

Homework Assignment #2 — Solutions

Textbook problems: Ch. 8: 8.6, 8.8

8.6 A resonant cavity of copper consists of a hollow, right circular cylinder of inner radius  $R$  and length  $L$ , with flat end faces.

- a) Determine the resonant frequencies of the cavity for all types of waves. With  $(1/\sqrt{\mu\epsilon R})$  as a unit of frequency, plot the lowest four resonant frequencies of each type as a function of  $R/L$  for  $0 < R/L < 2$ . Does the same mode have the lowest frequency for all  $R/L$ ?

This cavity is essentially covered in Section 8.7 of the textbook. It is also similar to the waveguide problem 8.4, but with endcaps to form a resonant cavity. The normal modes are either TM or TE modes. The TM modes are given by

$$\psi(\rho, \phi) = E_0 J_m(\gamma_{mn}\rho) e^{\pm im\phi}, \quad \gamma_{mn} = \frac{x_{mn}}{R} \quad (1)$$

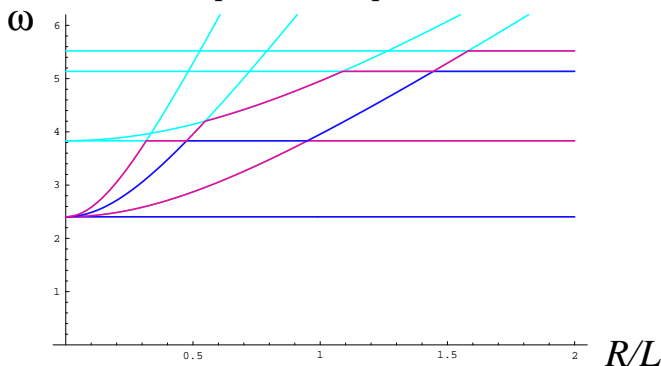
where  $x_{mn}$  are the zeros of the Bessel functions  $J_m$ . The resonant frequencies are thus

$$(TM) \quad \omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon R}} \sqrt{x_{mn}^2 + \left(\frac{p\pi R}{L}\right)^2} \quad (p \geq 0)$$

with

$$x_{01} = 2.405, \quad x_{11} = 3.832, \quad x_{21} = 5.136, \quad x_{02} = 5.520$$

The lowest four resonant frequencies are plotted as follows



Note that the  $p = 0$  modes are independent of  $R/L$ . Clearly the same mode does not always have the lowest frequency. The cross-over points are accidental degeneracies, and are not related to any particular symmetry of the cylinder.

The TE modes are given by

$$\psi(\rho, \phi) = H_0 J_m(\gamma_{mn} \rho) e^{\pm im\phi}, \quad \gamma_{mn} = \frac{x'_{mn}}{R}$$

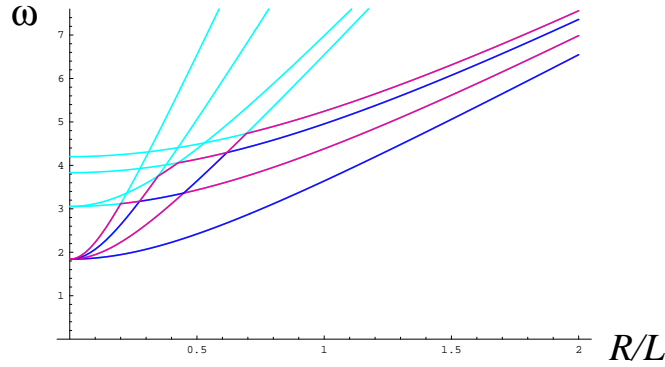
where  $x'_{mn}$  are the zeros of  $J'_m$ . The TE resonant frequencies are

$$(TE) \quad \omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon R}} \sqrt{x'^2_{mn} + \left(\frac{p\pi R}{L}\right)^2} \quad (p > 0)$$

with

$$x'_{11} = 1.841, \quad x'_{21} = 3.054, \quad x'_{01} = 3.832, \quad x'_{31} = 4.201$$

In this case, the lowest four resonant frequencies are



- b) If  $R = 2$  cm,  $L = 3$  cm, and the cavity is made of pure copper, what is the numerical value of  $Q$  for the lowest resonant mode?

For this geometry, it turns out the lowest mode is the  $TM_{010}$  mode. We thus calculate the  $Q$  factor for  $TM_{mn0}$  modes. We start with the stored energy

$$U = \frac{L\epsilon}{2} \int_A |\psi|^2 da \quad (2)$$

where  $\psi$  is given by (1). Note that this is double the  $p \neq 0$  result. The power loss expression for  $TM_{mn0}$  modes is

$$P_{\text{loss}} = \frac{\epsilon}{\sigma\delta\mu} \left(1 + \xi_{mn} \frac{CL}{2A}\right) \int_A |\psi|^2 da \quad (3)$$

where we have taken  $p = 0$  into account. Here  $C = 2\pi R$  and  $A = \pi R^2$  are the circumference and cross-sectional area of the cylinder. The geometrical factor  $\xi_{mn}$  is the same as the waveguide result, which was obtained in Problem 8.4 as  $\xi_{mn} = 1$ . More directly, we may start with the definition

$$\oint_C \left| \frac{d\psi}{dn} \right|^2 dl = \xi_{mn} \gamma_{mn}^2 \frac{C}{A} \int_A |\psi|^2 da$$

