Prof. G. Raithel

Problem Set 5

Total 40 Points

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1. Problem 10.2 **10 Points**

The partial-wave analysis presented in Chapter 10.4 applied to the case of a perfectly conducting sphere with radius $ka \ll 1$ leads to the result stated in Eq. 10.71, which applies to incident electric fields of either ϵ_+ (upper sign) or ϵ_- polarization (lower sign),

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{2\pi}{3} a^2 (ka)^4 \left| \mathbf{X}_{1,\pm 1} \mp 2\mathrm{i}\hat{\mathbf{n}} \times \mathbf{X}_{1,\pm 1} \right|^2 \tag{1}$$

The scattering cross section equals the radiated power per solid angle divided by the incident intensity,

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{dP_{sc}}{d\Omega} / I_{inc} = r^2 \mathbf{E}_{sc} \cdot \mathbf{E}^*_{sc} / \mathbf{E}_0 \cdot \mathbf{E}^*_0$$

where $\mathbf{E}_{sc}$ and $\mathbf{E}_0$ are the scattered and incident electric fields, respectively. Thus, up to a pre-factor including $\exp(\mathrm{i}kr)/r$ the term $\mathbf{X}_{1,\pm 1} \mp 2\mathrm{i}\hat{\mathbf{n}} \times \mathbf{X}_{1,\pm 1}$ represents the scattered electric field in the radiation zone for the case of either clean ϵ_+ or ϵ_- polarizations. Based on the superposition principle, for an incident field with a unit polarization vector

$$\epsilon = \frac{1}{\sqrt{1+r^2}} (\epsilon_+ + r \exp(\mathrm{i}\alpha) \epsilon_-) \tag{2}$$

the scattered electric field is obtained via a corresponding coherent superposition of the scattered fields of ϵ_+ and ϵ_- polarizations. Thus, for the incident polarization of Eq. 2 the scattering cross section is

$$\begin{aligned} \frac{d\sigma_{sc}}{d\Omega} &= \frac{2\pi}{3} a^2 (ka)^4 \frac{1}{1+r^2} \left| [\mathbf{X}_{1,1} - 2\mathrm{i}\hat{\mathbf{n}} \times \mathbf{X}_{1,1}] + r \exp(\mathrm{i}\alpha) [\mathbf{X}_{1,-1} + 2\mathrm{i}\hat{\mathbf{n}} \times \mathbf{X}_{1,-1}] \right|^2 \\ &=: \frac{2\pi}{3} a^2 (ka)^4 \frac{1}{1+r^2} |\mathbf{F}|^2 \end{aligned}$$

Using that

$$\begin{aligned} \mathbf{X}_{l,m} &= \frac{1}{\sqrt{l(l+1)}} \hat{\mathbf{L}} Y_{lm} \\ \hat{\mathbf{L}} &= \frac{1}{\mathrm{i}} \left(\hat{\phi} \partial_\theta - \frac{\hat{\theta}}{\sin\theta} \partial_\phi \right) \\ Y_{1,\pm 1} &= \mp \sqrt{\frac{3}{8\pi}} \sin\theta \exp(\pm \mathrm{i}\phi) \end{aligned}$$

it is found that

$$\mathbf{X}_{1,\pm1} = \mp\sqrt{\frac{3}{16\pi}}\left(\frac{\hat{\phi}}{\mathrm{i}}\cos\theta \mp \hat{\theta}\right)\exp(\pm \mathrm{i}\phi)$$

