

Homework Assignment #1 — Solutions

Textbook problems: Ch. 8: 8.2, 8.4, 8.5

8.2 A transmission line consisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ , as shown, is filled with a uniform lossless dielectric (μ, ϵ). A TEM mode is propagated along this line. Section 8.1 applies.

a) Show that the time-averaged power flow along the line is

$$P = \sqrt{\frac{\mu}{\epsilon}} \pi a^2 |H_0|^2 \ln \left(\frac{b}{a} \right)$$

where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor.

A TEM mode is essentially a two-dimensional electrostatic problem. Thus we start by finding the electric field between the two cylinders. By elementary means, it should be clear that

$$\vec{E}_t = \frac{A}{\rho} \hat{\rho}$$

where A is a constant that will be determined shortly. Using $\vec{B}_t = \sqrt{\mu\epsilon} \hat{k} \times \vec{E}_t$, and assuming wave propagation in the $+z$ direction, we find

$$\vec{H}_t = \sqrt{\frac{\epsilon}{\mu}} \frac{A}{\rho} \hat{\phi}$$

so that the magnitude of the magnetic field at the inner conductor is $H(a) = \sqrt{\epsilon/\mu}(A/a)$. Defining this as H_0 gives

$$\vec{E}_t = \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{a}{\rho} \hat{\rho}, \quad \vec{H}_t = H_0 \frac{a}{\rho} \hat{\phi} \quad (1)$$

The (harmonic) Poynting vector is then

$$\vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} |H_0|^2 \frac{a^2}{\rho^2} \hat{z}$$

so the power flow is

$$P = \int_A \hat{z} \cdot \vec{S} da = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} |H_0|^2 \int_a^b \frac{a^2}{\rho^2} 2\pi\rho d\rho = \pi \sqrt{\frac{\mu}{\epsilon}} |H_0|^2 a^2 \ln \left(\frac{b}{a} \right) \quad (2)$$

b) Show that the transmitted power is attenuated along the line as

$$P(z) = P_0 e^{-2\gamma z}$$

where

$$\gamma = \frac{1}{2\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(\frac{1}{a} + \frac{1}{b} \right)$$

We compute the attenuation coefficient according to

$$\gamma = -\frac{1}{2P} \frac{dP}{dz} \quad (3)$$

The power P was calculated in part a). For the power loss per unit length of the waveguide, we use

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \oint_C |\hat{n} \times \vec{H}|^2 dl = \frac{1}{2\sigma\delta} |H_0|^2 \oint_C \frac{a^2}{\rho^2} dl$$

Note that there are two boundaries, one at $\rho = a$ (with circumference $2\pi a$) and the other at $\rho = b$ (with circumference $2\pi b$). This gives

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} |H_0|^2 [2\pi a + (a/b)^2 2\pi b] = \frac{\pi}{\sigma\delta} |H_0|^2 \frac{a}{b} (a + b) \quad (4)$$

Inserting this power loss expression and the power (2) into (3) yields

$$\gamma = \frac{1}{2\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \frac{a + b}{ab \ln(b/a)} = \frac{1}{2\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(\frac{1}{a} + \frac{1}{b} \right)$$

c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position z . Show that for this line

$$Z_0 = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln \left(\frac{b}{a} \right)$$

Since $Z_0 = |\Delta V|/I$, we need to compute the voltage difference between the cylinders as well as the current. For the voltage difference, we have

$$\Delta V = - \int_a^b \vec{E} \cdot d\vec{l} = - \sqrt{\frac{\mu}{\epsilon}} H_0 \int_a^b \frac{a}{\rho} d\rho = - \sqrt{\frac{\mu}{\epsilon}} H_0 a \ln \left(\frac{b}{a} \right)$$

where we have used (1) for the electric field. In addition, the current is given by integrating the surface current density. For the inside conductor, we have

$$\vec{K} = \hat{n} \times \vec{H} = \hat{\rho} \times \left(H_0 \frac{a}{\rho} \hat{\phi} \right)_{\rho=a} = H_0 \hat{z}$$

