Phys. 506 | Electricity and Magnetism | Winter 2004

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Problem Set 8

Total 40 Points

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1. Problem 11.13 **10 Points**

a): A line charge with linear density q_0 is placed on the $\hat{\mathbf{z}}'$-axis of its rest frame K'. Then, in K'

$$\mathbf{B}' = 0 \quad \text{and} \quad \mathbf{E}' = \frac{2q_0}{\rho'}\hat{\rho}'$$

In SI units, $\mathbf{E}' = \frac{q_0}{2\pi\epsilon_0\rho'}\hat{\rho}'$. The fields in K are obtained from the inverse of Eqns. 11.149 of Jackson with $\underline{\beta} = \frac{\mathbf{v}}{c} = \hat{\mathbf{z}}\frac{v}{c}$. The inverse of Eqs. 11.149 is obtained by flipping the signs of all linear occurrences of $\underline{\beta}$ and swapping primed and unprimed field variables,

$$\mathbf{E} = \gamma(\mathbf{E}' - \underline{\beta} \times \mathbf{B}') - \frac{\gamma^2}{\gamma+1}\underline{\beta}(\underline{\beta}\cdot\mathbf{E}')$$
$$\mathbf{B} = \gamma(\mathbf{B}' + \underline{\beta} \times \mathbf{E}') - \frac{\gamma^2}{\gamma+1}\underline{\beta}(\underline{\beta}\cdot\mathbf{B}')$$

In the present case, the boost is in the z-direction. Since the directions transverse to the boost don't undergo length contraction, at event coordinates $(ct', \mathbf{x}')$ in K' and $(ct, \mathbf{x})$ in K related through the Lorentz transformation the unit vectors $\hat{\rho}'$ and $\hat{\phi}'$ in K' and the unit vectors $\hat{\rho}$ and $\hat{\phi}$ in K are identical. The same applies to the transverse coordinates, i.e. $\rho' = \rho$ and $\phi' = \phi$. Thus,

$$\begin{aligned}
\mathbf{E} &= \gamma\mathbf{E}' = \frac{2\gamma q_0}{\rho'}\hat{\rho}' = \frac{2\gamma q_0}{\rho}\hat{\rho} \\
\mathbf{B} &= \frac{2\gamma q_0}{\rho'}\frac{v}{c}(\hat{\mathbf{z}}\times\hat{\rho}') = \frac{2\gamma q_0}{\rho}\frac{v}{c}(\hat{\mathbf{z}}\times\hat{\rho}) = \frac{2\gamma v q_0}{c\rho}\hat{\phi}
\end{aligned}$$

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b): In K': The volume charge density $\sigma'(\mathbf{x}')$ in K' is

$$\sigma'(\rho') = \frac{q_0\delta(\rho')}{\pi\rho'}$$

because the transverse integral

$$\int_{\rho'=0}^{\infty}\sigma'(\rho')2\pi\rho' d\rho' = \frac{1}{2}\int_{\rho'=-\infty}^{\infty}\frac{\rho' q_0\delta(\rho')}{\pi\rho'}2\pi d\rho' = q_0 \quad ,$$

as required. The current in K' is zero. Thus, the four-current in K'

$$J'^{\alpha}(ct', \mathbf{x}') = (c\sigma'(ct', \mathbf{x}'), \mathbf{J}'(ct', \mathbf{x}')) = \left(\frac{q_0 c\delta(\rho')}{\pi\rho'}, 0\right)$$

In K:. To transform the four-current and coordinates, we use the coordinate-free form of the Lorentz transformation (inverse of Eq. 11.19 of Jackson),

$$
\begin{aligned}
x^0 &= \gamma(x'^0 + \underline{\beta}\cdot\mathbf{x}') \\
\mathbf{x} &= \mathbf{x}' + \gamma x'^0\underline{\beta} + \frac{\gamma-1}{\beta^2}\underline{\beta}(\underline{\beta}\cdot\mathbf{x}')
\end{aligned}
$$