Inserting into Eq. 3 we find, with $\hat{\mathbf{n}} \times \hat{\theta} = \hat{\phi}$ and $\hat{\mathbf{n}} \times \hat{\phi} = -\hat{\theta}$, that the components of the transverse field $\hat{\mathbf{F}}$ are

$$\begin{aligned}
F_\theta &= \sqrt{\frac{3}{16\pi}}\left[\exp(\mathrm{i}\phi)(1-2\cos\theta) + r\exp(-\mathrm{i}\phi+\mathrm{i}\alpha)(1-2\cos\theta)\right] \\
F_\phi &= \sqrt{\frac{3}{16\pi}}\left[\mathrm{i}\exp(\mathrm{i}\phi)(\cos\theta-2) + \mathrm{i}r\exp(-\mathrm{i}\phi+\mathrm{i}\alpha)(2-\cos\theta)\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\sigma_{sc}}{d\Omega} &= \frac{2\pi}{3}a^2(ka)^4\frac{1}{1+r^2}\left(F_\theta F_\theta^* + F_\phi F_\phi *\right) \\
&= \frac{k^4a^6}{8(1+r^2)}\left[(1-2\cos\theta)^2(1+r^2+2r\cos(2\pi-\alpha)) + (2-\cos\theta)^2(1+r^2-2r\cos(2\phi-\alpha))\right] \\
&= \frac{k^4a^6}{8(1+r^2)}\left[(1+r^2)(5(1+\cos^2\theta)-8\cos\theta) + 2r\cos(2\phi-\alpha)(3\cos^2\theta-3)\right] \\
&= k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3r}{4(1+r^2)}\sin^2\theta\cos(2\phi-\alpha)\right] \quad \text{q.e.d.}
\end{aligned}$$

A more basic but cumbersome approach is to calculate the scattered electric field in the far zone,

$$\mathbf{E}_{sc} = \frac{k^2}{4\pi}\frac{\exp(\mathrm{i}kr)}{r}\left(\frac{1}{\epsilon_0}(\hat{\mathbf{n}} \times \mathbf{p}) + Z_0\mathbf{m}\right) \times \hat{\mathbf{n}}$$

for the electric and magnetic dipoles $\mathbf{p} = 4\pi\epsilon_0 a^3\mathbf{E}_{in}$ and $\mathbf{m} = -\frac{2\pi}{\mu_0}a^3\mathbf{B}_{in}$ induced by an incident field with k-vector $\mathbf{k}_0 = k\hat{\mathbf{z}}$

$$\begin{aligned}
\mathbf{E}_{in} &= \frac{E_0}{\sqrt{1+\tilde{r}^2}}\left(\epsilon_+ + \tilde{r}\exp(\mathrm{i}\alpha)\epsilon_-\right) \\
\mathbf{B}_{in} &= \frac{1}{c}\hat{\mathbf{z}} \times \mathbf{E}_{in} \quad .
\end{aligned}$$

The scattering cross section follows from an elementary calculation of

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{r^2\mathbf{E}_{sc}\cdot\mathbf{E}_{sc}^*}{\mathbf{E}_{in}\cdot\mathbf{E}_{in}^*}$$

2. Problem 10.3 **10 Points**

a): Since the skin depth is much smaller than the radius, $\delta \ll R$, the magnetic field does not penetrate significantly into the sphere. Also, since $kR \ll 1$, the magnetic field in the vicinity of the sphere is essentially static (near-field limit). Because of both these facts, the H-field in the vicinity of the sphere is obtained by considering a sphere with radius R and $\mu = 0$ in an external homogeneous magnetic field $\mathbf{H}_0$. Due to the absence of free currents in this model, the magnetostatic potential may be used.

We first assume a magnetic field H_0 in the z-direction. The magnetic potentials inside and outside the sphere are then of the form

$$\begin{aligned} \Phi_i &= \sum_l a_l r^l P_l(\cos\theta) \\ \Phi_o &= \sum_l b_l r^{-l-1} P_l(\cos\theta) - H_0 r P_1(\cos\theta) \end{aligned}$$

where the second term in the last equation is added to match the boundary condition

$$\mathbf{H}(r \to \infty) = -\nabla\Phi_o(r \to \infty) = \mathbf{H}_0 = H_0 \hat{\mathbf{z}}$$

The radial boundary condition on the surface is $\mu H_{r,i} = 0 = \mu_0 H_{r,o}$, i.e.

