Phys. 506 Electricity and Magnetism Winter 2004

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Problem Set 6

Total 30 Points

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1. Problem 10.10a **10 Points**

We consider the Smythe-Kirchhoff integral,

$$E_{diff} = \frac{1}{2\pi} \nabla \times \int_{hole} \hat{\mathbf{n}} \times \mathbf{E} \frac{\exp(\mathrm{i}kR)}{R} da'$$

where $\hat{\mathbf{n}}$ is the normal of the conducting plane pointing into the volume of interest, and $\mathbf{R} = \mathbf{x} - \mathbf{x}'$. It is $\hat{\mathbf{n}} \times \mathbf{E} = \hat{\mathbf{n}} \times \mathbf{E}_{tan}$, where E_{tan} is the total electric field tangential with the conducting plane. Also, in the radiation zone

$$\frac{\exp(\mathrm{i}kR)}{R} = \frac{\exp(\mathrm{i}kr)}{r} \exp(-\mathrm{i}\mathbf{k}\mathbf{x}')$$

where $\mathbf{k} = k\hat{\mathbf{r}}$ is the k-vector pointing to the observation point.

Thus, in the radiation zone

$$E_{diff} = \frac{1}{2\pi} \nabla \times \frac{\exp(\mathrm{i}kr)}{r} \int_{hole} (\hat{\mathbf{n}} \times \mathbf{E}_{tan}) \exp(-\mathrm{i}\mathbf{k}\mathbf{x}') da'$$

Also, since for expressions of the kind "$E = \nabla \times \frac{\exp(\mathrm{i}kr)}{r} \mathbf{F}(\theta, \phi)$" in the radiation zone the usual replacement "$\nabla \times = \mathrm{i}\mathbf{k}\times$" applies, we obtain Eq. 10.109:

$$E_{diff} = \frac{\mathrm{i}}{2\pi} \mathbf{k} \times \frac{\exp(\mathrm{i}kr)}{r} \int_{hole} (\hat{\mathbf{n}} \times \mathbf{E}_{tan}) \exp(-\mathrm{i}\mathbf{k}\mathbf{x}') da'$$

Since the aperture is small, we can make the small-source approximation for the fields emanating from the hole, $\exp(-\mathrm{i}\mathbf{k}\mathbf{x}') = 1 - \mathrm{i}\mathbf{k} \cdot \mathbf{x}'$, and get:

$$E_{diff} = \frac{\mathrm{i}}{2\pi} \frac{\exp(\mathrm{i}kr)}{r} \mathbf{k} \times \left[\int_{hole} (\hat{\mathbf{n}} \times \mathbf{E}_{tan}) da' - \mathrm{i} \int_{hole} (\hat{\mathbf{n}} \times \mathbf{E}_{tan}) \mathbf{k} \cdot \mathbf{x}' da' \right]$$

As advertised in class, we employ the vector identity Eq. 9.31 with $\hat{\mathbf{n}} \times \mathbf{E}_{tan}$ in place of $\mathbf{J}$,

$$(\hat{\mathbf{n}} \times \mathbf{E}_{tan}) \mathbf{k} \cdot \mathbf{x}' = \frac{1}{2} \left[\mathbf{x}' \times (\hat{\mathbf{n}} \times \mathbf{E}_{tan}) \right] \times \mathbf{k} + \frac{1}{2} \left[(\mathbf{k} \cdot \mathbf{x}')(\hat{\mathbf{n}} \times \mathbf{E}_{tan}) + (\mathbf{k} \cdot (\hat{\mathbf{n}} \times \mathbf{E}_{tan}))\, \mathbf{x}' \right]$$

to get

$$\begin{aligned} E_{diff} &= \frac{\mathrm{i}}{2\pi} \frac{\exp(\mathrm{i}kr)}{r} \mathbf{k} \times \left[\int_{hole} (\hat{\mathbf{n}} \times \mathbf{E}_{tan}) \left\{ 1 - \frac{1}{2} \mathrm{i}\mathbf{k} \cdot \mathbf{x}' \right\} da' - \frac{\mathrm{i}}{2} \int_{hole} \mathbf{x}' \times (\hat{\mathbf{n}} \times \mathbf{E}_{tan}) da' \times \mathbf{k} \right. \\ &- \left. \frac{\mathrm{i}}{2} \int_{hole} \mathbf{k} \cdot (\hat{\mathbf{n}} \times \mathbf{E}_{tan}) \mathbf{x}' da' \right] \end{aligned}$$

