Physics 506 **Winter 2006**

Homework Assignment #3 — Due Thursday, January 26

Textbook problems: Ch. 8: 8.18, 8.19
Ch. 9: 9.3, 9.6

8.18 *a*) From the use of Green's theorem in two dimensions show that the TM and TE modes in a waveguide defined by the boundary-value problems (8.34) and (8.36) are orthogonal in the sense that

$$\int_A E_{z\,\lambda} E_{z\,\mu} da = 0 \qquad \text{for } \lambda \neq \mu$$

for TM modes, and a corresponding relation for H_z for TE modes.

b) Prove that the relations (8.131)–(8.134) form a consistent set of normalization conditions for the fields, including the circumstances when λ is a TM mode and μ is a TE mode.

8.19 The figure shows a cross-sectional view of an infinitely long rectangular waveguide with the center conductor of a coaxial line extending vertically a distance h into its interior at $z = 0$. The current along the probe oscillates sinusoidally in time with frequency ω, and its variation in space can be approximated as $I(y) = I_0 \sin[(\omega/c)(h - y)]$. The thickness of the probe can be neglected. The frequency is such that only the TE_{10} mode can propagate in the guide.

a) Calculate the amplitudes for excitation of both TE and TM modes for all (m, n) and show how the' amplitudes depend on m and n for $m, n \gg 1$ for a fixed frequency ω.

b) For the propagating mode show that the power radiated in the positive z direction is

$$P = \frac{\mu c^2 I_0^2}{\omega kab} \sin^2\left(\frac{\pi X}{a}\right) \sin^4\left(\frac{\omega h}{2c}\right)$$

with an equal amount in the opposite direction. Here k is the wave number for the TE_{10} mode.

c) Discuss the modifications that occur if the guide, instead of running off to infinity in both directions, is terminated with a perfectly conducting surface at $z = L$. What values of L with maximize the power flow for a fixed current I_0? What is the radiation resistance of the probe (defined as the ratio of power flow to one-half the square of the current at the base of the probe) at maximum?

9.3 Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V \cos\omega t$. In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

9.6 *a*) Starting from the general expression (9.2) for $\vec{A}$ and the corresponding expression for Φ, expand both $R = |\vec{x} - \vec{x}\,'|$ and $t' = t - R/c$ to first order in $|\vec{x}\,'|/r$ to obtain the electric dipole potentials for arbitrary time variation

$$\begin{aligned} \Phi(\vec{x},t) &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^2}\vec{n}\cdot\vec{p}_{\rm ret} + \frac{1}{cr}\vec{n}\cdot\frac{\partial\vec{p}_{\rm ret}}{\partial t}\right] \\ \vec{A}(\vec{x},t) &= \frac{\mu_0}{4\pi r}\frac{\partial\vec{p}_{\rm ret}}{\partial t} \end{aligned}$$

where $\vec{p}_{\rm ret} = \vec{p}(t' = t - r/c)$ is the dipole moment evaluated at the retarded time measured from the origin.

b) Calculate the dipole electric and magnetic fields directly from these potentials and show that

$$\begin{aligned} \vec{B}(\vec{x},t) &= \frac{\mu_0}{4\pi}\left[-\frac{1}{cr^2}\vec{n}\times\frac{\partial\vec{p}_{\rm ret}}{\partial t} - \frac{1}{c^2 r}\vec{n}\times\frac{\partial^2\vec{p}_{\rm ret}}{\partial t^2}\right] \\ \vec{E}(\vec{x},t) &= \frac{1}{4\pi\epsilon_0}\left\{\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\left[\frac{3\vec{n}(\vec{n}\cdot\vec{p}_{\rm ret}) - \vec{p}_{\rm ret}}{r^3}\right] + \frac{1}{c^2 r}\vec{n}\times\left(\vec{n}\times\frac{\partial^2\vec{p}_{\rm ret}}{\partial t^2}\right)\right\} \end{aligned}$$

c) Show explicitly how you can go back and forth between these results and the harmonic fields of (9.18) by the substitutions $-i\omega \leftrightarrow \partial/\partial t$ and $\vec{p}e^{ikr-i\omega t} \leftrightarrow \vec{p}_{\rm ret}(t')$.