Hence

$$I = \oint_C |K| dl = 2\pi a H_0$$

Taking the ratio $Z_0 = |\Delta V|/I$ results in

$$Z_0 = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln\left(\frac{b}{a}\right)$$

d) Show that the series resistance and inductance per unit length of the line are

$$R = \frac{1}{2\pi\sigma\delta} \left(\frac{1}{a} + \frac{1}{b} \right)$$

$$L = \left\{ \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) + \frac{\mu_c\delta}{4\pi} \left(\frac{1}{a} + \frac{1}{b} \right) \right\}$$

where μ_c is the permeability of the conductor. The correction to the inductance comes from the penetration of the flux into the conductors by a distance of order δ .

We may obtain the series resistance from the power loss

$$\frac{1}{2}|I|^2 R = -\frac{dP}{dz}$$

where R denotes the resistance per unit length. Using $-dP/dz$ from (4) as well as the current computed above, we find

$$R = \frac{2}{|I|^2} \left(-\frac{dp}{dz} \right) = \frac{1}{2\pi\sigma\delta} \frac{a+b}{ab}$$

For the inductance per unit length, we compute the energy per unit length stored in the magnetic field. Inside the volume of the waveguide, we have

$$U_{\text{vol}} = \int_A \frac{\mu}{4} |\vec{H}|^2 da = \frac{\mu}{4} |H_0|^2 \int_a^b \frac{a^2}{\rho^2} 2\pi\rho d\rho = \frac{\mu}{2} |H_0|^2 \pi a^2 \ln\left(\frac{b}{a}\right)$$

In addition, since some of the magnetic field penetrates the conducting walls, we use the approximation

$$H(\zeta) = H_{\parallel} e^{-\zeta/\delta} e^{i\zeta/\delta}$$

where ζ is the distance into the conductor. Assuming the skin depth is much less than the thickness of the conductor as well as the radius of curvature, we approximate

$$U_{\text{wall}} = C \int_0^{\infty} \frac{\mu_c}{4} |H(\xi)|^2 d\xi = \frac{\mu_c}{4} C |H_{\parallel}|^2 \int_0^{\infty} e^{-2\xi/\delta} d\xi = \frac{\mu_c}{8} C \delta |H_{\parallel}|^2$$

where C is the circumference of the wall. On the inside wall, we have $C = 2\pi a$ and $H_{\parallel} = H_0$, while on the outside wall, we have $C = 2\pi B$ and $H_{\parallel} = H_0(a/b)$. Hence

$$U_{\text{walls}} = \frac{\mu_c}{8} \delta |H_0|^2 [2\pi a + 2\pi b(a/b)^2] = \frac{\mu_c}{4} \pi \delta |H_0|^2 \frac{a}{b} (a + b)$$

Using

$$\frac{1}{4} L |I|^2 = U_{\text{vol}} + U_{\text{walls}}$$

we end up with

$$L = \frac{\mu}{2\pi} \ln \left(\frac{b}{a} \right) + \frac{\mu_c \delta}{4\pi} \frac{a + b}{ab}$$

8.4 Transverse electric and magnetic waves are propagated along a hollow, right circular cylinder with inner radius R and conductivity σ .

- a) Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (the dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part assume that the conductivity of the cylinder is infinite.

The eigenvalue equation for either TE or TM modes is

$$[\nabla_t^2 + \gamma^2] \psi(\rho, \phi) = 0$$

where $\psi(R, \phi) = 0$ for TM modes or $d\psi(\rho, \phi)/d\rho|_{\rho=R} = 0$ for TE modes. Writing $\psi(\rho, \phi) = \psi(\rho)e^{\pm im\phi}$, the cylindrical coordinates radial equation becomes

$$\left(\frac{1}{\rho} \partial_{\rho} \rho \partial_{\rho} + \gamma^2 - \frac{m^2}{\rho^2} \right) \psi(\rho) = 0$$

which is solved by Bessel functions. Avoiding the Neumann function which blows up at $\rho = 0$, we have

