Phys. 506 Electricity and Magnetism Winter 2004

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Problem Set 9

Total 60 Points

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1. Problem 12.1 **10 Points**

Note: In part a) of 12.1, it is implied that the action is obtained by integrating over proper time. In part b), consider Eqns. 12.33f in Jackson.

a): Since for this Lagrangian the action integral is over proper time, the Euler-Lagrange equations are

$$\frac{d}{d\tau}\frac{\partial L}{\partial U^\gamma} = \frac{\partial L}{\partial x^\gamma}$$

or, $\frac{d}{d\tau}\frac{\partial L}{\partial U^\gamma} = \partial_\gamma L$. Detailed calculation:

$$\begin{aligned}
L &= -\frac{m}{2}U_\alpha U^\alpha - \frac{q}{c}U_\alpha A^\alpha \\
&= -\frac{m}{2}g_{\alpha\beta}U^\beta U^\alpha - \frac{q}{c}g_{\alpha\beta}U^\beta A^\alpha \\
\frac{\partial L}{\partial U^\gamma} &= -\frac{m}{2}g_{\alpha\beta}\left[\delta^\beta{}_\gamma U^\alpha + U^\beta \delta^\alpha{}_\gamma\right] - \frac{q}{c}g_{\alpha\beta}\delta^\beta{}_\gamma A^\alpha \\
&= -\frac{m}{2}\left[g_{\alpha\gamma}U^\alpha + g_{\gamma\beta}U^\beta\right] - \frac{q}{c}g_{\alpha\gamma}A^\alpha \\
&= -\frac{m}{2}\left[U_\gamma + U_\gamma\right] - \frac{q}{c}A_\gamma = -mU_\gamma - \frac{q}{c}A_\gamma \\
\frac{\partial L}{\partial x^\gamma} &= -\frac{q}{c}g_{\alpha\beta}U^\beta \partial_\gamma A^\alpha = -\frac{q}{c}U_\alpha \partial_\gamma A^\alpha \\
&= -\frac{q}{c}U^\alpha \partial_\gamma A_\alpha
\end{aligned}$$

Thus, the Euler-Lagrange equations are

$$\begin{aligned}
\frac{d}{d\tau}\left[mU_\gamma + \frac{q}{c}A_\gamma\right] &= \frac{q}{c}U^\alpha \partial_\gamma A_\alpha \\
m\frac{d}{d\tau}U_\gamma &= -\frac{q}{c}\frac{d}{d\tau}A_\gamma + \frac{q}{c}U^\alpha \partial_\gamma A_\alpha \\
m\frac{d}{d\tau}U_\gamma &= \frac{q}{c}\left[U^\alpha \partial_\gamma A_\alpha - \frac{d}{d\tau}A_\gamma\right] \\
m\frac{d}{d\tau}U_\gamma &= \frac{q}{c}\left[U^\alpha \partial_\gamma A_\alpha - \frac{dx^\alpha}{d\tau}\frac{\partial}{\partial x^\alpha}A_\gamma\right] \\
m\frac{d}{d\tau}U_\gamma &= \frac{q}{c}\left[U^\alpha \partial_\gamma A_\alpha - U^\alpha \partial_\alpha A_\gamma\right] \\
m\frac{d}{d\tau}U_\gamma &= \frac{q}{c}\left[\partial_\gamma A_\alpha - \partial_\alpha A_\gamma\right]U^\alpha \\
m\frac{d}{d\tau}U_\gamma &= \frac{q}{c}F_{\gamma\alpha}U^\alpha
\end{aligned}$$

$$m\frac{d}{d\tau}U^\gamma = \frac{q}{c}F^{\gamma\alpha}U_\alpha$$

The last two lines are equivalent forms of the covariant Lorentz force equation.

b): Following 12.33 of Jackson, it is $P^\alpha = -\frac{\partial L}{\partial U_\alpha}$. From the above result for $\frac{\partial L}{\partial U^\gamma}$, which equals $-P_\gamma$, we see

$$P^\alpha = mU^\alpha + \frac{q}{c}A^\alpha$$

and thus, by inserting into

$$\begin{aligned} H &= P^\alpha U_\alpha + L \\ &= \frac{m}{2}U^\alpha U_\alpha \\ &= \frac{1}{2m}(P^\alpha - \frac{q}{c}A^\alpha)(P_\alpha - \frac{q}{c}A_\alpha) \end{aligned}$$

The last line is the Hamiltonian in correct coordinates (position coordinates in the argument of A and conjugate momenta). The second line shows the value of H is the Lorentz invariant $H = \frac{m}{2}U^\alpha U_\alpha = \frac{mc^2}{2}$.