Thus,

$$
\begin{aligned}
c\sigma(ct,\mathbf{x}) &= \gamma\, c\sigma'(ct'[ct,\mathbf{x}], \mathbf{x}'[ct,\mathbf{x}]) = \gamma\frac{q_0 c\delta(\rho'[ct,\mathbf{x}])}{\pi\rho'[ct,\mathbf{x}]} = \frac{q_0 c\gamma\delta(\rho)}{\pi\rho} \\
\mathbf{J}(ct,\mathbf{x}) &= \gamma\,\underline{\beta}c\sigma'(ct'[ct,\mathbf{x}], \mathbf{x}'[ct,\mathbf{x}]) = \hat{\mathbf{z}}\frac{q_0\gamma v\delta(\rho)}{\pi\rho}
\end{aligned}
$$

and the current 4-vector in K is

$$
J^\alpha(ct,\mathbf{x}) = J^\alpha(\rho) = \left(\frac{q_0 c\gamma\delta(\rho)}{\pi\rho}, \hat{\mathbf{z}}\frac{q_0\gamma v\delta(\rho)}{\pi\rho}\right) = \frac{q_0\gamma\delta(\rho)}{\pi\rho}(c, \hat{\mathbf{z}}v)
$$

Note the considerable simplification of the present problem arising from the fact that the transverse coordinates in K and K' are the same. Also, note that the 4-current in K looks just like what one would obtain from a Galilean transformation, except the additional factor γ in the charge and current density. This factor is due to the non-Galilean effect of the length contraction of the wire. Due to charge conservation, the length-contracted wire in K has a charge density that is larger than the proper charge density by γ (i.e. the inverse of the length contraction factor).

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c): Since the charge density in K is enhanced by the factor γ, take result from a) and multiply with γ,

$$
\mathbf{E}(\rho) = \frac{2\gamma q_0}{\rho}\hat{\rho}
$$

(In SI units, $\mathbf{E}(\rho) = \frac{\gamma q_0}{2\pi\epsilon_0\rho}\hat{\rho}$.)

For the magnetic field, you may use symmetry and Ampere's law in integral form with a circle of radius ρ and integration direction $\hat{\phi}$,

$$
\begin{aligned}
\oint \mathbf{B}\cdot\mathbf{dl} &= \frac{4\pi}{c}\int \mathbf{J}\cdot\mathbf{da} \\
B(\rho)2\pi\rho &= \frac{4\pi}{c}\int_{\rho=0}^{\infty}\frac{q_0\gamma v\delta(\rho)}{\pi\rho}\hat{\mathbf{z}}\cdot[\hat{\mathbf{z}}2\pi\rho d\rho] \\
B(\rho)\rho &= \frac{4q_0\gamma v}{c}\int_{\rho=0}^{\infty}\delta(\rho)d\rho = \frac{2q_0\gamma v}{c}\int_{\rho=-\infty}^{\infty}\delta(\rho)d\rho = \frac{2q_0\gamma v}{c} \\
B(\rho) &= \frac{2q_0\gamma v}{c\rho} \\
\mathbf{B}(\rho) &= \hat{\phi}\frac{2q_0\gamma v}{c\rho}
\end{aligned}
$$

(To convert to SI-units, replace $\frac{4\pi}{c}\rightarrow\mu_0$, yielding $\mathbf{B}(\rho) = \hat{\phi}\frac{q_0\gamma v}{2\pi\rho}$.)

2. Problem 11.14 **10 Points**

a): One Lorentz scalar is the contraction of the antisymmetric contravariant field tensor $F^{\alpha\beta}$, the matrix form of which we denote $\mathcal{F}u$, with the covariant field tensor $F_{\alpha\beta}$, the matrix form of which we denote $\mathcal{F}d$:

$$\begin{aligned}
F^{\alpha\beta}F_{\alpha\beta} &= -F^{\alpha\beta}F_{\beta\alpha} = -\mathrm{Trace}(\mathcal{F}u \circ \mathcal{F}d) \\
&= -\mathrm{Trace}\left[\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}\right] \\
&= 2(B^2 - E^2) \quad .
\end{aligned}$$