$$\psi(\rho, \phi) \sim J_m(\gamma\rho) e^{\pm im\phi}$$

The boundary conditions then place conditions on γ . For TM modes (Dirichlet conditions), we demand $J_m(\gamma R) = 0$. Hence

$$(TM) \quad \gamma_{mn} = \frac{x_{mn}}{R} \quad \text{or} \quad \omega_{mn} = \frac{x_{mn}}{\sqrt{\mu\epsilon}R}$$

where x_{mn} is the n -th zero of J_m . For TE modes (Neumann conditions), on the other hand, we demand $J'_m(\gamma R) = 0$. Hence

$$(TE) \quad \gamma_{mn} = \frac{x'_{mn}}{R} \quad \text{or} \quad \omega_{mn} = \frac{x'_{mn}}{\sqrt{\mu\epsilon}R}$$

where x'_{mn} is the n -th zero of J'_m . Sorting through the zeros of J_m and J'_m , the lowest five modes are given by

mode	$\sqrt{\mu\epsilon}R\omega_{mn}$	$\omega_{mn}/\omega_{\text{dominant}}$
TE ₁₁	1.841	1
TM ₀₁	2.405	1.306
TE ₂₁	3.054	1.659
TE ₀₂ and TM ₁₁	3.832	2.081

Note that the TE₀₂ and TM₁₁ modes are degenerate. This is a special case where the Bessel identity $J'_0(\zeta) = -J_1(\zeta)$ demonstrates that $x'_{0,n+1} = x_{1n}$.

- b) Calculate the attenuation constants of the waveguide as a function of frequency for the lowest two distinct modes and plot them as a function of frequency.

The computation of the attenuation coefficients involves computing both power P and power loss $-dP/dz$. We first consider TM modes. The power is given by

$$P = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \left(\frac{\omega}{\omega_{mn}} \right)^2 \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{1/2} \int_A |\psi|^2 da \quad (5)$$

Using $\psi = J_m(\gamma\rho)e^{\pm im\phi}$ gives

$$\int_A |\psi|^2 da = 2\pi \int_0^R J_m(x_{mn}\rho/R)^2 \rho d\rho = 2\pi \left[\frac{1}{2} R^2 J_{m+1}(x_{mn})^2 \right] = \pi R^2 J_{m+1}(x_{mn})^2$$

where the expression in the square brackets comes from Bessel function orthogonality/normalization. Hence

$$P = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \left(\frac{\omega}{\omega_{mn}} \right)^2 \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{1/2} \pi R^2 J_{m+1}(x_{mn})^2 \quad (6)$$

For a TM mode, the power loss is given by

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \left(\frac{\omega}{\omega_{mn}} \right)^2 \oint_C \frac{1}{\mu^2 \omega_{mn}^2} \left| \frac{\partial\psi}{\partial n} \right|^2 dl$$

In this case

$$\frac{\partial\psi}{\partial n} = - \frac{\partial\psi}{\partial\rho} \Big|_{\rho=R} = -\gamma_{mn} J'_m(x_{mn}) e^{\pm im\phi}$$

Using $\gamma_{mn}^2 = \mu\epsilon\omega_{mn}^2$, we obtain

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \frac{\epsilon}{\mu} (2\pi R) J'_m(x_{mn})^2$$

We now have some fun with Bessel functions. Using the recursion relation

$$J_{m+1}(\zeta) = \frac{m}{\zeta} J_m(\zeta) - J'_m(\zeta)$$

as setting $\zeta = x_{mn}$ to be a zero of J_m , we obtain

$$J_{m+1}(x_{mn}) = -J'_m(x_{mn})$$

This allows us to rewrite the power loss as

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \frac{\epsilon}{\mu} (2\pi R) J_{m+1}(x_{mn})^2 \quad (7)$$

Given (6) and (7), the TM_{mn} attenuation coefficient is obtained by setting

$$\beta_{mn} = -\frac{1}{2P} \frac{dP}{dz} = \frac{1}{2\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2} \frac{2\pi R}{\pi R^2} = \frac{1}{\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2} \frac{1}{R}$$

Note that $1/R = C/(2A)$ were $C = 2\pi R$ and $A = \pi R^2$ are the circumference and area of the cylindrical waveguide. Since $\delta = \delta_{mn} \sqrt{\omega_{mn}/\omega}$, we get the standard TM expression with the geometric factor $\xi_{mn} = 1$.