Using (1), this statement is equivalent to

$$\begin{aligned} C (\gamma_{mn}^2 J'_m(x_{mn})^2) &= \xi_{mn} \gamma_{mn}^2 \frac{C}{A} \left( 2\pi \int_0^R J_m(x_{mn}\rho/R)^2 \rho d\rho \right) \\ &= \xi_{mn} \gamma_{mn}^2 \frac{C}{A} (\pi R^2 J_{m+1}(x_{mn})^2) \end{aligned}$$

where the second line is obtained by the Bessel function normalization condition. This results in

$$\xi_{mn} = \left( \frac{J'_m(x_{mn})}{J_{m+1}(x_{mn})} \right)^2$$

However, using the Bessel recursion relation  $J_{m+1}(\xi) = (m/\xi)J_m(\xi) - J'_m(\xi)$  and letting  $\xi = x_{mn}$  be a zero of  $J_m$ , we obtain simply  $J_{m+1}(x_{mn}) = -J'_m(x_{mn})$ . This proves that the geometrical factor is simply  $\xi_{mn} = 1$ . Finally, using (2) and (3) with  $\xi_{mn} = 1$  gives

$$Q_{mn0} = \omega_{mn} \frac{U}{P_{\text{loss}}} = \frac{\mu}{\mu_c} \frac{L}{\delta} \left( 1 + \frac{CL}{2A} \right)^{-1} = \frac{\mu}{\mu_c} \frac{L}{\delta} \left( 1 + \frac{L}{R} \right)^{-1}$$

Since copper is non-ferromagnetic, we may take  $\mu_c = \mu_0$ . Furthermore, we assume the interior of the cavity has  $\mu = \mu_0$  and  $\epsilon = \epsilon_0$ . Substituting in  $R = 2$  cm,  $L = 3$  cm then yields

$$Q_{mn0} = \frac{1.2 \times 10^{-2} \text{ m}}{\delta}$$

We calculate the lowest resonant frequency to be

$$\omega_{010} = \frac{x_{01c}}{R} = \frac{2.405c}{R} = 3.61 \times 10^{10} \text{ s}^{-1}$$

or  $\nu_{010} = 5.94$  GHz, where we have used  $R = 3$  cm. At this frequency, the skin depth for copper is

$$\delta = \frac{6.52 \times 10^{-2} \text{ m}}{\sqrt{\nu_{\text{mnp}}(\text{Hz})}} = 8.6 \times 10^{-7} \text{ m}$$

This gives a cavity  $Q$  of

$$Q_{010} = 1.4 \times 10^4$$

8.8 For the Schumann resonances of Section 8.9 calculate the  $Q$  values on the assumption that the earth has a conductivity  $\sigma_e$  and the ionosphere has a conductivity  $\sigma_i$ , with corresponding skin depths  $\delta_e$  and  $\delta_i$ .

- a) Show that to lowest order in  $h/a$  the  $Q$  value is given by  $Q = Nh/(\delta_e + \delta_i)$  and determine the numerical factor  $N$  for all  $l$ .

The long-wavelength Schumann resonances have electric and magnetic fields approximately given by

$$E_r \approx -\frac{i}{\epsilon_0 \omega_l a} l(l+1) H_0 P_l(\cos \theta), \quad H_\phi \approx H_0 P_l^1(\cos \theta) \quad (4)$$

where

$$\omega_l \approx \sqrt{l(l+1)} \frac{c}{a}$$

To calculate the  $Q$  value, we begin with the stored energy.

$$\begin{aligned} U &= \int_V \left[ \frac{\epsilon_0}{4} |\vec{E}|^2 + \frac{\mu_0}{4} |\vec{H}|^2 \right] d^3x \\ &\approx ha^2 \int d\Omega \left[ \frac{\epsilon_0}{4} |E_r|^2 + \frac{\mu_0}{4} |H_\phi|^2 \right] \\ &= \frac{\mu_0 ha^2}{4} |H_0|^2 \int d\Omega \left[ \frac{c^2}{\omega_l^2 a^2} l^2(l+1)^2 P_l(\cos \theta)^2 + P_l^1(\cos \theta)^2 \right] \\ &= \frac{\mu_0 ha^2}{4} |H_0|^2 2\pi \int_{-1}^1 d\cos \theta [l(l+1) P_l(\cos \theta)^2 + P_l^1(\cos \theta)^2] \end{aligned}$$