Similarly, it is found that the contraction of the covariant field tensor $F_{\alpha\beta}$ with the contravariant dual field tensor $DF^{\alpha\beta}$, the matrix form of which we denote $\mathcal{DF}u$, is

$$\begin{aligned}
DF^{\alpha\beta}F_{\alpha\beta} &= -DF^{\alpha\beta}F_{\beta\alpha} = -\mathrm{Trace}(\mathcal{DF}u \circ \mathcal{F}d) \\
&= -\mathrm{Trace}\left[\begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}\right] \\
&= -4\mathbf{E}\cdot\mathbf{B} \quad .
\end{aligned}$$

The contraction of the dual field tensor, which is also antisymmetric, with itself yields no new invariant, because

$$\begin{aligned}
DF^{\alpha\beta}DF_{\alpha\beta} &= -DF^{\alpha\beta}DF_{\beta\alpha} = -\mathrm{Trace}(\mathcal{DF}u \circ \mathcal{DF}d) \\
&= -\mathrm{Trace}\left[\begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}\right] \\
&= 2(E^2 - B^2) = -F^{\alpha\beta}F_{\alpha\beta} \quad .
\end{aligned}$$

Also, $F^{\alpha\beta}DF_{\alpha\beta} = -4\mathbf{E}\cdot\mathbf{B} = DF^{\alpha\beta}F_{\alpha\beta}$. Horizontal permutations of indices don't yield new invariants, because it is $F^{\alpha\beta}F_{\beta\alpha} = -F^{\alpha\beta}F_{\alpha\beta}$, etc. Pairwise, matched vertical flips also don't yield new invariants, because it is, for instance

$$F^{\alpha}_{\ \beta}F_{\alpha}^{\ \beta} = g_{\beta\gamma}g^{\beta\delta}F^{\alpha\gamma}F_{\alpha\delta} = \delta_{\gamma}^{\ \delta}F^{\alpha\gamma}F_{\alpha\delta} = F^{\alpha\gamma}F_{\alpha\gamma} = F^{\alpha\beta}F_{\alpha\beta} \quad ,$$

Expressions such as $F^{\alpha\beta}F_{\alpha}^{\ \beta}$ are garbage. Higher-order products, such as $F^{\alpha\beta}F_{\alpha}^{\ \delta}F_{\beta\delta}$ will give results that are at least cubic in the fields. Thus, the only independent Lorentz scalars quadratic in the fields are $E^2 - B^2$ and $\mathbf{E}\cdot\mathbf{B}$.

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b): Since $E^2 - B^2$ is invariant, there exist no fields that are purely electric in one frame and purely magnetic in another (with the trivial exception $E = B = 0$).

Assume fields $\mathbf{E}$ and $\mathbf{B}$ in some frame. Due to the invariants found in a), the conditions that the electric field can be eliminated by a Lorentz transformation into another frame are

$$E < B \quad \text{and} \quad \mathbf{E} \cdot \mathbf{B} = 0$$

The fields also need to be homogeneous. Explicit transformation equations are given by Eqs. 12.43f in Jackson.

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c): We form new scalar combinations with the field tensors of the auxiliary fields (see page 557 of Jackson). Two independent combinations between auxiliary-field tensors are

$$\begin{aligned}
G^{\alpha\beta}G_{\alpha\beta} &= -G^{\alpha\beta}G_{\beta\alpha} = -\mathrm{Trace}(\mathcal{G}u \circ \mathcal{G}d) \\
&= -\mathrm{Trace}\left[\begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & -H_z & H_y \\ -D_y & H_z & 0 & -H_x \\ -D_z & -H_y & H_x & 0 \end{pmatrix}\right] \\
&= 2(H^2 - D^2) \quad .
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}G^{\alpha\beta}G_{\alpha\beta} &= -\mathcal{D}G^{\alpha\beta}G_{\beta\alpha} = -\mathrm{Trace}(\mathcal{D}\mathcal{G}u \circ \mathcal{G}d) \\
&= -\mathrm{Trace}\left[\begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & D_z & -D_y \\ H_y & -D_z & 0 & D_x \\ H_z & D_y & -D_x & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & -H_z & H_y \\ -D_y & H_z & 0 & -H_x \\ -D_z & -H_y & H_x & 0 \end{pmatrix}\right] \\
&= -4\mathbf{D} \cdot \mathbf{H} \quad .
\end{aligned}$$

All other scalar, quadratic combinations between auxiliary-field tensors depend on those.