$$0 = -\mu_0 \sum_l b_l(-l-1) R^{-l-2} P_l - H_0 P_1$$

yielding $b_l = 0$ for $l \neq 1$ and

$$b_1 = -\frac{1}{2} R^3 H_0 \quad .$$

The θ-boundary condition on the surface is $H_{r,i} = H_{r,o}$, i.e.

$$\sum_l a_l R^{l-1} P_l' = \sum_l b_l R^{-l-2} P_l' - H_0 P_1' \quad .$$

With the previous result for the b_l, we find $a_l = 0$ for $l \neq 1$, and

$$a_1 = -\frac{3}{2} H_0 \quad .$$

<u>Result inside:</u>

$$\begin{aligned} \Phi_i &= -\frac{3}{2} H_0 r \cos\theta \\ H_{r,i} &= \frac{3}{2} H_0 \cos\theta \\ H_{\theta,i} &= -\frac{3}{2} H_0 \sin\theta \\ B_{r,i} &= B_{\theta,i} = 0 \end{aligned}$$

Result outside:

$$
\begin{aligned}
\Phi_o &= -\left(\frac{R^3}{2r^2} + r\right) H_0 \cos\theta \\
H_{r,o} &= \left(-\frac{R^3}{r^3} + 1\right) H_0 \cos\theta \\
H_{\theta,o} &= -\left(\frac{R^3}{2r^3} + 1\right) H_0 \sin\theta \\
B_{r,o} &= \mu_0 \left(-\frac{R^3}{r^3} + 1\right) H_0 \cos\theta \\
B_{\theta,o} &= -\mu_0 \left(\frac{R^3}{2r^3} + 1\right) H_0 \sin\theta
\end{aligned}
$$

Immediately outside the surface, $H_{r,s} = 0$ and $H_{\theta,s} = -\frac{3}{2} H_0 \sin\theta$. These results hold for the specialized case of $\mathbf{H}_0 = H_0 \hat{\mathbf{z}}$; in this case, all ϕ-components of the fields are zero. Also, from the form of Φ_o it is seen that the outside field equals $\mathbf{H}_0$ plus that of a magnetic dipole with moment $\mathbf{m} = -2\pi R^3 \mathbf{H}_0$.

Next, we consider the field for general polarization. The outside field in the near zone equals $\mathbf{H}_0$ plus the field of a magnetic dipole $\mathbf{m} = -2\pi R^3 \mathbf{H}_0$. Based on the superposition principle, we can - without further analysis - state that for a magnetic field $\mathbf{H}_0$ of the general form

$$
\mathbf{H}_0 = H_0 \epsilon_H = H_0 \hat{\mathbf{k}}_0 \times \epsilon
$$

the induced magnetic moment is

$$
\mathbf{m} = -2\pi R^3 H_0 \epsilon_H
$$

There, ϵ_H is the polarization vector of the magnetic field of the incident wave, ϵ the (usual) polarization vector of the electric field, and $\hat{\mathbf{k}}_0$ a unit vector in the direction of propagation of the incident wave. In a linear-polarization basis, a general polarization state is characterized by the equivalent forms

$$
\begin{aligned}
\epsilon &= c_1 \epsilon_1 + c_2 \epsilon_2 \\
\epsilon_H &= c_1 \epsilon_2 - c_2 \epsilon_1 \quad ,
\end{aligned}
$$

where $c_1 c_1^* + c_2 c_2^* = 1$. The magnetic field in the near-zone outside the sphere then is

$$
\mathbf{H} = \frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{m}) - \mathbf{m}}{4\pi r^3} + H_0 \epsilon_H = \left[-\frac{R^3}{2r^3} (3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \epsilon_H) - \epsilon_H) + \epsilon_H \right] H_0
$$

where $\hat{\mathbf{n}}$ is a radial unit vector. The surface field $\mathbf{H}_s$ is obtained by setting $r = R$,

$$
\mathbf{H}_s = \frac{3}{2} H_0 \left[\epsilon_H - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \epsilon_H) \right]
$$

