Phys. 506 Electricity and Magnetism Winter 2004

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Problem Set 7

Total 40 Points

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1. Problem 11.3 **10 Points**

Since the boosts are in parallel directions, we may consider - without loss of generality - two Lorentz transformations in the x-direction with velocities v_1 and v_2, described by matrices

$$A_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 & 0 & 0 \\ -\beta_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\beta_i = \frac{v_i}{c}$ and $\gamma_i = \frac{1}{\sqrt{1-\beta_i^2}}$ with i=1,2. The net transformation then is

$$A_2 \circ A_1 = \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & -\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

The $\neq 1$ diagonal terms can be written as

$$\begin{aligned} \gamma_1\gamma_2(1+\beta_1\beta_2) &= \frac{1+\beta_1\beta_2}{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}} = \frac{1}{\sqrt{\frac{(1-\beta_1^2)(1-\beta_2^2)}{(1+\beta_1\beta_2)^2}}} \\ &= \frac{1}{\sqrt{1-\frac{(1+\beta_1\beta_2)^2-(1-\beta_1^2)(1-\beta_2^2)}{(1+\beta_1\beta_2)^2}}} = \frac{1}{\sqrt{1-\left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)^2}} \\ &= \frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}\right)^2}} =: \frac{1}{\sqrt{1-\frac{v_3^2}{c^2}}} =: \frac{1}{\sqrt{1-\beta_3^2}} =: \gamma_3 \end{aligned}$$

with $v_3 = \frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}$. Then, the non-zero off-diagonal terms can be written in the form

$$-\gamma_1\gamma_2(\beta_1+\beta_2) = -\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\left[\gamma_1\gamma_2(1+\beta_1\beta_2)\right] = -\frac{v_3}{c}\gamma_3 = -\beta_3\gamma_3$$

Thus,

$$A_2 \circ A_1 = \begin{pmatrix} \gamma_3 & -\beta_3\gamma_3 & 0 & 0 \\ -\beta_3\gamma_3 & \gamma_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is a net Lorentz transformation with boost velocity $v_3 = \frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}$, q.e.d.

2. Problem 11.5 **10 Points**

Inertial frame K' moves wrt. frame K with a constant velocity $\mathbf{v}$. Observed in K', a particle has a trajectory $(ct', \mathbf{x}'(t'))$. The trajectory observed in K, $(ct, \mathbf{x}(t))$, is the inverse Lorentz transform of $(ct', \mathbf{x}'(t'))$. The components of the particle velocities parallel to $\mathbf{v}$ in K' and K, $u'_{||} = \frac{dx'(t')}{dt'}$ and $u_{||} = \frac{dx(t)}{dt}$, are related as shown in Eq. 11.31 of Jackson. The longitudinal acceleration components observed in K' and K, denoted by $a'_{||}$ and $a_{||}$, are then related via

$$\begin{aligned}
a_{||}(t) &= \frac{du_{||}}{dt} = \frac{d}{dt}\left(\frac{u'_{||}+v}{1+\frac{u'_{||}v}{c^2}}\right) \\
&= \frac{1}{(1+\frac{u'_{||}v}{c^2})^2}\left[(1+\frac{u'_{||}v}{c^2})\frac{du'_{||}}{dt} - (u'_{||}+v)\frac{v}{c^2}\frac{du'_{||}}{dt}\right] \\
&= \frac{1}{(1+\frac{u'_{||}v}{c^2})^2}\left[1+\frac{u'_{||}v}{c^2} - \frac{u'_{||}v}{c^2} - \frac{v^2}{c^2}\right]\left[\frac{dt'}{dt}\right]\frac{du'_{||}}{dt'} \\
&= \frac{1}{(1+\frac{u'_{||}v}{c^2})^2}\frac{1}{\gamma^2}\left[\frac{dt'}{dt}\right]a'_{||} \\
&= \frac{a'_{||}}{\gamma^2(1+\frac{u'_{||}v}{c^2})^2}\left[\frac{d}{dt}\left(\gamma t - \frac{\beta\gamma x_{||}(t)}{c}\right)\right] \\
&= \frac{a'_{||}}{\gamma(1+\frac{u'_{||}v}{c^2})^2}\left[1 - \frac{v}{c^2}\frac{dx_{||}}{dt}\right] \\
&= \frac{a'_{||}}{\gamma(1+\frac{u'_{||}v}{c^2})^2}\left[1 - \frac{v}{c^2}u_{||}\right] \\
&= \frac{a'_{||}}{\gamma(1+\frac{u'_{||}v}{c^2})^2}\left[1 - \frac{v}{c^2}\left(\frac{u'_{||}+v}{1+\frac{u'_{||}v}{c^2}}\right)\right] \\
&= \frac{a'_{||}}{\gamma(1+\frac{u'_{||}v}{c^2})^3}\left[1+\frac{u'_{||}v}{c^2} - \frac{v(u'_{||}+v)}{c^2}\right] \\
&= \frac{a'_{||}}{\gamma^3(1+\frac{u'_{||}v}{c^2})^3}
\end{aligned}$$