A complete set of independent invariants involving a fundamental-field and an auxiliary-field tensor are:

$$\begin{aligned}
F^{\alpha\beta}G_{\alpha\beta} &= -F^{\alpha\beta}G_{\beta\alpha} = -\mathrm{Trace}(\mathcal{F}u \circ \mathcal{G}d) \\
&= -\mathrm{Trace}\left[\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & -H_z & H_y \\ -D_y & H_z & 0 & -H_x \\ -D_z & -H_y & H_x & 0 \end{pmatrix}\right] \\
&= 2(\mathbf{B} \cdot \mathbf{H} - \mathbf{E} \cdot \mathbf{D}) \quad .
\end{aligned}$$

$$
\begin{aligned}
DF^{\alpha\beta}G_{\alpha\beta} &= -DF^{\alpha\beta}G_{\beta\alpha} = -\mathrm{Trace}(\mathcal{DF}u \circ \mathcal{G}d) \\
&= -\mathrm{Trace}\left[\begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & -H_z & H_y \\ -D_y & H_z & 0 & -H_x \\ -D_z & -H_y & H_x & 0 \end{pmatrix}\right] \\
&= -2\,(\mathbf{B}\cdot\mathbf{D} + \mathbf{E}\cdot\mathbf{H}) \quad .
\end{aligned}
$$

3. Problem 11.18 **10 Points**

a): In the rest frame of the decaying particle with initial (rest) mass M, the total relativistic momentum $\mathbf{P} = 0$ and the relativistic energy $E = M$ (we set $c = 1$). Both are conserved in the decay process. Thus, after the decay and in the rest frame, the particles have energy-momentum 4-vectors $(E_1, \mathbf{p}_1)$ and $(E_2, \mathbf{p}_2) = (M - E_1, -\mathbf{p}_1)$. Equating the corresponding Lorentz invariants and using $E_i^2 = m_i^2 + p_i^2$, $i = 1, 2$, and using $p_1^2 = p_2^2$, we find

$$\begin{aligned}
E_2^2 - p_2^2 &= (M - E_1)^2 - p_1^2 \\
E_2^2 &= M^2 - 2ME_1 + E_1^2 \\
m_2^2 + p_2^2 &= M^2 - 2ME_1 + m_1^2 + p_1^2 \\
m_2^2 &= M^2 - 2ME_1 + m_1^2 \\
E_1 &= \frac{M^2 + m_1^2 - m_2^2}{2M}
\end{aligned}$$

Also,

$$\begin{aligned}
E_1^2 - p_1^2 &= (M - E_2)^2 - p_2^2 \\
m_1^2 + p_1^2 &= M^2 - 2ME_2 + m_2^2 + p_2^2 \\
E_2 &= \frac{M^2 + m_2^2 - m_1^2}{2M}
\end{aligned}$$

Note that conservation of relativistic energy and relativistic momentum in the decay process is sufficient to obtain this result (i.e. the first lines in the above proofs can be skipped).

b): To prove this, in the following we define j to be the opposite of i ($j = 2$ when $i = 1$, for instance) and use the result of a),

$$\begin{aligned}
(M - m_1 - m_2)\left(1 - \frac{m_i}{M} - \frac{M - m_1 - m_2}{2M}\right) &= (M - m_1 - m_2)\left(\frac{2M - 2m_i - M + m_1 + m_2}{2M}\right) \\
&= \frac{(M - m_1 - m_2)(M - m_i + m_j)}{2M} \\
&= \frac{(M - m_i - m_j)(M - m_i + m_j)}{2M} \\
&= \frac{(M^2 + m_i^2 - m_j^2) - 2Mm_i}{2M} \\
\text{by a)} \quad &= E_i - m_i \\
&= T_i \quad \text{q.e.d.}
\end{aligned}$$

c): Say 1 is the μ-meson and 2 the neutrino. Use a) to find $E_1 = 109.8MeV$. Then,

$$T_1 = E_1 - M_1 = 4.1MeV$$

Then, due to energy conservation

$$T_2 = E - E_1 - M_2 = E - E_1 = 29.8MeV$$

4. Problem 11.19 **10 Points**

a): For a particle moving along the z-axis, equations 11.152 of Jackson are equivalent to