As a test, we can verify that this field is entirely tangential by seeing that $\hat{\mathbf{n}} \cdot \mathbf{H}_s = 0$.

b): The absorbed power equals, by Eq. 8.15 of Jackson,

$$
\begin{aligned}
P_{abs} &= \frac{1}{2\sigma\delta}\oint \left|\hat{\mathbf{n}}\times\mathbf{H}_{||}\right|^2 da = \frac{R^2}{2\sigma\delta}\oint \mathbf{H}_s\cdot\mathbf{H}_s^* d\cos\theta d\phi \\
&= \frac{R^2}{2\sigma\delta}\frac{9|H_0|^2}{4}\int [\epsilon_H - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\epsilon_H)]\,[\epsilon_H^* - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\epsilon_H^*)]\,d\Omega \\
&= \frac{9|H_0|^2R^2}{8\sigma\delta}\int \left[\epsilon_H\cdot\epsilon_H^* - |\hat{\mathbf{n}}\cdot\epsilon_H|^2\right] d\Omega \\
&= \frac{9|H_0|^2R^2}{8\sigma\delta}\left[4\pi - c_1c_1^*\int |\hat{\mathbf{n}}\cdot\epsilon_2|^2\,d\Omega - c_2c_2^*\int |\hat{\mathbf{n}}\cdot\epsilon_1|^2\,d\Omega + 2\mathrm{Re}\left(c_1c_2^*\int (\hat{\mathbf{n}}\cdot\epsilon_1)(\hat{\mathbf{n}}\cdot\epsilon_2)d\Omega\right)\right] \\
&= \frac{9|H_0|^2R^2}{8\sigma\delta}\left[4\pi - c_1c_1^*\frac{4\pi}{3} - c_2c_2^*\frac{4\pi}{3} + 2\mathrm{Re}\left(c_1c_2^*\times 0\right)\right] \\
&= \frac{9|H_0|^2R^2}{8\sigma\delta}\left[4\pi - \frac{4\pi}{3}(c_1c_1^* + c_2c_2^*)\right] \\
&= \frac{9|H_0|^2R^2}{8\sigma\delta}\left[4\pi - \frac{4\pi}{3}\right] \\
&= \frac{3|H_0|^2R^2\pi}{\sigma\delta}
\end{aligned}
$$

Since the incident intensity

$$I_{in} = \frac{1}{2Z_0}\mathbf{E}_0\cdot\mathbf{E}_0^* = \frac{Z_0}{2}\mathbf{H}_0\cdot\mathbf{H}_0^* = \frac{Z_0}{2}|H_0|^2\epsilon_H\cdot\epsilon_H^* = \frac{Z_0}{2}|H_0|^2$$

the absorption cross section is

$$\sigma_{abs} = \frac{P_{abs}}{I_{in}} = \frac{6R^2\pi}{Z_0\sigma\delta} = \frac{6R^2\pi}{Z_0}\sqrt{\frac{\mu_0\omega}{2\sigma}} = 6R^2\pi\sqrt{\frac{\epsilon_0\omega}{2\sigma}} \propto \sqrt{\omega}$$

We find that P_{abs} and σ_{abs} are independent of the polarization state of the incident wave. Therefore, the results also apply for unpolarized light.

3. Problem 10.8 **10 Points**

a): According to Eq. 8.11 of Jackson, the tangential electric and magnetic fields on the surface of a non-ideal conductor with $\mu = \mu_0$ follow

$$\mathbf{E}_{tan} = \sqrt{\frac{\mu_0 \omega}{2\sigma}}(1-\mathrm{i})(\hat{\mathbf{n}} \times \mathbf{H}_{tan}) = \sqrt{\frac{\mu_0 \omega}{2\sigma}}(1-\mathrm{i})(\hat{\mathbf{n}} \times \mathbf{H})$$

This is of the form of Eq. 10.64,

$$\mathbf{E}_{tan} = Z_s(\hat{\mathbf{n}} \times \mathbf{H})$$

with surface impedance

$$Z_s = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-\mathrm{i}) = \frac{Z_0 k\delta}{2}(1-\mathrm{i}) \quad ,$$

where we have used the skin depth $\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$.