and thus, written as a vector, $$\mathbf{a}_{||}(t) = \frac{\mathbf{a}'_{||}}{\gamma^3(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2})^3} \quad \text{q.e.d.}$$

(Note that we use $\gamma = \gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.) Also, beginning with Eq. 11.31 we find for the transverse accelerations in K' and K, denoted $\mathbf{a}'_\perp$ and $\mathbf{a}_\perp$

$$\begin{aligned}
\mathbf{a}_\perp(t) &= \frac{1}{\gamma}\frac{d}{dt}\left(\frac{\mathbf{u}'_\perp}{1+\frac{u'_{||}v}{c^2}}\right) \\
&= \frac{1}{\gamma(1+\frac{u'_{||}v}{c^2})^2}\left[(1+\frac{u'_{||}v}{c^2})\frac{d}{dt}\mathbf{u}'_\perp - \mathbf{u}'_\perp\frac{d}{dt}(1+\frac{u'_{||}v}{c^2})\right] \\
&= \frac{1}{\gamma(1+\frac{u'_{||}v}{c^2})^2}\left[\frac{dt'}{dt}\right]\left[(1+\frac{u'_{||}v}{c^2})\frac{d}{dt'}\mathbf{u}'_\perp - \mathbf{u}'_\perp\frac{v}{c^2}\frac{d}{dt'}u'_{||}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\gamma^2(1+\frac{u'_{||}v}{c^2})^3}\left[(1+\frac{u'_{||}v}{c^2})\frac{d}{dt'}\mathbf{u}'_\perp - \mathbf{u}'_\perp\frac{v}{c^2}\frac{d}{dt'}u'_{||}\right] \\
&= \frac{1}{\gamma^2(1+\frac{u'_{||}v}{c^2})^3}\left[(1+\frac{u'_{||}v}{c^2})\mathbf{a}'_\perp - \mathbf{u}'_\perp\frac{v}{c^2}a'_{||}\right] \\
&= \frac{1}{\gamma^2(1+\frac{u'_{||}v}{c^2})^3}\left[\mathbf{a}'_\perp + \mathbf{a}'_\perp\frac{u'_{||}v}{c^2} - \mathbf{u}'_\perp\frac{v}{c^2}a'_{||}\right] \\
&= \frac{1}{\gamma^2(1+\frac{u'_{||}v}{c^2})^3}\left[\mathbf{a}'_\perp + \left(\mathbf{a}' - a'_{||}\hat{\mathbf{v}}\right)\frac{u'_{||}v}{c^2} - \mathbf{u}'_\perp\frac{v}{c^2}a'_{||}\right] \\
&= \frac{1}{\gamma^2(1+\frac{u'_{||}v}{c^2})^3}\left[\mathbf{a}'_\perp + \mathbf{a}'\frac{u'_{||}v}{c^2} + \frac{va'_{||}}{c^2}\left(-\mathbf{u}'_\perp - \hat{\mathbf{v}}u'_{||}\right)\right] \\
&= \frac{1}{\gamma^2(1+\frac{u'_{||}v}{c^2})^3}\left[\mathbf{a}'_\perp + \frac{1}{c^2}\left(\mathbf{a}'u'_{||}v - va'_{||}\mathbf{u}'\right)\right] \\
&= \frac{1}{\gamma^2(1+\frac{u'_{||}v}{c^2})^3}\left[\mathbf{a}'_\perp + \frac{1}{c^2}\left(\mathbf{a}'(\mathbf{u}'\cdot\mathbf{v}) - (\mathbf{a}'\cdot\mathbf{v})\mathbf{u}'\right)\right] \\
&= \frac{1}{\gamma^2(1+\frac{\mathbf{u}'\cdot\mathbf{v}}{c^2})^3}\left[\mathbf{a}'_\perp + \frac{1}{c^2}\mathbf{v}\times(\mathbf{a}'\times\mathbf{u}')\right] \quad \text{q.e.d.}
\end{aligned}
$$