$$\begin{aligned}
\mathbf{E}(ct,x,y,z) &= -\hat{\mathbf{z}}\frac{q\gamma(vt-z)}{\sqrt{r_\perp^2+\gamma^2(vt-z)^2}^3}+\mathbf{r}_\perp\frac{q\gamma}{\sqrt{r_\perp^2+\gamma^2(vt-z)^2}^3}\\
\mathbf{B}(ct,x,y,z) &= \hat{\mathbf{z}}\times\mathbf{r}_\perp\frac{q\gamma}{\sqrt{r_\perp^2+\gamma^2(vt-z)^2}^3}
\end{aligned}$$

where $\mathbf{r}_\perp=(x,y,0)$. To see the equivalence, perform a suitable translation and a rotation about the z-axis to get back to Eqns. 11.152.

To obtain the limit $\gamma\to\infty$, we first consider the electric field. Considering the denominator, wee see that the field generally only is appreciable if $|vt-z|$ is of order $r_\perp/\gamma$ or less. Thus, in the limit $\gamma\to\infty$ non-zero fields only exist if $|vt-z|\ll r_\perp$. Thus, in the limit $\gamma\to\infty$ the z-component of the electric field is negligible. Next, we observe that

$$\frac{\gamma}{\sqrt{r_\perp^2+\gamma^2(vt-z)^2}^3}=\begin{cases}\frac{\gamma}{r_\perp^3}\to\infty & , \quad vt-z=0\\ \frac{1}{\gamma^2(vt-z)^3}\to 0 & , \quad vt-z\neq 0\end{cases}\quad \text{in the limit}\quad \gamma\to\infty\quad .$$

Further, at fixed time the integral over z is

$$\int_{z=-\infty}^{\infty}\frac{\gamma}{\sqrt{r_\perp^2+\gamma^2(vt-z)^2}^3}dz=\int_{-\infty}^{\infty}\frac{\gamma}{\sqrt{r_\perp^2+\gamma^2z^2}^3}dy=\gamma\frac{1}{r_\perp^2}\left[\frac{z}{\sqrt{r_\perp^2+\gamma^2z^2}}\right]_{-\infty}^{\infty}=\frac{2}{r_\perp^2}$$

This result can, of course, also obtained by considering a fixed position and integrating over ct. Thus, in the limit $\gamma\to\infty$ it is

$$\frac{\gamma}{\sqrt{r_\perp^2+\gamma^2(vt-z)^2}^3}=\frac{2}{r_\perp^2}\delta(ct-z)$$

and therefore

$$\begin{aligned}
\mathbf{E}(ct,x,y,z) &= \mathbf{r}_\perp\frac{2q}{r_\perp^2}\delta(ct-z)\\
\mathbf{B}(ct,x,y,z) &= \hat{\mathbf{z}}\times\mathbf{r}_\perp\frac{2q}{r_\perp^2}\delta(ct-z)\quad \text{q.e.d.}
\end{aligned}$$

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b): $\nabla\cdot\mathbf{E}=4\pi\rho$: For the above $\mathbf{E}$, it is

$$\nabla\cdot\mathbf{E}=\nabla\cdot\left[\mathbf{r}_\perp\frac{2q}{r_\perp^2}\delta(ct-z)\right]=2q\delta(ct-z)\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]=0$$

unless $\mathbf{r}_\perp = 0$ and $z = ct$. Thus, $\nabla \cdot \mathbf{E}$ is of the form

$$\nabla \cdot \mathbf{E} = 4\pi f \delta^2(\mathbf{r}_\perp)\delta(ct - z)$$

with a constant f that we can determine by integrating this equation over an infinitesimal spherical volume centered around the particle location $(0, 0, ct)$:

$$\begin{aligned} \int \nabla \cdot \mathbf{E} \, dx \, dy \, d(ct - z) &= \int 4\pi f \delta^2(\mathbf{r}_\perp)\delta(ct - z) \, dx \, dy \, d(ct - z) \\ \oint \mathbf{E} \cdot d\mathbf{a} &= 4\pi f \end{aligned}$$