For the TE mode, the power loss calculation is somewhat lengthier, as it involves both H_z and \vec{H}_t . We begin with the power, which is given by a similar expression as (5), however with a factor of $\sqrt{\mu/\epsilon}$ instead. The Bessel normalization integral is now

$$\int_0^R J_m(x'_{mn}\rho/R)^2 \rho d\rho = \frac{1}{2} R^2 (1 - m^2/x'^2_{mn}) J_m(x'_{mn})^2$$

which gives

$$P = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} \left(\frac{\omega}{\omega_{mn}}\right)^2 \left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{1/2} \pi R^2 \left(1 - \frac{m^2}{x'^2_{mn}}\right) J_m(x'_{mn})^2 \quad (8)$$

This time, the power loss expression is

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \left(\frac{\omega}{\omega_{mn}}\right)^2 \oint_C \left[\frac{1}{\gamma_{mn}^2} \left(1 - \frac{\omega_{mn}^2}{\omega^2}\right) |\hat{n} \times \vec{\nabla}_t \psi|^2 + \frac{\omega_{mn}^2}{\omega^2} |\psi|^2 \right] dl$$

There are two terms to evaluate. The simple one is

$$\oint_C |\psi|^2 dl = (2\pi R) J_m(x'_{mn})^2$$

For the gradient term, we note that $\hat{n} = -\hat{\rho}$ on the inside of the cylinder. And $\vec{\nabla}_t = \hat{\rho}\partial_\rho + (1/\rho)\hat{\phi}\partial_\phi$. Hence

$$\oint_C |\hat{n} \times \vec{\nabla}_t \psi|^2 dl = (2\pi R) \left| \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \right|^2 = (2\pi R) \frac{m^2}{R^2} J_m(x'_{mn})^2$$

Combining these two terms yields

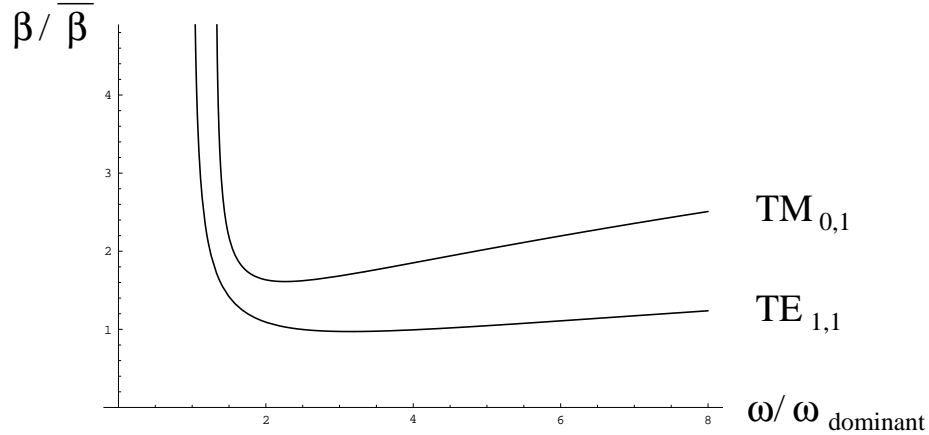
$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \left(\frac{\omega}{\omega_{mn}} \right)^2 (2\pi R) \left[\frac{m^2}{x'_{mn}{}^2} \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right) + \frac{\omega_{mn}^2}{\omega^2} \right]$$

Using this for the power loss and (8) for the power itself gives an attenuation coefficient

$$\begin{aligned} \beta_{mn} &= -\frac{1}{2P} \frac{dP}{dz} \\ &= \frac{1}{2\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{-1/2} \frac{2\pi R}{\pi R^2} \left[\frac{m^2}{x'_{mn}{}^2} \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right) + \frac{\omega_{mn}^2}{\omega^2} \right] \left[1 - \frac{m^2}{x'_{mn}{}^2} \right]^{-1} \\ &= \frac{1}{\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{-1/2} \frac{1}{R} \left[\frac{m^2}{x'_{mn}{}^2 - m^2} + \frac{\omega_{mn}^2}{\omega^2} \right] \end{aligned}$$

This demonstrates that the TE geometric factors are $\xi_{mn} = m^2/(x'_{mn}{}^2 - m^2)$ and $\eta_{mn} = 1$.