There, we have used the unit vector $\hat{\mathbf{v}} = \frac{\mathbf{v}}{v}$.

3. Problem 11.6 **10 Points**

a): To calculate the time interval in the earth frame K along the first acceleration leg,

$$T_1 = \int_0^{5a} \gamma(\tau) d\tau = \int_0^{5a} \frac{1}{\sqrt{1 - \frac{u^2(\tau)}{c^2}}} d\tau \quad , \tag{1}$$

we require $u(\tau)$. To find $u(\tau)$, we use the parallel-component result of Problem 11.5 for the case that K' is the instantaneously co-moving frame of the rocket and K is the earth frame. Then, $dt' = d\tau$ and $u'_{||} = 0$, and

$$\frac{du}{dt} = \frac{1}{\gamma_u^3} \frac{du'}{dt} = \frac{1}{\gamma_u^3} \frac{du'}{d\tau} = \frac{g}{\gamma_u^3} \tag{2}$$

Also, due to time delation between the earth and the instantaneously co-moving rocket frame it is $dt = \gamma_u d\tau$, and therefore

$$\begin{aligned}
\frac{du}{dt} &= \frac{du}{\gamma_u d\tau} = \frac{g}{\gamma_u^3} \\
\gamma_u^2 du &= g d\tau \\
\int_{u=0}^{u(\tau)} \frac{1}{1 - u^2/c^2} du &= \int_{\tau=0}^{\tau} g d\tau = g\tau \\
\frac{u(\tau)}{c} &= \tanh \frac{g\tau}{c}
\end{aligned}$$

Insertion into Eq. 1 allows us to calculate the travel time of the first leg observed in K,

$$T_1 = \int_{\tau=0}^{5a} \frac{1}{\sqrt{1 - \tanh^2\left(\frac{g\tau}{c}\right)}} d\tau = \int_{\tau=0}^{5a} \cosh\left(\frac{g\tau}{c}\right) d\tau = \frac{c}{g} \sinh\left(\frac{g \times 5a}{c}\right) = 84a$$

By symmetry, all other legs give the same result, yielding a travel time observed in K of $T = 4\,T_1 = 336a$. Thus, the year of return to earth is 2436.

Note that in the analysis the instantaneously co-moving frame (ICMF) of the rocket K' is an inertial frame; for that reason the presented analysis is valid. As the rocket moves along, it marks the origins of an infinite sequence of different ICMFs. The rocket itself is not an inertial frame, of course, but the rocket frame never enters in our analysis.

b): The travel distance in K along the first leg

$$L_1 = \int_{t=0}^{T_1} u(t) dt = \int_{t=0}^{84a} u(t) dt$$

requires knowledge of the rocket velocity $u(t)$ observed in K. From Eq. 2 it follows

$$\begin{aligned}
\gamma^3(u)du &= gdt \\
\int_{u=0}^{u(t)} \frac{1}{\sqrt{1-\frac{u^2}{c^2}}^3} du &= \int_0^t gdt \\
u(t)\frac{1}{\sqrt{1-\frac{u(t)^2}{c^2}}} &= gt \\
u(t) &= \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}}
\end{aligned}$$

Insertion into the previous equation yields

$$L_1 = \frac{c^2}{g}\int_{t=0}^{T_1\times g/c} \frac{gt/c}{\sqrt{1+\frac{g^2t^2}{c^2}}} d(gt/c) = \frac{c^2}{g}\left[\sqrt{1+\frac{g^2T_1^2}{c^2}}-1\right]$$

Since $\frac{gT_1}{c} \gg 1$, the square-root can be developed, yielding with $T_1 = 84a$

$$L_1 \approx cT_1 + \frac{c^2}{g}\left(\frac{c}{2gT_1}-1\right) \approx cT_1 - \frac{c^2}{g} = (84-0.969)\, lightyears$$

Since all legs have, for symmetry, the same length, we find a total travel distance of $L = 2\,L_1 = 166\, lightyears$, which is almost as far as a beam of light would travel (which would be $168\, lightyears$).