Since the hole is small, in the first integral we may set $1-\frac{1}{2}\mathrm{i}\mathbf{k}\cdot\mathbf{x}' = 1$. In the second integral, $\mathbf{x}'\times(\hat{\mathbf{n}}\times\mathbf{E}_{tan}) = (\mathbf{x}'\cdot\mathbf{E}_{tan})\hat{\mathbf{n}} - (\mathbf{x}'\cdot\hat{\mathbf{n}})\mathbf{E}_{tan} = (\mathbf{x}'\cdot\mathbf{E}_{tan})\hat{\mathbf{n}}$, because $\mathbf{x}'\cdot\hat{\mathbf{n}} = 0$. Thus,

$$E_{diff} = \frac{\mathrm{i}}{2\pi}\frac{\exp(\mathrm{i}kr)}{r}\mathbf{k}\times\left[\int_{hole}(\hat{\mathbf{n}}\times\mathbf{E}_{tan})da' - \frac{\mathrm{i}}{2}\left(\hat{\mathbf{n}}\int_{hole}\mathbf{x}'\cdot\mathbf{E}_{tan}da'\right)\times\mathbf{k} - \frac{\mathrm{i}}{2}\int_{hole}\mathbf{k}\cdot(\hat{\mathbf{n}}\times\mathbf{E}_{tan})\mathbf{x}'da'\right]$$

The first term can be re-written as

$$E_{diff,1} = -\frac{Z_0k^2}{4\pi}\frac{\exp(\mathrm{i}kr)}{r}\left(\hat{\mathbf{k}}\times\left[\frac{-2\mathrm{i}}{Z_0k}\int_{hole}(\hat{\mathbf{n}}\times\mathbf{E}_{tan})da'\right]\right)$$

This field can be compared with with Eq. 9.36. Thereby, the term in the square bracket can be identified with an effective magnetic dipole

$$\mathbf{m} = \frac{-2\mathrm{i}}{Z_0k}\int_{hole}(\hat{\mathbf{n}}\times\mathbf{E}_{tan})da' = \frac{2}{\mathrm{i}\omega\mu}\int_{hole}(\hat{\mathbf{n}}\times\mathbf{E}_{tan})da'$$

(note in vacuum $k = \omega/c = \omega\mu/Z_0$). The second term,

$$E_{diff,2} = \frac{Z_0ck^2}{4\pi}\frac{\exp(\mathrm{i}kr)}{r}\hat{\mathbf{k}}\times\left(\left[\frac{1}{Z_0c}\hat{\mathbf{n}}\int_{hole}\mathbf{x}'\cdot\mathbf{E}_{tan}da'\right]\times\mathbf{k}\right)$$

can be, by comparison with Eq. 9.19, identified with the electric field of an electric dipole

$$\mathbf{p} = \epsilon\hat{\mathbf{n}}\int_{hole}\mathbf{x}'\cdot\mathbf{E}_{tan}da'$$

Both the first and the third term have an $\hat{\mathbf{n}}\times\mathbf{E}_{tan}$ under the integral. Further, the third term is of the order of the first term times $kx' << 1$. Thus, the third term can be neglected.

2. Problem 10.12 **10 Points**

We start with the Smythe-Kirchhoff formula in the radiation zone,

$$E_{diff} = \frac{\mathrm{i}}{2\pi}\frac{\exp(\mathrm{i}kr)}{r}\mathbf{k}\times\int_{hole}(\hat{\mathbf{n}}\times\mathbf{E}_{tan})\exp(-\mathrm{i}\mathbf{k}\mathbf{x}')da'$$