The attenuation constants are plotted as follows



where

$$\bar{\beta} = \frac{1}{\sigma\delta_{mn}} \sqrt{\frac{\epsilon}{\mu}} \frac{1}{R}$$

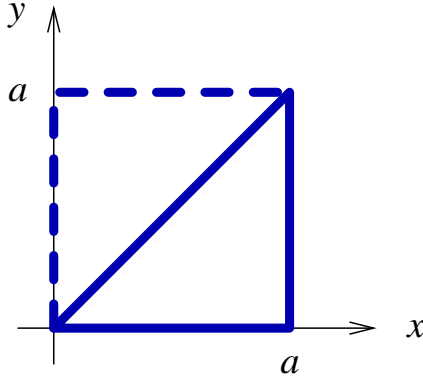
8.5 A waveguide is constructed so that the cross section of the guide forms a right triangle with sides of length a , a , $\sqrt{2}a$, as shown. The medium inside has $\mu_r = \epsilon_r = 1$.

- a) Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.

In general, to solve a problem like this, we need to consider the Dirichlet or Neumann problem for a boundary without any ‘standard’ (ie rectangular or circular) symmetry. In particular, this means there is no natural coordinate system to use for the two-dimensional Helmholtz equation $[\nabla_t^2 + \gamma^2]\psi = 0$ that both allows for separation of variables *and* respects the symmetry of the boundary surface (which

would allow a simple specification of the boundary data). A general problem of this form (with no simple boundary symmetry) is quite unpleasant to solve.

In this case, however, we can think of the triangle as ‘half’ of a square.



In particular, the key step to this problem is to note that the triangle may be obtained from the square by imposing reflection symmetry along the $x = y$ diagonal. This symmetry is a \mathbb{Z}_2 reflection on the coordinates of the form

$$\mathbb{Z}_2 : \quad x \rightarrow y, \quad y \rightarrow x$$

Eigenfunctions $\psi(x, y)$ can then be classified as either \mathbb{Z}_2 -even or \mathbb{Z}_2 -odd

$$\mathbb{Z}_2 : \quad \psi(x, y) \rightarrow \pm\psi(y, x)$$

The odd functions vanish along the diagonal, so they automatically satisfy Dirichlet conditions $\psi(x = y) = 0$ on the diagonal. Similarly, the even functions have vanishing normal derivative on the diagonal and hence automatically satisfy Neumann conditions. We will use this fact to construct TM and TE modes for the triangle.

We begin with the TM modes. Using rectangular coordinates, it is natural to write solutions of the Helmholtz equation $[\partial_x^2 + \partial_y^2 + \gamma^2]\psi = 0$ as $\psi \sim e^{i(k_x x + k_y y)}$ where $k_x^2 + k_y^2 = \gamma^2$. This means we may expand the eigenfunctions in terms of sines and cosines. For TM modes satisfying the Dirichlet condition $\psi_S = 0$, we start with eigenfunctions on the square

$$\psi \sim \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a}$$

which automatically satisfy the boundary conditions on the four walls of the square. This gives

$$\gamma_{mn} = \frac{\pi}{a} \sqrt{m^2 + n^2}$$

so the cutoff frequencies are

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu_0 \epsilon_0} a} \sqrt{m^2 + n^2} = \frac{\pi c}{a} \sqrt{m^2 + n^2} \quad (9)$$

In order to satisfy the Dirichlet condition on the diagonal, we take the \mathbb{Z}_2 -odd combination

$$(TM) \quad \psi_{mn} = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} - \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a}$$

It is simple to verify that $\psi(x, 0) = \psi(a, y) = \psi(x, x) = 0$, so that all boundary conditions on the triangle are indeed satisfied. The cutoff frequencies are given by (9). Note here that the \mathbb{Z}_2 projection removes the $m = n$ modes and also antisymmetrizes m with n . As a result, the integer labels m and n may be taken to satisfy the condition $m > n > 0$.