In space-time coordinates, use

$$P^\alpha - \frac{q}{c}A^\alpha = \begin{pmatrix} p^0 & - & \frac{q}{c}\Phi(\mathbf{x},t) \\ & & \\ \mathbf{p} & - & \frac{q}{c}\mathbf{A}(\mathbf{x},t) \end{pmatrix}$$

to see

$$H = \frac{1}{2m}\left((p^0)^2 - \mathbf{p}^2 + \frac{q^2}{c^2}\left[\Phi^2 - \mathbf{A}^2\right] + \frac{2q}{c}\left[\mathbf{p}\cdot\mathbf{A} - p^0\,\Phi\right]\right)$$

The relation between conjugate momenta and velocities is

$$\begin{pmatrix} p^0 \\ \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \gamma mc & + & \frac{q}{c}\Phi(\mathbf{x},t) \\ & & \\ \gamma m\mathbf{u} & + & \frac{q}{c}\mathbf{A}(\mathbf{x},t) \end{pmatrix}$$

2. Problem 12.5 **10 Points**

a): For $E < B$ we boost into a frame K' in which E' vanishes using a Lorentz transformation with boost velocity

$$\mathbf{u} = c\frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{cE}{B}\hat{\mathbf{z}}$$

in the given geometry. In K' we have $E' = 0$ and $\mathbf{B}' = \gamma^{-1}\mathbf{B} = \gamma^{-1}B\hat{\mathbf{y}}$ with $\gamma = \sqrt{1-(u/c)^2}^{-1}$. Then, the trajectory in K' is

$$\begin{aligned} x'(t') &= a\cos(\omega_B t') \\ y'(t') &= v_{||}t' \\ z'(t') &= a\sin(\omega_B t') \end{aligned}$$

where $\omega_B = \frac{qB'}{\gamma_a mc}$ and $\gamma_a = \frac{1}{\sqrt{1-\frac{v_{||}^2+\omega_B^2 a^2}{c^2}}}$. Transformation into K yields

$$\begin{aligned} x(t') &= x'(t') = a\cos(\omega_B t') \\ y(t') &= y'(t') = v_{||}t' \\ z(t') &= \gamma(z'(t') + ut') = \gamma(a\sin(\omega_B t') + ut') \\ t(t') &= \gamma(t' + \frac{u}{c^2}z'(t')) = \gamma(t' + \frac{u}{c^2}a\sin(\omega_B t')) \end{aligned}$$

The first three lines give the trajectory in K as a function of the time t' in K'. This is the most convenient form of a result. Note that the parameter in this result is t', i.e. the time in K'. The result can, in principle, be written as a function of t, the time in K, by inverting the fourth equation The result $t'(t)$ could be inserted into the first three equations. Since the fourth equation is transcendental equation, we don't do it.

Note. For $\gamma \approx 1$, the trajectory is a cycloid. In highly relativistic cases, however, the trajectory "cycloids" are stretched in the boost direction $\mathbf{E}\times\mathbf{B}$, and the whole thing isn't a cycloid any more.

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b): For $E > B$ we boost into a frame K' in which B' vanishes using a Lorentz transformation with boost velocity

$$\mathbf{u} = c\frac{\mathbf{E}\times\mathbf{B}}{E^2} = \frac{cB}{E}\hat{\mathbf{z}}$$

in the given geometry. In K' we have $B' = 0$ and $\mathbf{E}' = \gamma^{-1}\mathbf{E} = \gamma^{-1}E\hat{\mathbf{x}}$ with $\gamma = \sqrt{1-(u/c)^2}^{-1}$.