The plane normal $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, the incident wavevector $\mathbf{k}_0 = k(\cos\alpha\hat{\mathbf{z}}+\sin\alpha\hat{\mathbf{x}})$, and the wavevector pointing to the observation point, $\mathbf{k} = k(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})$. The incident electric field is linearly polarized transverse to the plane of incidence (the xz-plane), i.e. $\mathbf{E}_0 = E_0\hat{\mathbf{y}}$. The circular hole over which we integrate extends in the $x'y'$-plane. Thus, using 2-dimensional cylindrical coordinates ρ' and β' in the $x'y'$-plane,

$$\begin{aligned}E_{diff} &= \frac{\mathrm{iE}_0}{2\pi}\frac{\exp(\mathrm{i}kr)}{r}\mathbf{k}\times\int_{\rho'=0}^{a}\int_{\beta=0}^{2\pi}(\hat{\mathbf{z}}\times\hat{\mathbf{y}})\exp(\mathrm{i}(\mathbf{k_0}-\mathbf{k})\mathbf{x}')\rho' d\rho' d\beta' \\ &= \frac{-\mathrm{iE}_0}{2\pi}\frac{\exp(\mathrm{i}kr)}{r}(\mathbf{k}\times\hat{\mathbf{x}})\int_{\rho'=0}^{a}\left\{\int_{\beta'=0}^{2\pi}\exp(\mathrm{i}k\rho'(\sin\alpha\cos\beta' - \sin\theta\cos(\phi-\beta')))d\beta'\right\}\rho' d\rho'\end{aligned}$$

The angular function in the exponent can be rewritten,

$$\begin{aligned}\sin\alpha\cos\beta' - \sin\theta\cos(\phi-\beta') &= \cos\beta\,[\sin\alpha - \sin\theta\cos\phi] + \sin\beta\,[-\sin\theta\sin\phi] \\ &= \xi\cos(\beta'+\delta)\end{aligned}$$

where the amplitude ξ is the square-root of the sum of the squares of the terms in square-brackets, and δ is a constant phase shift. Thus,

$$\xi = \sqrt{[\sin\alpha - \sin\theta\cos\phi]^2 + [-\sin\theta\sin\phi]^2} = \sqrt{\sin^2\theta + \sin^2\alpha - 2\sin\alpha\sin\theta\cos\phi}$$

In the angular integral the phase shift δ is irrelevant, because the angular integral is over a full circle:

$$\begin{aligned}\int_{\beta'=0}^{2\pi}\exp(\mathrm{i}k\rho'(\sin\alpha\cos\beta' - \sin\theta\cos(\phi-\beta')))d\beta' &= \int_0^{2\pi}\exp(\mathrm{i}k\rho'\xi\cos(\beta'+\delta))d\beta' \\ = \int_0^{2\pi}\exp(\mathrm{i}k\rho'\xi\cos\beta')d\beta' = \int_0^{2\pi}\exp(\mathrm{i}k\rho'\xi\sin\beta')d\beta' &= 2\pi J_0(k\rho'\xi)\end{aligned}$$

and the diffracted fields

$$\begin{aligned}\mathbf{E}_{diff}(r,\alpha,\theta,\phi) &= -\mathrm{i}E_0\frac{\exp(\mathrm{i}kr)}{r}(\mathbf{k}\times\hat{\mathbf{x}})\int_{\rho'=0}^{a}J_0(k\rho'\xi)\rho' d\rho' \\ &= -\mathrm{i}E_0 a^2\frac{\exp(\mathrm{i}kr)}{r}(\mathbf{k}\times\hat{\mathbf{x}})\frac{J_1(k\xi a)}{ak\xi} \\ \mathbf{H}_{diff}(r,\alpha,\theta,\phi) &= \frac{1}{Z_0}\hat{\mathbf{k}}\times\mathbf{E}_{diff}(\mathbf{x})\end{aligned}$$

The diffracted power per solid angle

$$\begin{aligned}
\frac{dP}{d\Omega} &= r^2 \frac{1}{2Z_0} \mathbf{E}_{diff} \cdot \mathbf{E}^*_{diff} \\
&= \frac{|E_0|^2}{2Z_0} a^4 \left(\frac{J_1(k\xi a)}{ak\xi} \right)^2 |(\mathbf{k} \times \hat{\mathbf{x}})|^2 \\
&= \frac{|E_0|^2}{2Z_0} a^4 k^2 \left(\frac{J_1(k\xi a)}{ak\xi} \right)^2 (\cos^2\theta + \sin^2\theta \sin^2\phi)
\end{aligned}$$