Using the (associated) Legendre polynomial normalization

$$\int_{-1}^1 P_l(x) P_{l'}(x) dx = \frac{2}{2l+1} \delta_{ll'}, \quad \int_{-1}^1 P_l^m(x) P_{l'}^m(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}$$

gives

$$U = 2\mu_0 h \pi a^2 |H_0|^2 \frac{l(l+1)}{2l+1} \quad (5)$$

For the power lost, there are two contributions, one due to the conductivity at the surface of the Earth, and the other due to the conductivity at the ionosphere boundary. For a uniform tangential magnetic field at either boundary, we approximate

$$\begin{aligned} P_{\text{lost}} &= \frac{1}{2\sigma\delta} \int_S |\hat{n} \times \vec{H}|^2 da \approx \frac{1}{2\sigma\delta} |H_0|^2 R^2 \int d\Omega P_l^1(\cos \theta)^2 \\ &= \frac{\pi R^2 |H_0|^2}{\sigma\delta} \int_{-1}^1 d\cos \theta P_l^1(\cos \theta)^2 \\ &= \frac{2\pi R^2 |H_0|^2}{\sigma\delta} \frac{l(l+1)}{2l+1} \end{aligned}$$

where  $R$  is the radius of the sphere. Since  $h \ll a$ , we use  $R = a$  for the Earth as well as  $R = a + h \approx a$  for the ionosphere. Furthermore, we note that

$$\frac{1}{\sigma\delta} = \frac{1}{2} \mu_c \omega \delta$$

We may use  $\mu_c \approx \mu_0$ . Hence the sum of the power lost at the surface of the Earth and the ionosphere is given by

$$P_{\text{lost}} \approx \mu_0 \omega (\delta_e + \delta_i) \pi a^2 |H_0|^2 \frac{l(l+1)}{2l+1}$$

Combining this with (5) gives

$$Q = \omega \frac{U}{P_{\text{lost}}} \approx \frac{2h}{\delta_e + \delta_i}$$

This demonstrates that the numerical factor  $N$  is simply  $N = 2$  for all  $l$ .

- b) For the lowest Schumann resonance evaluate the  $Q$  value assuming  $\sigma_e = 0.1 (\Omega\text{m})^{-1}$ ,  $\sigma_i = 10^{-5} (\Omega\text{m})^{-1}$ ,  $h = 10^2 \text{ km}$ .

For the lowest Schumann resonance, we use  $\nu_1 = 10.6 \text{ Hz}$  or  $\omega_1 = 66.6 \text{ s}^{-1}$ . The skin depths are

$$\delta_e = \sqrt{\frac{2}{\mu_0 \sigma_e \omega_1}} \approx 500 \text{ m}$$

$$\delta_i = \sqrt{\frac{2}{\mu_0 \sigma_i \omega_1}} \approx 5 \times 10^4 \text{ m}$$

This gives a  $Q$  of

$$Q \approx \frac{2 \times 10^5 \text{ m}}{5 \times 10^2 \text{ m} + 5 \times 10^4 \text{ m}} \approx 4$$

- c) Discuss the validity of the approximations used in part a) for the range of parameters used in part b).

There are several issues to worry about, mainly related to the poor conductivity of the ionosphere. Firstly, the power loss expression

$$\frac{dP_{\text{loss}}}{da} = \frac{1}{2\sigma\delta} |\vec{K}_{\text{eff}}|^2$$

depended on extrapolating the tangential magnetic field  $\vec{H}_{\parallel}$  (which gives rise to the effective surface current density  $\vec{K}_{\text{eff}} = \hat{n} \times \vec{H}_{\parallel}$ ) into the conductor. The expressions that was used were made in the 'excellent conductor' approximation,  $\sigma \gg \omega\epsilon_0$ . Using the above parameters, we compute

$$\frac{\sigma}{\omega\epsilon_0} \approx 1.7 \times 10^4$$

so this is actually a very good approximation, despite the poor conductivity of the ionosphere. In reality, of course, a constant conductivity  $\sigma_i$  (independent of height) is not particularly realistic. But that is outside the scope of this problem.

The more important issue, however, is that the calculation of part *a*) involved using a perfect conductor approximation for the resonant mode. The ‘perfect conductor’ fields (4) are then substituted in to the energy and power loss expressions. So long as the the skin-depth is small compared to the size of the cavity, this is a reasonable approximation. However, here, we see that the ‘skin-depth’ of the ionosphere is about 50 km, while the height of the ionosphere is 100 km. Thus the penetration of the fields into the ionosphere is hardly small compared to the (vertical) size of the cavity. Since  $\delta_i/h \approx 1/2$ , we can expect such corrections to be on the order of 50%. Note, however, that the correction to the Schumann modes is not as severe as it could have been, as these modes are essentially independent of height. The penetration of the fields into the ionosphere give rise to an increased ‘effective height’, but otherwise do not drastically modify the resonances.

This increased effective height, however, indicates that the fields penetrate quite a bit into the ionosphere. In particular, this means there is a substantial amount of field energy that was unaccounted for in the calculation of the stored energy  $U$  of part *a*). From this point of view, an ‘improved’ expression for  $Q$  may be of the form

$$Q \approx \frac{2h_{\text{eff}}}{\delta_c + \delta_i}$$

where  $h_{\text{eff}} = h + \delta_i/2$ . Nevertheless, whatever expression we use for  $Q$  is essentially a reasonable result up to a factor of order unity.