Since the field is localized to the plane $ct = z$, the area integral only yields contributions from a thin azimuthal band in the $ct = z$ plane. We can therefore write the area integral in the form

$$\begin{aligned} \int_{ct-z=-\epsilon}^{\epsilon} \int_{\phi=0}^{2\pi} \mathbf{r}_\perp \frac{2q}{r_\perp^2} \delta(ct - z) \cdot \mathbf{r}_\perp \, d(ct - z) \, d\phi &= 4\pi f \\ q &= f \end{aligned}$$

There, ϵ is an infinitesimal length. Thus, from the given field alone we have derived that

$$\nabla \cdot \mathbf{E} = 4\pi q \delta^2(\mathbf{r}_\perp)\delta(ct - z)$$

By Gauss's law, it must also be $\nabla \cdot \mathbf{E} = 4\pi\rho$. Thus, the charge density for the given field is $\rho(\mathbf{x}) = q\delta^2(\mathbf{r}_\perp)\delta(ct - z)$. The zero-th component of the four-current producing the field given in part a) must therefore be

$$J^0 = cq\delta^2(\mathbf{r}_\perp)\delta(ct - z) \quad .$$

This is in agreement with the 0-component of the current specified in the problem.

$\nabla \cdot \mathbf{B} = 0$: The validity can be verified explicitly for locations $\mathbf{x} \neq (0, 0, ct)$. It is then concluded that $\nabla \cdot \mathbf{B} = 4\pi g \delta^2(\mathbf{r}_\perp)\delta(ct - z)$. The constant g is determined via a small volume integral,

$$\begin{aligned} \int \nabla \cdot \mathbf{B} \, dx \, dy \, d(ct - z) &= \int 4\pi g \delta^2(\mathbf{r}_\perp)\delta(ct - z) \, dx \, dy \, d(ct - z) \\ \oint \mathbf{B} \cdot d\mathbf{a} &= 4\pi g \end{aligned}$$

Since the $\mathbf{B}$-field is also localized to the plane $ct = z$, the area integral is, with an infinitesimal ϵ,

$$\oint \mathbf{B} \cdot d\mathbf{a} = \int_{ct-z=-\epsilon}^{\epsilon} \int_{\phi=0}^{2\pi} (r_\perp \hat{\phi}) \frac{2q}{r_\perp^2} \delta(ct - z) \cdot \mathbf{r}_\perp \, d(ct - z) \, d\phi = 0$$

Thus, it is $g = 0$, and it is, as required, $\nabla \cdot \mathbf{B} = 0$ everywhere. We conclude that the B-field given in a) is consistent with Gauss's law for B.

$\underline{\nabla \times \mathbf{B} - \frac{\partial}{\partial ct}\mathbf{E} = \frac{4\pi}{c}\mathbf{J}}$: By direct calculation using the given fields, it is found that

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}}\frac{x}{r_\perp^2}\delta'(ct - z) + \hat{\mathbf{y}}\frac{y}{r_\perp^2}\delta'(ct - z) + \hat{\mathbf{z}}\delta(ct - z) \cdot 0$$

where $\delta'(ct - z) = \frac{d}{dx}\delta(x)|_{x=ct-z}$.

Also, it is found that

$$\frac{\partial}{\partial ct}\mathbf{E} = \hat{\mathbf{x}}\frac{x}{r_\perp^2}\delta'(ct - z) + \hat{\mathbf{y}}\frac{y}{r_\perp^2}\delta'(ct - z)$$

so that $\nabla \times \mathbf{B} - \frac{\partial}{\partial ct}\mathbf{E} = 0$, unless $\mathbf{r}_\perp = 0$ and $z = ct$. Thus, $\nabla \times \mathbf{B} - \frac{\partial}{\partial ct}\mathbf{E}$ must be of the form

$$\nabla \times \mathbf{B} - \frac{\partial}{\partial ct}\mathbf{E} = \mathbf{h}\delta^2(\mathbf{r}_\perp)\delta(ct - z)$$

with a vector constant $\mathbf{h}$ to be determined. We note that due to the cylindrical symmetry of the fields on the left side of the equation, the right side must have cylindrical symmetry as well. We conclude that $\mathbf{h}$ can only point in the z-direction, and thus