The analysis for TE modes is similar. However, for Neumann conditions, we take cosine combinations as well as a \mathbb{Z}_2 -even eigenfunction. This gives

$$(TE) \quad \psi_{mn} = \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{a} + \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{a}$$

with identical cutoff frequencies as in (9). This time, however, the labels m and n may be taken to satisfy $m \geq n \geq 0$ (except $m = n = 0$ is not allowed).

- b) For the lowest modes of each type calculate the attenuation constant, assuming that the walls have large, but finite, conductivity. Compare the result with that for a square guide of side a made from the same material.

The attenuation coefficients are determined by power and power loss. We begin with TM modes. For the power, we need to compute

$$\int_A |\psi|^2 da = \int_A [\sin k_m x \sin k_n y - \sin k_n x \sin k_m y]^2 da \quad (10)$$

It is perhaps easiest to compute this by integrating over the square and then dividing by two for the triangle. This is because the integration separates into x and y integrals, and we may use orthogonality

$$\int_0^a \sin k_i x \sin k_j x dx = \frac{a}{2} \delta_{i,j} \quad \left(\text{where } k_j = \frac{j\pi}{a} \right)$$

This gives

$$\int_A |\psi|^2 da = \frac{1}{2} \times 2 \left(\frac{a}{2} \right)^2 = \frac{a^2}{4}$$

The factor of $1/2$ is for the triangle, while the factor of 2 is because two non-vanishing terms arise when squaring the integrand in (10). (Recall that $m \neq n$ for TM modes.) This gives an expression for the power

$$\begin{aligned} P &= \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \left(\frac{\omega}{\omega_{mn}} \right)^2 \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{1/2} \int_A |\psi|^2 da \\ &= \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \left(\frac{\omega}{\omega_{mn}} \right)^2 \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{1/2} \frac{A}{2} \end{aligned}$$

where $A = a^2/2$ is the area of the triangle. Calculating the power loss involves integrating a normal derivative

$$\oint_C \left| \frac{\partial \psi}{\partial n} \right|^2 dl$$

We break this into three parts: along $y = 0$, along $x = a$ and along the diagonal $x = y$. Along the $y = 0$ wall, we have $\hat{n} = \hat{y}$ and

$$\left. \frac{\partial \psi}{\partial y} \right|_{y=0} = \frac{\pi}{a} [n \sin k_m x - m \sin k_n x]$$

As a result

$$\int_0^a \left| \frac{\partial \psi}{\partial y} \right|^2 dx = \left(\frac{\pi}{a} \right)^2 \frac{a}{2} (m^2 + n^2) = \frac{\pi^2}{2a} (m^2 + n^2) \quad (11)$$

A similar calculation, or use of symmetry, will result in an identical expression for the integral along the $x = a$ wall. For the diagonal, we use $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{y})$ to compute

$$\begin{aligned} \frac{\partial \psi}{\partial n} &= \frac{1}{\sqrt{2}} \left[\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} \right]_{y=x} = \sqrt{2} \frac{\pi}{a} [m \cos k_m x \sin k_n x - n \cos k_n x \sin k_m x] \\ &= \frac{\sqrt{2}}{2} \frac{\pi}{a} [(m - n) \sin(m + n)x - (m + n) \sin(m - n)x] \end{aligned}$$

This gives

$$\begin{aligned} \int_0^{\sqrt{a}} \left| \frac{\partial \psi}{\partial n} \right|^2 dl &= \sqrt{2} \int_0^a \left| \frac{\partial \psi}{\partial n} \right|^2 dx = \sqrt{2} \frac{1}{2} \left(\frac{\pi}{a} \right)^2 \frac{a}{2} [(m - n)^2 + (m + n)^2] \\ &= \sqrt{2} \frac{\pi^2}{2a} (m^2 + n^2) \end{aligned}$$