4. Problem 11.9 **10 Points**

The objective of the problem is to find the generators of the Lorentz transformation in a simplified way (compared with Chapter 11.7 of Jackson). The simplification is achieved by considering finite Lorentz transformations as a sequence of infinitesimal ones. The generators of the latter are obtained in the following.

a): We consider the effect of the consecutive application of the transformation given in the problem and its inverse,

$$\begin{aligned}
x''^{\alpha} &= (g^{\alpha\beta} + \epsilon'^{\alpha\beta})\, x'_{\beta} \\
&= (g^{\alpha\beta} + \epsilon'^{\alpha\beta})\, g_{\beta\gamma}\, x'^{\gamma} \\
&= (g^{\alpha\beta} + \epsilon'^{\alpha\beta})\, g_{\beta\gamma}\, (g^{\gamma\delta} + \epsilon^{\gamma\delta})\, x_{\delta} \\
&= (g^{\alpha\beta} + \epsilon'^{\alpha\beta})\, g_{\beta\gamma}\, (g^{\gamma\delta} + \epsilon^{\gamma\delta})\, g_{\delta\eta}\, x^{\eta}
\end{aligned}$$

Since the two consecutive transformations are the inverse of each other, it also is $x''^{\alpha} = \delta^{\alpha}_{\ \eta} x^{\eta}$ for all x. By comparison with the previous equation, we can write

$$\begin{aligned}
\delta^{\alpha}_{\ \eta} &= (g^{\alpha\beta} + \epsilon'^{\alpha\beta})\, g_{\beta\gamma}\, (g^{\gamma\delta} + \epsilon^{\gamma\delta})\, g_{\delta\eta} \\
&= \left[\delta^{\alpha}_{\ \gamma}\, (g^{\gamma\delta} + \epsilon^{\gamma\delta}) + \epsilon'^{\alpha\beta}\, g_{\beta\gamma}\, (g^{\gamma\delta} + \epsilon^{\gamma\delta})\right] g_{\delta\eta} \\
&= \left[g^{\alpha\delta} + \epsilon^{\alpha\delta} + \epsilon'^{\alpha\beta}\, (\delta_{\beta}^{\ \delta} + g_{\beta\gamma}\, \epsilon^{\gamma\delta})\right] g_{\delta\eta} \\
&= \left[g^{\alpha\delta} + \epsilon^{\alpha\delta} + \epsilon'^{\alpha\delta} + \epsilon'^{\alpha\beta}\, g_{\beta\gamma}\, \epsilon^{\gamma\delta}\right] g_{\delta\eta} \\
&= \left[g^{\alpha\delta} + \epsilon^{\alpha\delta} + \epsilon'^{\alpha\delta}\right] g_{\delta\eta} \\
&= \delta^{\alpha}_{\ \eta} + g_{\delta\eta}(\epsilon^{\alpha\delta} + \epsilon'^{\alpha\delta}) = \delta^{\alpha}_{\ \eta} + (\epsilon^{\alpha}_{\ \eta} + \epsilon'^{\alpha}_{\ \eta})
\end{aligned}$$

Note that due to the infinitesimal character of the elements of the ϵ-tensors, we were allowed to drop terms quadratic in them. From the last line it follows that $\epsilon^{\alpha\delta} = -\epsilon'^{\alpha\delta}$, q.e.d.