$$\nabla \times \mathbf{B} - \frac{\partial}{\partial ct}\mathbf{E} = \hat{\mathbf{z}}h\delta^2(\mathbf{r}_\perp)\delta(ct - z) \tag{1}$$

with a scalar constant h to be determined. To find h, we consider the area integral of Eq. 1 over a small disk centered around the location $(0, 0, ct)$ with area vector in the $+\hat{\mathbf{z}}$-direction. Using Stokes's theorem, the left side yields, with the given electric and magnetic fields,

$$\int (\nabla \times \mathbf{B} - \frac{\partial}{\partial ct}\mathbf{E}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} - \int \frac{\partial}{\partial ct}\mathbf{E} \cdot (\hat{\mathbf{z}}da) = 4\pi q\delta(ct - z)$$

(The Stokes loop is in the $+\hat{\phi}$-direction). The area integral of the right side of Eq. 1 yields,

$$\int \hat{\mathbf{z}}h\delta^2(\mathbf{r}_\perp)\delta(ct - z) \cdot \hat{\mathbf{z}}da = h\delta(ct - z)$$

Comparing the last two equations, we see $h = 4\pi q$, and therefore

$$\nabla \times \mathbf{B} - \frac{\partial}{\partial ct}\mathbf{E} = \hat{\mathbf{z}}4\pi q\delta^2(\mathbf{r}_\perp)\delta(ct - z)$$

Note that this result is obtained solely from the given fields. By Maxwell-Ampere's law, it must in addition be $\nabla \times \mathbf{B} - \frac{\partial}{\partial ct}\mathbf{E} = \frac{4\pi}{c}\mathbf{J}$. By comparison we see that the current density must be

$$\mathbf{J} = \hat{\mathbf{z}} qc\delta^2(\mathbf{r}_\perp)\delta(ct - z) \quad .$$

This is in agreement with the spatial components of the current specified in the problem.

$\underline{\nabla \times \mathbf{E} + \frac{\partial}{\partial ct}\mathbf{B} = 0}$: For locations $\mathbf{x} \neq (0, 0, ct)$, validity of Faraday's law can be shown by direct calculation. To verify consistency at the particle location, consider the area integral of the field-side of Faraday's law over a small disk centered around the location $(0, 0, ct)$ with area vector in the $+\hat{\mathbf{z}}$-direction. Using Stokes's theorem, from the given electric and magnetic fields it is, finally and thankfully, found that

$$\oint \mathbf{E} \cdot d\mathbf{l} + \int \frac{\partial}{\partial ct}\mathbf{B} \cdot (\hat{\mathbf{z}} da) = 0$$

Combining the above results, the four-current J^α that is consistent with the given fields and with Maxwell's equations is

$$J^\alpha(\rho c, \mathbf{J}) = (J^0, \mathbf{J}) = qc\delta^2(\mathbf{r}_\perp)\delta(ct - z)(1, \hat{\mathbf{v}}) \quad \text{, q.e.d.}$$

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c): To derive the fields from the potentials, use

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -\frac{\partial}{\partial ct}\mathbf{A} - \nabla \cdot A^0$$

or, equivalently,

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = (\frac{\partial}{\partial ct}, -\nabla) \cdot (A^0, \mathbf{A}) \qquad \text{and} \qquad F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

For $A^\alpha = -2q\delta(ct - z)\ln(\lambda r_\perp)(1, 0, 0, 1) = -2q\delta(ct - z)\ln(\lambda\sqrt{x^2 + y^2})(1, 0, 0, 1)$ we find