Combining this diagonal with (11) for the sides, we obtain

$$\oint_C \left| \frac{\partial \psi}{\partial n} \right|^2 dl = C \frac{\pi^2}{2a} (m^2 + n^2) = \frac{C}{2} \gamma_{mn}^2$$

where $C = a + a + \sqrt{2}a$ is the circumference of the triangle. This gives a TM mode power loss of

$$\begin{aligned} -\frac{dP}{dz} &= \frac{1}{2\sigma\delta} \left(\frac{\omega}{\omega_{mn}^2} \right)^2 \frac{1}{\mu^2 \omega_{mn}^2} \oint_C \left| \frac{\partial \psi}{\partial n} \right|^2 dl \\ &= \frac{1}{2\sigma\delta} \left(\frac{\omega}{\omega_{mn}^2} \right)^2 \frac{1}{\mu^2 \omega_{mn}^2} \frac{C}{2} \gamma_{mn}^2 = \frac{1}{2\sigma\delta} \left(\frac{\omega}{\omega_{mn}^2} \right)^2 \frac{\epsilon C}{\mu 2} \end{aligned}$$

The attenuation coefficient is thus

$$\beta_{mn} = -\frac{1}{2P} \frac{dP}{dz} = \frac{1}{\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2} \frac{C}{2A}$$

so that the geometrical factor $\xi_{mn} = 1$ is trivial. Note that, the energy loss calculation along the diagonal of the triangle gives the same result as along the square edges. As a result, the geometrical factor $\xi_{mn} = 1$ does not care whether the waveguide is square or right triangular. As a result, the triangular TM result is identical to the square TM result, at least up to the ratios $C/A = 2(2+\sqrt{2})/a \approx 6.83/a$ for the triangle and $C/A = 4/a$ for the square.

The power loss for the TE mode is somewhat harder to deal with because of the possibility of special cases. Consider

$$\psi = \cos k_m x \cos k_n y + \cos k_n x \cos k_m y \quad (12)$$

where $m \geq n \geq 0$. If $n = 0$, we end up with

$$\psi = \cos k_m x + \cos k_m y \quad (m > 0)$$

In this case

$$\int_A |\psi|^2 da = \frac{1}{2} \int_0^a dx \int_0^a dy [\cos k_m x + \cos k_m y]^2 = \frac{1}{2} \times 2(\frac{1}{2}a^2) = \frac{a^2}{2} = A$$

while the perimeter integrals are

$$\begin{aligned} \int_0^a dx |\psi(y=0)|^2 &= \int_0^a dx [1 + \cos k_m x]^2 = a(1 + \frac{1}{2}) = \frac{3a}{2} \\ \sqrt{2} \int_0^a dx |\psi(y=x)|^2 &= \sqrt{2} \int_0^a dx [2 \cos k_m x]^2 = 4\sqrt{2}(\frac{1}{2}a) = 2\sqrt{2}a \end{aligned}$$

which gives

$$\oint_C |\psi|^2 dl = (3 + 2\sqrt{2})a$$

and

$$\begin{aligned} \int_0^a dx |\hat{n} \times \vec{\nabla}_t \psi|^2 &= \int_0^a dx |\hat{y} \times \vec{\nabla}_t \psi|^2 = \int_0^a dx |-\hat{z} \partial_x \psi|_{y=0}^2 \\ &= \int_0^a dx \frac{\pi^2}{a^2} m^2 |\sin k_m x|^2 = \frac{\pi^2}{2a} m^2 \\ \sqrt{2} \int_0^a dx |\hat{n} \times \vec{\nabla}_t \psi|_{y=x}^2 &= \sqrt{2} \int_0^a dx |\frac{1}{\sqrt{2}} \hat{z} (\partial_y + \partial_x) \psi|_{y=x}^2 \\ &= \frac{\sqrt{2}}{2} \int_0^a dx \frac{\pi^2}{a^2} m^2 |2 \sin k_m x|^2 = \sqrt{2} \frac{\pi^2}{a} m^2 \end{aligned}$$

which gives

$$\oint_C |\hat{n} \times \vec{\nabla}_t \psi|^2 dl = (1 + \sqrt{2}) \frac{\pi^2}{a} m^2 = (1 + \sqrt{2}) a \gamma_{m0}^2$$