The trajectory in K' is found as follows. Call the velocity in the $y'z'$-plane of K' $v_\perp$ and the x'-component $v_{||}$. Then, the relativistic version of Newton's II law in K' reads

$$m\frac{d}{dt}\gamma_a(t')v_\perp(t') = 0$$
$$m\frac{d}{dt}\gamma_a(t')v_{||}(t') = qE'$$

Choosing a suitable space-time origin in K', the initial position and the initial longitudinal velocity are zero, and the initial transverse velocity is $v_\perp(t' = 0) = v_0$. Thus, without loss of generality and with $\gamma_0 := \sqrt{1-\frac{v_0^2}{c^2}}^{-1}$ it is

$$\begin{aligned} m\gamma_a(t')v_\perp(t') &= m\gamma_0 v_0 \\ m\gamma_a(t')v_{||}(t') &= qE't' \end{aligned}$$

Add the squares of these equations and note $\gamma_a^{-2}(t') = 1 - \frac{v^2(t')}{c^2}$. You find

$$\begin{aligned} v^2(t') &= \frac{c^2(\gamma_0^2 v_0^2 + q^2E'^2t'^2/m^2)}{\gamma_0^2 c^2 + q^2E'^2t'^2/m^2} \\ \gamma_a^2(t') &= \gamma_0^2 + \frac{q^2E'^2t'^2}{m^2c^2} \\ v_\perp(t') &= \frac{\gamma_0 v_0}{\gamma_a(t')} = \frac{\gamma_0 v_0}{\sqrt{\gamma_0^2 + \frac{q^2E'^2t'^2}{m^2c^2}}} \\ v_{||}(t') &= \frac{qE't'}{m\sqrt{\gamma_0^2 + \frac{q^2E'^2t'^2}{m^2c^2}}} \end{aligned}$$

With initial position at the origin, this integrates to

$$\begin{aligned} x_\perp(t') &= \frac{\gamma_0 v_0 mc}{qE'}\sinh^{-1}\left(\frac{qE't'}{\gamma_0 mc}\right) \\ x_{||}(t') &= \frac{\gamma_0 mc^2}{qE'}\left(\sqrt{1+\left(\frac{qE't'}{\gamma_0 mc}\right)^2} - 1\right) \end{aligned}$$

or, in terms of the primed coordinates of frame K' and with a fixed angle ϕ_0 describing the initial direction of motion in the $y'z'$-plane,

$$\begin{aligned} x'(t') &= \frac{\gamma_0 mc^2}{qE'}\left(\sqrt{1+\left(\frac{qE't'}{\gamma_0 mc}\right)^2} - 1\right) \\ y'(t') &= \cos\phi_0\frac{\gamma_0 v_0 mc}{qE'}\sinh^{-1}\left(\frac{qE't'}{\gamma_0 mc}\right) \\ z'(t') &= \sin\phi_0\frac{\gamma_0 v_0 mc}{qE'}\sinh^{-1}\left(\frac{qE't'}{\gamma_0 mc}\right) \end{aligned}$$

This can be transformed into frame K. With constant $\delta := \frac{qE'}{\gamma_0 mc}$, we find

$$\begin{aligned} x(t') &= \frac{c}{\delta}\left(\sqrt{1+(\delta t')^2}-1\right) \\ y(t') &= \cos\phi_0 \frac{v_0}{\delta}\sinh^{-1}(\delta t') \\ z(t') &= \gamma(z'(t')+ut') = \gamma\left(\sin\phi_0\frac{v_0}{\delta}\sinh^{-1}(\delta t')+ut'\right) \end{aligned}$$

Again, it's best to just leave the time in K' as trajectory parameter.