b): Beginning with norm conservation, we find by application of the infinitesimal transformation law specified in the problem,

$$\begin{aligned}
x^{\alpha}\, x_{\alpha} &= x'^{\alpha}\, x'_{\alpha} \\
&= (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, x_{\beta}\, x'_{\alpha} = (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, x_{\beta}\, g_{\alpha\gamma} x'^{\gamma} \\
&= (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, x_{\beta}\, g_{\alpha\gamma}(g^{\gamma\delta} + \epsilon^{\gamma\delta})\, x_{\delta} \\
&= (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, \left[g_{\alpha\gamma}(g^{\gamma\delta} + \epsilon^{\gamma\delta})\right]\, g_{\delta\eta}\, x_{\beta}\, x^{\eta} \\
&= (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, (\delta_{\alpha}^{\ \delta} + g_{\alpha\gamma}\, \epsilon^{\gamma\delta})\, g_{\delta\eta}\, x_{\beta}\, x^{\eta} \\
&= (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, \left[(\delta_{\alpha}^{\ \delta} + g_{\alpha\gamma}\, \epsilon^{\gamma\delta})\, g_{\delta\eta}\right]\, x_{\beta}\, x^{\eta} \\
&= (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, (g_{\alpha\eta} + g_{\alpha\gamma}\, \epsilon^{\gamma\delta}\, g_{\delta\eta})\, x_{\beta}\, x^{\eta} \\
&= (\delta^{\beta}_{\ \eta} + \delta^{\beta}_{\ \gamma}\, \epsilon^{\gamma\delta}\, g_{\delta\eta} + \epsilon^{\alpha\beta}\, g_{\alpha\eta})\, x_{\beta}\, x^{\eta}
\end{aligned}$$

$$
\begin{aligned}
&= x_\beta\, x^\beta + (\epsilon^{\beta\delta}\, g_{\delta\eta} + \epsilon^{\alpha\beta}\, g_{\alpha\eta})\; x_\beta\, x^\eta \\
&= x_\alpha\, x^\alpha + (\epsilon^{\beta\alpha}\, g_{\alpha\eta} + \epsilon^{\alpha\beta}\, g_{\alpha\eta})\; x_\beta\, x^\eta \\
&= x_\alpha\, x^\alpha + (\epsilon^{\beta\alpha} + \epsilon^{\alpha\beta})\, g_{\alpha\eta}\, x_\beta\, x^\eta \\
&= x_\alpha\, x^\alpha + (\epsilon^{\beta\alpha} + \epsilon^{\alpha\beta})\, g_{\alpha\eta}\, x_\beta\, x^\eta \\
&= x^\alpha\, x_\alpha + (\epsilon^{\beta\alpha} + \epsilon^{\alpha\beta})\, x_\alpha\, x_\beta
\end{aligned}
$$

Again, due to the infinitesimal character of the elements of the ϵ-tensors we were allowed to drop terms quadratic in them. Since the result must be valid for all x, it follows that $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$, q.e.d.

c): It is

$$
x'^\alpha = (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, x_\beta = (g^{\alpha\beta} + \epsilon^{\alpha\beta})\, g_{\beta\gamma}\, x^\gamma = (\delta^\alpha_{\ \gamma} + \epsilon^{\alpha\beta}\, g_{\beta\gamma})\, x^\gamma
$$

Also, an infinitesimal Lorentz transformation matrix with generator $\mathcal{L}$ is of the form $\mathcal{A} = \exp(\mathcal{L}) = 1 + \mathcal{L}$. In index notation, the effect of such a transformation is

$$
x'^\alpha = A^\alpha_{\ \gamma} x^\gamma = (\delta^\alpha_{\ \gamma} + L^\alpha_{\ \gamma}) x^\gamma
$$

Comparison of the last two equations shows

$$
L^\alpha_{\ \gamma} = \epsilon^{\alpha\beta}\, g_{\beta\gamma}
$$

which is equivalent to $L^\alpha_{\ \gamma} g^{\gamma\delta} = \epsilon^{\alpha\beta}\, g_{\beta\gamma} g^{\gamma\delta} = \epsilon^{\alpha\beta}\, \delta_\beta^{\ \delta} = \epsilon^{\alpha\delta}$. Written in matrix form, $\mathcal{L} \circ \mathrm{g} = \epsilon$ with an antisymmetric matrix ϵ. This is equivalent to Eq. 11.89 of Jackson, which directly leads to Eqs. 11.90-11.93.