}}{\partial t}\right) = \frac{\mu_0}{4\pi}\hat{n}\times\left(-\frac{1}{r^2}\frac{\partial\vec{p}}{\partial t} - \frac{1}{cr}\frac{\partial^2\vec{p}}{\partial t^2}\right) \tag{12}$$

The expression for the electric field is a bit more involved. Using (10) and (11), we obtain

$$
\begin{aligned}
\vec{E} &= -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \\
&= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{\vec{x}}{r^3}\cdot\vec{p} + \frac{\vec{x}}{cr^2}\cdot\frac{\partial\vec{p}}{\partial t}\right) - \frac{\mu_0}{4\pi r}\frac{\partial^2\vec{p}}{\partial t^2} \\
&= -\frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}}{r^3} - 3\frac{\vec{x}(\vec{x}\cdot\vec{p})}{r^5} - \frac{\vec{x}}{cr^4}\left(\vec{x}\cdot\frac{\partial\vec{p}}{\partial t}\right) + \frac{1}{cr^2}\frac{\partial\vec{p}}{\partial t} - \frac{2\vec{x}}{cr^4}\left(\vec{x}\cdot\frac{\partial\vec{p}}{\partial t}\right)\right. \\
&\qquad\qquad \left. - \frac{\vec{x}}{c^2r^3}\left(\vec{x}\cdot\frac{\partial^2\vec{p}}{\partial t^2}\right)\right] - \frac{1}{4\pi\epsilon_0}\frac{1}{c^2 r}\frac{\partial^2\vec{p}}{\partial t^2} \\
&= -\frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p} - 3\hat{n}(\hat{n}\cdot\vec{p})}{r^3} + \frac{1}{cr^2}\frac{\partial}{\partial t}(\vec{p} - 3\hat{n}(\hat{n}\cdot\vec{p})) + \frac{1}{c^2 r}\frac{\partial^2}{\partial t^2}(\vec{p} - \hat{n}(\hat{n}\cdot\vec{p}))\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\frac{3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}}{r^3} + \frac{1}{c^2 r}\hat{n}\times\left(\hat{n}\times\frac{\partial^2\vec{p}}{\partial t^2}\right)\right]
\end{aligned} \tag{13}
$$

*c*) Show explicitly how you can go back and forth between these results and the harmonic fields of (9.18) by the substitutions $-i\omega \leftrightarrow \partial/\partial t$ and $\vec{p}e^{ikr-i\omega t} \leftrightarrow \vec{p}_{\rm ret}(t')$.

Making the substitution

$$
\vec{p}_{\rm ret} \longrightarrow \vec{p}e^{ikr} \qquad \text{and} \qquad \frac{\partial}{\partial t} \longrightarrow -i\omega
$$

the magnetic field (12) becomes

$$
\begin{aligned}
\vec{H} &= \frac{1}{4\pi}\hat{n}\times\left(-\frac{1}{r^2}(-i\omega)\vec{p} - \frac{1}{cr}(-\omega^2)\vec{p}\right)e^{ikr} \\
&= \frac{\omega^2}{4\pi cr}(\hat{n}\times\vec{p})\left(1 - \frac{c}{i\omega r}\right)e^{ikr} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)
\end{aligned}
$$

while the electric field (13) becomes

$$
\begin{aligned}
\vec{E} &= \frac{1}{4\pi\epsilon_0}\left[\left(1 + \frac{r}{c}(-i\omega)\right)\frac{3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}}{r^3} + \frac{1}{c^2 r}(-\omega^2)\hat{n}\times(\hat{n}\times\vec{p})\right]e^{ikr} \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{e^{ikr}}{r^3}(1 - ikr)(3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}) - k^2\frac{e^{ikr}}{r}\hat{n}\times(\hat{n}\times\vec{p})\right]
\end{aligned}
$$

To go in the other direction, we simply read these equations backwards.