3. Problem 12.9 **10 Points**

a): In Gaussian units, the magnetic field of a dipole $\mathbf{m} = -m\hat{\mathbf{z}}$ is

$$\mathbf{B} = -\frac{m}{r^3}(\hat{\mathbf{r}}\, 2\cos\theta + \hat{\theta}\sin\theta)$$

We consider the contour line $f(r,\theta) = 0$ for the function $f(r,\theta) = r - r_0 \sin^2\theta$. On that line, it is $r = r_0 \sin^2\theta$. Also, on the contour line the gradient

$$\nabla f = \hat{\mathbf{r}} - \hat{\theta}\frac{2r_0}{r}\sin\theta\cos\theta = \hat{\mathbf{r}} - \hat{\theta}\frac{2r_0}{r_0\sin^2\theta}\sin\theta\cos\theta = \hat{\mathbf{r}} - \hat{\theta}\frac{2\cos\theta}{\sin\theta}$$

We then see that on the contour line

$$\mathbf{B}\cdot\nabla f = 0$$

Thus, the contour line $f = 0$ is a magnetic-field line, and

$$r(\theta) = r_0 \sin^2\theta$$

describes the radial coordinate of that line as a function of θ. Insertion of $r(\theta)$ into the equation for $\mathbf{B}$ yields, along a given magnetic-field line,

$$\mathbf{B}(\theta) = -\frac{m}{r_0^3\sin^6\theta}(\hat{\mathbf{r}}\, 2\cos\theta + \hat{\theta}\sin\theta)$$

and, for the magnitude

$$B(\theta) = \frac{m}{r_0^3\sin^6\theta}\sqrt{4 - 3\sin^2\theta}$$

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b): From $\nabla B = -\frac{3m}{r^4}\left(\hat{\mathbf{r}}\sqrt{4-3\sin^2\theta} + \hat{\theta}\frac{\sin\theta\cos\theta}{\sqrt{4-3\sin^2\theta}}\right)$ it follows that in the equatorial plane $\theta = \pi/2$

$$\mathbf{B}\times\nabla B = \mathbf{B}\times\nabla_\perp B = -\hat{\phi}\frac{3m^2}{r^7} = -\hat{\phi}\frac{3B^2}{r}$$

and the gradient drift velocity

$$\mathbf{v}_G = \omega_B\frac{a^2}{2B^2}\mathbf{B}\times\nabla_\perp B = -\hat{\phi}\omega_B\frac{3a^2}{2r}$$

There, $\omega_B = \frac{qB}{\gamma mc}$ is the cyclotron frequency and a the cyclotron radius. For an average radial coordinate of the particle $R \gg a$ it then is

$$\mathbf{v}_G = \hat{\phi} R \dot{\phi} = -\hat{\phi} \omega_B \frac{3a^2}{2R}$$

and therefore $\dot{\phi} = -\omega_B \frac{3a^2}{2R^2}$. This integrates to

$$\phi(t) = \phi_0 - \frac{3a^2}{2R^2} \omega_B (t - t_0) \quad \text{q.e.d.}$$

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c): Since $v_{||}^2 = v_0^2 - v_{\perp 0}^2 \frac{B}{B_0}$ and $v_{||} = R\dot{\theta}$ it is

$$R^2 \dot{\theta}^2 = v_0^2 - v_{\perp 0}^2 \frac{B(\theta)}{B_0}$$

Taking the time derivative,

$$2R^2 \dot{\theta} \ddot{\theta} = -\frac{v_{\perp 0}^2}{B_0} \frac{dB(\theta)}{d\theta} \dot{\theta}$$

Redefining $\theta = \pi/2 + \alpha$ and noting that the problem statement implies $\alpha \ll 1$, we see

$$\ddot{\alpha} = -\frac{v_{\perp 0}^2}{2R^2 B_0} \frac{dB(\pi/2 + \alpha)}{d\alpha}$$