This can be normalized with the power incident on the hole,

$$P_{in} = \frac{1}{2Z_0} |E_0|^2 a^2 \pi \cos\alpha$$

yielding

$$\frac{dP}{d\Omega} / P_{in} = \frac{a^2 k^2}{\pi \cos\alpha} \left(\frac{J_1(k\xi a)}{ak\xi} \right)^2 (\cos^2\theta + \sin^2\theta \sin^2\phi)$$

b): The result we have obtained equals that of Eq. 10.114 (case of polarization in plane of incidence) times a factor

$$\frac{1}{\cos^2\alpha} \frac{(\cos^2\theta + \sin^2\theta \sin^2\phi)}{(\cos^2\theta + \sin^2\theta \cos^2\phi)}$$

It is also somewhat similar with the result of the scalar calculation, given in Eq. 10.119. In fact, all three results share the essential dependence

$$\propto k^2 a^2 \left(\frac{J_1(k\xi a)}{ak\xi} \right)^2$$

It is also noted that for the case of normal incidence $\alpha = 0$ the two vectorial results are identical, as required. To see this, take the polarization directions into account. Then, note that in the case of normal incidence in both calculations - polarization perpendicular to and in the plane of incidence - the respective terms $\sin\phi$ and $\cos\phi$ are equal to the sine of the angle between the laser polarization and the projection of $\mathbf{k}$ into the xy-plane.

3. Problem 10.16 **10 Points**

a): Using Eq. 10.125 of Jackson, the scattering cross section for incident field $\mathbf{E}_0 = E_0\epsilon_0$ with incident polarization ϵ_0, summed over exit polarizations, is

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \sum_i \frac{(\epsilon_i^* \cdot F_{sh})(\epsilon_i \cdot F_{sh}^*)}{E_0 E_0^*} \\ &= \frac{k^2}{4\pi^2}\sum_i |\epsilon_i^* \cdot \epsilon_0|^2 \left(\int_{shadow} \exp(-\mathrm{i}\mathbf{k}_\perp \cdot \mathbf{x}_\perp) d^2x_\perp\right)\left(\int_{shadow} \exp(\mathrm{i}\mathbf{k}_\perp \cdot \mathbf{x}'_\perp) d^2x'_\perp\right)\end{aligned}$$

where the integrals go over the shadow of the object in the xy-plane. As orthonormal basis for the exit polarizations we can use

$$\epsilon_1 = \hat{\phi}_k = \begin{pmatrix} -\sin\phi_k \\ \cos\phi_k \\ 0 \end{pmatrix} \quad \text{and} \quad \epsilon_2 = \hat{\theta}_k = \begin{pmatrix} \cos\theta_k\cos\phi_k \\ \cos\theta_k \sin\phi_k \\ -\sin\theta_k \end{pmatrix}$$

To cover the case of arbitrary incident polarization, we use $\epsilon_0 = c_1\hat{\mathbf{x}} + c_2\hat{\mathbf{y}}$ with complex numbers $c_1c_1^* + c_2c_2^* = 1$. Then,

$$\sum_i |\epsilon_i^* \cdot \epsilon_0|^2 = |c_1|^2\left(\sin^2\phi_k + \cos^2\theta_k\cos^2\phi_k\right) + |c_2|^2\left(\cos^2\phi_k + \cos^2\theta_k \sin^2\phi_k\right) =: A(\theta_k, \phi_k)$$

Then,

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{k^2}{4\pi^2}\int_{sh}\int_{sh} \exp(-\mathrm{i}\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)) A(\theta_k,\phi_k) d^2x_\perp d^2x'_\perp \\ \sigma &= \frac{k^2}{4\pi^2}\int_{\theta_k,\phi_k}\int_{sh}\int_{sh} \exp(-\mathrm{i}\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)) A(\theta_k,\phi_k) d^2x_\perp d^2x'_\perp \sin\theta_k d\theta_k d\phi_k\end{aligned}$$