$$\begin{aligned}
E_x &= \partial^1 A^0 - \partial^0 A^1 = 2q\delta(ct - z)\frac{\partial}{\partial x}\ln(\lambda\sqrt{x^2 + y^2}) = 2q\delta(ct - z)\frac{x}{r_\perp^2} \\
E_y &= \partial^2 A^0 - \partial^0 A^2 = 2q\delta(ct - z)\frac{\partial}{\partial y}\ln(\lambda\sqrt{x^2 + y^2}) = 2q\delta(ct - z)\frac{y}{r_\perp^2} \\
E_z &= \partial^3 A^0 - \partial^0 A^3 = 2q\ln(\lambda\sqrt{x^2 + y^2})\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial ct}\right]\delta(ct - z) = 0 \\
B_x &= \partial^3 A^2 - \partial^2 A^3 = -2q\delta(ct - z)\frac{\partial}{\partial y}\ln(\lambda\sqrt{x^2 + y^2}) = -2q\delta(ct - z)\frac{y}{r_\perp^2} \\
B_y &= \partial^1 A^3 - \partial^3 A^1 = 2q\delta(ct - z)\frac{\partial}{\partial x}\ln(\lambda\sqrt{x^2 + y^2}) = 2q\delta(ct - z)\frac{x}{r_\perp^2} \\
B_z &= \partial^2 A^1 - \partial^1 A^2 = 0
\end{aligned}$$

which agrees with the fields specified in part a).

For $A^\alpha = -2q\Theta(ct-z)(0, \nabla_\perp \ln(\lambda r_\perp)) = -2q\Theta(ct-z)(0, \frac{x}{r_\perp^2}, \frac{y}{r_\perp^2}, 0) = -2q\Theta(ct-z)(0, \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0)$ we find

$$
\begin{aligned}
E_x &= \partial^1 A^0 - \partial^0 A^1 = 2q\frac{x}{r_\perp^2}\frac{\partial}{\partial c t}\Theta(ct-z) = 2q\delta(ct-z)\frac{x}{r_\perp^2} \\
E_y &= \partial^2 A^0 - \partial^0 A^2 = 2q\frac{y}{r_\perp^2}\frac{\partial}{\partial c t}\Theta(ct-z) = 2q\delta(ct-z)\frac{y}{r_\perp^2} \\
E_z &= \partial^3 A^0 - \partial^0 A^3 = 0 \\
B_x &= \partial^3 A^2 - \partial^2 A^3 = 2q\frac{y}{r_\perp^2}\frac{\partial}{\partial z}\Theta(ct-z) = -2q\delta(ct-z)\frac{y}{r_\perp^2} \\
B_y &= \partial^1 A^3 - \partial^3 A^1 = -2q\frac{x}{r_\perp^2}\frac{\partial}{\partial z}\Theta(ct-z) = 2q\delta(ct-z)\frac{x}{r_\perp^2} \\
B_z &= \partial^2 A^1 - \partial^1 A^2 = 2q\Theta(ct-z)\left[\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)\right] = 0
\end{aligned}
$$

which also agrees with the fields specified in part a).

The two gauges must necessarily be related via a gauge transformation. To show this, we find the scalar function Λ that effects the gauge transformation

$$
A'^\alpha(x) = A^\alpha(x) - \partial^\alpha \Lambda(x) \quad \Leftrightarrow \quad \begin{pmatrix} A'^0(ct,\mathbf{x}) \\ \mathbf{A}'(ct,\mathbf{x}) \end{pmatrix} = \begin{pmatrix} A'^0(ct,\mathbf{x}) - \frac{\partial}{\partial c t}\Lambda(ct,\mathbf{x}) \\ \mathbf{A}(ct,\mathbf{x}) + \nabla\Lambda(ct,\mathbf{x}) \end{pmatrix}
$$

In the given case, it must thus be

$$
\begin{pmatrix} \frac{\partial}{\partial c t}\Lambda(ct,\mathbf{x}) \\ \nabla\Lambda(ct,\mathbf{x}) \end{pmatrix} = \begin{pmatrix} A^0 - A'^0 \\ \mathbf{A}' - \mathbf{A} \end{pmatrix} = -2q \begin{pmatrix} \delta(ct-z)\ln(\lambda r_\perp) \\ \Theta(ct-z)\frac{x}{r_\perp^2} \\ \Theta(ct-z)\frac{y}{r_\perp^2} \\ -\delta(ct-z)\ln(\lambda r_\perp) \end{pmatrix}
$$

From the first (time) line we may guess that $\Lambda = -2q\Theta(ct-z)\ln(\lambda r_\perp)$, and then verify that this also satisfies the three spatial equations. Thus, the gauge transformation is effected by the function

$$
\Lambda = -2q\Theta(ct-z)\ln(\lambda r_\perp) \quad .
$$