Since the expansion of $B(\pi/2 + \alpha) = \frac{m}{r_0^3 \cos^6 \alpha} \sqrt{4 - 3\cos^2 \alpha}$ for small α yields

$$B(\alpha) \approx \frac{m}{r_0^3} + \frac{9m\alpha^2}{2r_0^3}$$

it is $\frac{dB(\pi/2+\alpha)}{d\alpha} = \frac{9m\alpha}{r_0^3}$. Since also $r_0 = R$ for small α, we conclude

$$\ddot{\alpha} = -\frac{v_{\perp 0}^2}{2R^2 B_0} \frac{9m\alpha}{R^3} = -\Omega^2 \alpha$$

with $\Omega = 3v_{\perp 0} \sqrt{\frac{m}{2R^5 B_0}}$. Also, $B_0 = m/R^3$ and $v_{\perp 0} = \omega_B a$ with the cyclotron frequency ω_B and initial cyclotron radius a. Thus,

$$\Omega = 3\omega_B a \sqrt{\frac{1}{2R^2}} = \frac{3\omega_B a}{R\sqrt{2}} \quad \text{, q.e.d.}$$

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d): $\underline{E_{kin} = 10\text{MeV electron}}$:

$$
\begin{aligned}
\gamma &= \frac{E_{kin} + mc^2}{mc^2} = \frac{10.511MeV}{0.511MeV} = 19.6 \\
v_\perp &\approx v \approx c \\
B_0 &= \frac{m}{R^3} = 3mGauss \\
\omega_B &= \frac{4.8 \times 10^{-10} statcoulomb \cdot 3 \times 10^{-3} Gauss}{19.6 \cdot 9.1 \times 10^{-28} grams \cdot 3 \times 10^{10} cm/s} = 2\pi \times 408 s^{-1} \\
a &= v_\perp / \omega_B = 117km \\
T_\phi &= \frac{4\pi R^2}{3\omega_B a^2} = 107s \\
T_\theta &= \frac{2\pi\sqrt{2}R}{3v_\perp} = 0.3s
\end{aligned}
$$

$E_{kin} = 10$keV electron:

$$
\begin{aligned}
\gamma &= 1.0196 \\
v_\perp &\approx v = 0.195c \\
B_0 &= 3mGauss \\
\omega_B &= 2\pi \times 8230 s^{-1} \\
a &= 1.13km \\
T_\phi &= 15.9h \\
T_\theta &= 1.52s
\end{aligned}
$$

4. Problem 12.14 **10 Points**

$$
\begin{aligned}
L &= -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c}J_\alpha A^\alpha \\
&= -\frac{1}{8\pi} g_{\alpha\gamma} g_{\beta\delta} \partial^\gamma A^\delta \partial^\alpha A^\beta - \frac{1}{c}J_\alpha A^\alpha \\
\frac{\partial L}{\partial^\epsilon A^\eta} &= -\frac{1}{8\pi} g_{\alpha\gamma} g_{\beta\delta} \left[\delta^\gamma{}_\epsilon \delta^\delta{}_\eta \partial^\alpha A^\beta + \partial^\gamma A^\delta \delta^\alpha{}_\epsilon \delta^\beta{}_\eta\right] \\
&= -\frac{1}{8\pi}\left[g_{\alpha\epsilon} g_{\beta\eta} \partial^\alpha A^\beta + g_{\epsilon\gamma} g_{\eta\delta} \partial^\gamma A^\delta\right] \\
&= -\frac{1}{8\pi}\left[\partial_\epsilon A_\eta + \partial_\epsilon A_\eta\right] = -\frac{1}{4\pi}\partial_\epsilon A_\eta \\
\frac{\partial L}{\partial A^\eta} &= -\frac{1}{c} J_\alpha \delta^\alpha{}_\eta = -\frac{1}{c} J_\eta
\end{aligned}
$$

Thus, the Euler-Lagrange equations are

$$
\begin{aligned}
\partial^\epsilon \frac{\partial L}{\partial^\epsilon A^\eta} &= \frac{\partial L}{\partial A^\eta} \\
-\frac{1}{4\pi}\partial^\epsilon \partial_\epsilon A_\eta &= -\frac{1}{c} J_\eta \\
\partial^\epsilon \partial_\epsilon A_\eta &= \frac{4\pi}{c} J_\eta
\end{aligned}
$$

These are equivalent to the inhomogeneous Maxwell equations in the Lorentz gauge, $\partial_\alpha A^\alpha = 0$.