Since $\hat{\mathbf{x}} \cdot \mathbf{k} = \hat{\mathbf{x}} \cdot \mathbf{k}_\perp = k_x = k\sin\theta_k\cos\phi_k$ and $\hat{\mathbf{y}} \cdot \mathbf{k} = \hat{\mathbf{y}} \cdot \mathbf{k}_\perp = k_y = k\sin\theta_k \sin\phi_k$, in the angular integration we can substitute

$$d\theta_k d\phi_k = \left|\frac{\partial(\theta_k,\phi_k)}{\partial(k_x,k_y)}\right| dk_x dk_y = \left|\frac{\partial(k_x,k_y)}{\partial(\theta_k,\phi_k)}\right|^{-1} d^2k_\perp = \frac{1}{k^2\sin\theta_k\cos\theta_k} d^2k_\perp$$

and

$$\sigma = \frac{1}{4\pi^2}\int_{|\mathbf{k}_\perp|<k}\int_{sh}\int_{sh} \exp(-\mathrm{i}\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)) \frac{A(\theta_k,\phi_k)}{\cos\theta_k} d^2x_\perp d^2x'_\perp d^2k_\perp$$

Since the shadow region is much larger than the wavelength, in the double-integration over the area the phase term is rapidly oscillating unless $k_\perp \ll k$, that is unless $\theta_k \approx 0$. Angles θ_k substantially different form 0 will not significantly contribute to the integral. We are, essentially, restating the fact that short-wavelength shadow scattering mostly occurs into the forward directions. Thus, in the angle-dependent term $\frac{A(\theta_k,\phi_k)}{\cos\theta_k}$ we may set $\theta_k = 0$, and we may extend the integration range over $k_\perp$ to infinity:

$$\begin{aligned}
\sigma &= \frac{1}{4\pi^2}\int_{|\mathbf{k}_\perp|<\infty}\int_{sh}\int_{sh} \exp(-\mathrm{i}\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\mathbf{x}'_\perp))\frac{A(0,\phi_k)}{\cos(0)}d^2x_\perp d^2x'_\perp d^2k_\perp \\
&= \frac{1}{4\pi^2}\int_{|\mathbf{k}_\perp|<\infty}\int_{sh}\int_{sh} \exp(-\mathrm{i}\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\mathbf{x}'_\perp))(|c_1|^2+|c_2|^2)d^2x_\perp d^2x'_\perp d^2k_\perp \\
&= \frac{1}{4\pi^2}\int_{sh}\int_{sh}\left\{\int_{|\mathbf{k}_\perp|<\infty} \exp(-\mathrm{i}\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\mathbf{x}'_\perp))d^2k_\perp\right\}d^2x_\perp d^2x'_\perp \\
&= \frac{1}{4\pi^2}\int_{sh}\int_{sh}(2\pi)^2\delta^2(\mathbf{x}_\perp-\mathbf{x}'_\perp)d^2x_\perp d^2x'_\perp \\
&= \int_{sh} d^2x_\perp = A_{shadow}
\end{aligned}$$

b): According to the optical theorem, the total cross section (= the sum of scattering and absorption cross section) is

$$\begin{aligned}
\sigma_t &= \sigma+\sigma_{abs} = \frac{4\pi}{k}\mathrm{Im}\left[\epsilon_0^*\cdot\frac{\mathbf{F}(\mathbf{k}_0\cdot\mathbf{k}_0)}{E_0}\right] \\
&\approx \frac{4\pi}{k}\mathrm{Im}\left[\epsilon_0^*\cdot\frac{\mathbf{F}_{sh}(\mathbf{k}_0\cdot\mathbf{k}_0)}{E_0}\right] \\
&= \frac{4\pi}{k}\mathrm{Im}\left[\frac{\mathrm{i}k}{2\pi}(\epsilon_0^*\cdot\epsilon_0)\frac{E_0}{E_0}\int_{shadow}\exp(-\mathrm{i}\mathbf{k}_\perp\cdot\mathbf{x}_\perp)d^2x_\perp\right]_{\mathbf{k}_\perp=0} \\
&= 2A_{shadow}
\end{aligned} \tag{1}$$

This result makes sense because of the following. As seen in part a), small-angle shadow scattering has a cross section of A_{shadow}, independent of what happens to the radiation that actually hits the target. Since the radiation that hits the target either gets absorbed or re-scattered into directions $\mathbf{k} \neq \mathbf{k}_0$, absorption and scattering of the illuminated portion of the target also have a cross section of A_{shadow}. The total cross section thus is $2A_{shadow}$.