Using

$$P = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} \left(\frac{\omega}{\omega_{mn}} \right)^2 \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{1/2} \int_A |\psi|^2 da$$

and

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \left(\frac{\omega}{\omega_{mn}} \right)^2 \oint_C \left[\frac{1}{\gamma_{mn}^2} \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right) |\hat{n} \times \vec{\nabla}_t \psi|^2 + \frac{\omega_{mn}^2}{\omega^2} |\psi|^2 \right] dl$$

with the above integrals gives an attenuation coefficient

$$\begin{aligned} \beta_{m0} &= -\frac{1}{2P} \frac{dP}{dz} \\ &= \frac{1}{2\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(1 - \frac{\omega_{m0}^2}{\omega^2} \right)^{-1/2} \left[(1 + \sqrt{2}) \left(1 - \frac{\omega_{m0}^2}{\omega^2} \right) + \frac{\omega_{m0}^2}{\omega^2} (3 + 2\sqrt{2}) \right] \frac{a}{A} \\ &= \frac{1}{\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(1 - \frac{\omega_{m0}^2}{\omega^2} \right)^{-1/2} \left[\frac{1 + \sqrt{2}}{2 + \sqrt{2}} + \frac{\omega_{m0}^2}{\omega^2} \right] \frac{C}{2A} \end{aligned}$$

where $C = (2 + \sqrt{2})a$ and $A = a^2/2$. Here the geometrical factors are

$$\xi_{m0} = \frac{1 + \sqrt{2}}{2 + \sqrt{2}}, \quad \eta_{m0} = 1 \quad (m > n = 0)$$

For the rectangular waveguide, one has instead

$$\xi_{m0} = \frac{a}{a+b} \rightarrow \frac{1}{2}, \quad \eta_{m0} = \frac{2b}{a+b} \rightarrow 1 \quad \text{when } b \rightarrow a$$

This is different because the power loss calculation is no longer universal, giving different coefficients along the diagonal as along the square edges. The remaining TE cases to consider are modes (12) where $m = n > 0$ and $m > n > 0$. Here we simply state the results. For $m = n > 0$ we have

$$\psi = \cos k_m x \cos k_m y$$

(we have removed an unimportant factor of two) so that

$$\begin{aligned} \int_A |\psi|^2 da &= \frac{a^2}{8} = \frac{A}{4} \\ \oint_C |\psi|^2 dl &= \left(1 + \frac{3\sqrt{2}}{8} \right) a \\ \oint_C |\hat{n} \times \vec{\nabla}_t \psi|^2 dl &= \left(1 + \frac{\sqrt{2}}{4} \right) \frac{\pi^2}{a} m^2 = \left(\frac{1}{2} + \frac{\sqrt{2}}{8} \right) a \gamma_{mm}^2 \end{aligned}$$

This gives

$$\xi_{mm} = \frac{4 + \sqrt{2}}{4 + 2\sqrt{2}}, \quad \eta_{mm} = 1 \quad (m = n > 0)$$

On the other hand, for the general case $m > n > 0$ we find

$$\begin{aligned} \int_A |\psi|^2 da &= \frac{a^2}{4} = \frac{A}{2} \\ \oint_C |\psi|^2 dl &= (2 + \sqrt{2})a = C \\ \oint_C |\hat{n} \times \vec{\nabla}_t \psi|^2 dl &= (2 + \sqrt{2}) \frac{\pi^2}{2a} (m^2 + n^2) = \frac{C}{2} \gamma_{mn}^2 \end{aligned}$$

which yields

$$\xi_{mn} = 1, \quad \eta_{mn} = 1 \quad (m > n > 0)$$

In all cases, $\eta_{mn} = 1$, which is the same for the triangle or the square waveguide. For ξ_{mn} , the factor is essentially a geometric combination of contributions along the perimeter of either 1 or 1/2 depending on the particular mode and its degeneracies.