We take the difference of the two Lagrangian densities in question,

$$
\begin{aligned}
L - L' &= -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta \\
&= -\frac{1}{16\pi}\left[\partial_\alpha A_\beta \partial^\alpha A^\beta + \partial_\beta A_\alpha \partial^\beta A^\alpha - \partial_\beta A_\alpha \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha\right] + \frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta \\
&= \frac{1}{8\pi}\partial_\beta A_\alpha \partial^\alpha A^\beta
\end{aligned}
$$

where we mean, as usual, $L - L' = \frac{1}{8\pi}(\partial_\beta A_\alpha)(\partial^\alpha A^\beta)$. <u>Under the condition of the Lorentz gauge, $\partial_\alpha A^\alpha = 0$</u>, we may write

$$
\begin{aligned}
L - L' &= \frac{1}{8\pi}(\partial_\beta A_\alpha)(\partial^\alpha A^\beta) \\
&= \frac{1}{8\pi}\partial_\beta (A_\alpha \partial^\alpha A^\beta) - \frac{1}{8\pi} A_\alpha \partial_\beta \partial^\alpha A^\beta \\
&= \frac{1}{8\pi}\partial_\beta (A_\alpha \partial^\alpha A^\beta) - \frac{1}{8\pi} A_\alpha \partial^\alpha (\partial_\beta A^\beta) \\
&= \frac{1}{8\pi}\partial_\beta (A_\alpha \partial^\alpha A^\beta) - \frac{1}{8\pi} A_\alpha \partial^\alpha (0) \\
&= \frac{1}{8\pi}\partial_\beta (A_\alpha \partial^\alpha A^\beta) = \partial_\beta \Lambda^\beta
\end{aligned}
$$

which is the four-divergence of the 4-vector $\Lambda^\beta = \frac{1}{8\pi}(A_\alpha \partial^\alpha A^\beta)$, q.e.d.

Then, using the four-dimensional generalization of the divergence theorem the difference in the corresponding actions is

$$\begin{aligned} A - A' &= \int_{4-volmue} (L - L')d^4x = \int \partial_\beta \Lambda^\beta d^4x \\ &= \int_{4-volmue} (\frac{\partial}{\partial x^0}\Lambda^0 + \nabla \cdot \underline{\Lambda})dx^0 d^3x \\ &= \int_{4-surface} (\Lambda^0 n^0 + \underline{\Lambda} \cdot \mathbf{n})d^3a \end{aligned}$$

There, n is a 4-dimensional unit vector on the 4-surface containing the fields. Note that n is a unit vector with the usual cartesian norm of 1, i.e. $(n^0)^2 + \mathbf{n} \cdot \mathbf{n} = 1$. The 4-vector Λ^β is defined only through the potentials and their derivatives. Further, the variation principle is such that the potential and the field values (the field values are essentially the derivatives of the potentials) are not varied on the 4-surface. Thus, Λ^β is not varied on the surface, and

$$\begin{aligned} A - A' &= constant \\ \delta A &= \delta A' \end{aligned}$$

The added four-divergence changes the action merely by a constant, and the variations of the actions are the same. In particular, both actions become minimal for the same potentials A^α. The equations of motion for A^α must therefore also be the same in both cases, so as to produce identical solutions.

The equivalence of the equations of motion was seen explicitly in part a).

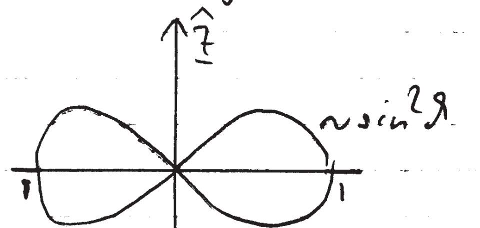
5. Problem 14.4 10 points

14.4a) $z(t) = a\cos(\omega_0 t)$, **Use non-relativistic equations, $v \ll c$**

Use Eq. 14.21, $\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\dot{v}|^2 \sin^2\vartheta$, where for the given geometry the angle ϑ is the usual polar angle ϑ.

Thus, $\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} a^2\omega_0 \sin^2\vartheta \cos^2(\omega_0 t)_{\text{retarded}}$. The cycle average is

$$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2 a^2 \omega_0^4}{8\pi c^3}\sin^2\vartheta}$$



$\langle dP/d\Omega \rangle$ in relative units.

$$P = \frac{e^2 a^2 \omega_0^4}{8\pi c^3} 2\pi \underbrace{\int_{-1}^{1} \sin^2\vartheta \, d\cos\vartheta}_{=4/3}$$

$$\boxed{P = \frac{e^2 a^2 \omega_0^4}{3c^3}}$$

The radiation pattern is that of an electric dipole in the $\hat{z}$-direction.

b) $\underline{r}(t) = \hat{x} R\cos\omega_0 t + \hat{y} R\sin\omega_0 t$

We may use complex notation, $\underline{r}(t) = (\hat{x} + i\hat{y})e^{-i\omega_0 t} R$

$$\dot{\underline{\beta}} = -\frac{R\omega_0^2}{c}(\hat{x} + i\hat{y})e^{-i\omega_0 t}$$

Eq. 14.18 => the electric radiation field is

$$\boxed{\underline{E}_a = \frac{e}{c}\frac{1}{r}\hat{n}\times(\hat{n}\times\dot{\underline{\beta}})\Big|_{\text{ret.}}}$$, where r is the distance of observer.

The cycle-averaged Poynting vector is

$$\boxed{\underline{S} = \frac{c}{8\pi}\hat{n}|\underline{E}_a|^2}$$ (note factor $\frac{1}{2}$ wrt. to Eq. 14.19)
(factor is due to complex notation)

The radiated power per $d\Omega$ is

$$(*) \quad \boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = r^2(\hat{n}\cdot\underline{S}) = \frac{r^2 c}{8\pi}|\underline{E}_a|^2}$$ ($\underline{E}_a$ in complex notation)

the total radiated power is $\boxed{P = \int_\Omega \langle \frac{dP}{d\Omega} \rangle d\Omega}$

To find $|E_a|^2$ we use $\hat{n} \times (\hat{n} \times \dot{\beta}) = (\hat{n} \cdot \dot{\beta})\hat{n} - (\hat{n} \cdot \hat{n})\dot{\beta}$ with

$$\dot{\beta} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \left(-\frac{R\omega_0^2}{c}\right) \quad \text{and} \quad \hat{n} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$

Then, $$\left|\hat{n} \times (\hat{n} \times \dot{\beta})\right|^2 = \frac{R^2\omega_0^4}{c^2} \left| \sin\vartheta\, e^{i\varphi} \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} - \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \right|^2 =$$

$$= \frac{R^2\omega_0^4}{c^2} \Big((\sin^2\vartheta\, e^{i\varphi}\cos\varphi - 1)(\sin^2\vartheta\, e^{-i\varphi}\cos\varphi - 1) + (\sin^2\vartheta\, e^{i\varphi}\sin\varphi - i) \times$$

$$\times (\sin^2\vartheta\, e^{-i\varphi}\sin\varphi + i) + \sin^2\vartheta\cos^2\vartheta \Big)$$

$$= \frac{R^2\omega_0^4}{c^2} \left(\sin^4\vartheta\cos^2\varphi + 1 - 2\sin^2\vartheta\cos^2\varphi + \sin^4\vartheta\sin^2\varphi + 1 - 2\sin^2\vartheta\sin^2\varphi + \sin^2\vartheta\cos^2\vartheta \right)$$

$$= \frac{R^2\omega_0^4}{c^2} (1 + \cos^2\vartheta)$$ Thus, $$\underline{|E_a|^2 = \frac{e^2 R^2 \omega_0^4}{c^4 r^2} (1 + \cos^2\vartheta)}$$

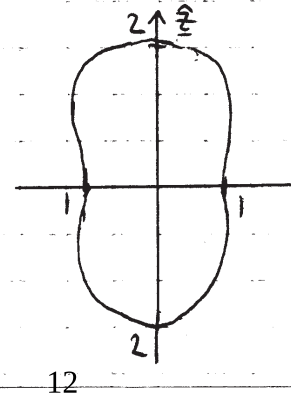
Insert into ⊛: $$\boxed{\langle \frac{dP}{d\Omega} \rangle = \frac{e^2 R^2 \omega_0^4}{c^3\, 8\pi} (1 + \cos^2\vartheta)}$$

Total radiated power: $$P = \int \langle \frac{dP}{d\Omega} \rangle d\Omega = 2\pi \cdot \frac{8}{3} \cdot \frac{e^2 R^2 \omega_0^4}{c^3 \cdot 8\pi} =$$

$$\boxed{P = \frac{2}{3} \frac{e^2 R^2 \omega_0^4}{c^3}}$$

Note that the power P in b) is twice that of problem part a) (with $a = R$).

This is to be expected, because the circular oscillator can be viewed as a superposition of two linear oscillators in the $\hat{x}$ and $\hat{y}$-directions with a $\frac{\pi}{2}$ phase difference.

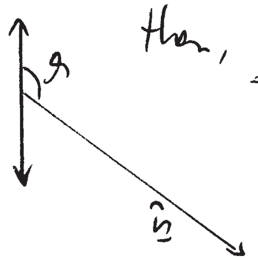


$\langle dP/d\Omega \rangle$ in relative units

14.12a) Since the problem implies relativistic motion, we use Eq. 14.38,

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c} \frac{|\underline{n} \times ((\underline{n}-\underline{\beta}) \times \dot{\underline{\beta}})|^2}{(1-\hat{n}\cdot\underline{\beta})^5}$$

where $\underline{r}(t') = \hat{z}\, a \cos(\omega_0 t')$, $\underline{\beta}(t') = -\frac{\omega_0 a}{c} \sin(\omega_0 t')\hat{z}$, $\dot{\underline{\beta}}(t') = -\frac{\omega_0^2 a}{c} \cos(\omega_0 t')\hat{z}$



then, $1-\hat{n}\cdot\underline{\beta} = 1+(\cos\vartheta)\frac{\omega_0 a}{c}\sin(\omega_0 t')$

also, $\underline{\beta} \times \dot{\underline{\beta}} = 0$, and $|\underline{n} \times (\underline{n} \times \dot{\underline{\beta}})|^2 = (\sin^2\vartheta)\frac{\omega_0^4 a^2}{c^2}\cos^2(\omega_0 t')$

Thus, using $\beta = \frac{a\omega_0}{c}$, it is

$$\boxed{\frac{dP(t')}{d\Omega} = \frac{e^2 c}{4\pi a^2}(\sin^2\vartheta)\ \beta^4 \cos^2(\omega_0 t') \Big/ (1+\beta\cos\vartheta \sin(\omega_0 t'))^5}$$ q.e.d.

14.12b)

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2 c \beta^4 \sin^2\vartheta}{8\pi^2 a^2} \int_0^{2\pi} \frac{\cos^2\varphi}{(1+\beta\cos\vartheta\sin\varphi)^5}\, d\varphi$$

$$= \frac{e^2 c \beta^4 \sin^2\vartheta}{8\pi^2 a^2}\ \frac{\pi}{4}\ \frac{4+\beta^2\cos^2\vartheta}{(1-\beta^2\cos^2\vartheta)^{7/2}} = \boxed{\frac{e^2 c\beta^4}{32\pi a^2}\left[\frac{4+\beta^2\cos^2\vartheta}{(1-\beta^2\cos^2\vartheta)^{7/2}}\right]\sin^2\vartheta}$$

q.e.d.

14.12c) Non-relativistic limit:

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2 c\,\beta^4}{8\pi a^2}\sin^2\vartheta = \frac{e^2 a^2 \omega_0^4}{8\pi c^3}\sin^2\vartheta,$$ which is the (result of problem 14.4.)

relativistic limit:

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2 c}{32\pi a^2}\ \frac{(4+\cos^2\vartheta)\sin^2\vartheta}{(1-[1-\frac{1}{\gamma^2}]\cos^2\vartheta)^{7/2}}$$