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b): The long-wavelength limit for $l = 1$ is obtained from Eq. 10.69 (set $x = ka$):

$$\alpha_{\pm}(1) \approx -\frac{2\mathrm{i}(ka)^3}{3}\left(\frac{ka - 2\mathrm{i}\frac{k\delta}{2}(1-\mathrm{i})}{ka + \mathrm{i}\frac{k\delta}{2}(1-\mathrm{i})}\right) = -\frac{2\mathrm{i}(ka)^3}{3}\left(\frac{(1-\frac{\delta}{a}) - \mathrm{i}\frac{\delta}{a}}{(1+\frac{\delta}{2a}) + \mathrm{i}\frac{\delta}{2a}}\right) \quad \text{q.e.d.}$$

$$\beta_{\pm}(1) \approx -\frac{2\mathrm{i}(ka)^3}{3}\left(\frac{ka - 2\mathrm{i}\frac{2}{k\delta(1-\mathrm{i})}}{ka + \mathrm{i}\frac{2}{k\delta(1-\mathrm{i})}}\right)$$

Since $k\delta < ka \ll 1$ and, consequently, $\frac{1}{k\delta} \gg 1$, we can drop the ka in the large parentheses, and

$$\beta_{\pm}(1) \approx -\frac{2\mathrm{i}(ka)^3}{3}\left(\frac{-2\mathrm{i}\frac{2}{k\delta(1-\mathrm{i})}}{\mathrm{i}\frac{2}{k\delta(1-\mathrm{i})}}\right) = \frac{4\mathrm{i}(ka)^3}{3} \quad \text{q.e.d.}$$

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c): With

$$\mathbf{X}_{1,\pm 1} = \mp\sqrt{\frac{3}{16\pi}}\left(\frac{\hat{\phi}}{\mathrm{i}}\cos\theta \mp \hat{\theta}\right)\exp(\pm \mathrm{i}\phi)$$

$$t := \frac{(1-\frac{\delta}{a}) - \mathrm{i}\frac{\delta}{a}}{(1+\frac{\delta}{2a}) + \mathrm{i}\frac{\delta}{2a}}$$

and $\hat{\mathbf{n}} \times \hat{\theta} = \hat{\phi}$ and $\hat{\mathbf{n}} \times \hat{\phi} = -\hat{\theta}$, from Eq. 10.63 it follows for the given case

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{3\pi}{2k^2}\left|\alpha\mathbf{X}_{1,\pm1} \pm \mathrm{i}\beta\hat{\mathbf{n}}\times\mathbf{X}_{1,\pm1}\right|^2 \\
&= \frac{3\pi}{2k^2}\frac{4(ka)^6}{9}\left|-t\mathbf{X}_{1,\pm1} \pm 2\mathrm{i}\beta\hat{\mathbf{n}}\times\mathbf{X}_{1,\pm1}\right|^2 \\
&= \frac{3\pi}{2k^2}\frac{4(ka)^6}{9}\frac{3}{16\pi}\left|\pm\hat{\phi}\left[\frac{t}{\mathrm{i}}\cos\theta + 2\mathrm{i}\right] + \hat{\theta}\left[-t + 2\cos\theta\right]\right|^2 \\
&= \frac{(ka)^6}{8k^2}\left[(t\cos\theta - 2)(t^*\cos\theta - 2) + (t - 2\cos\theta)(t^* - 2\cos\theta)\right] \\
&= \frac{(ka)^6}{8k^2}\left[(tt^* + 4)(1+\cos^2\theta) - 4(t+t^*)\cos\theta\right]
\end{aligned}
$$

In first order of $\frac{\delta}{a}$, we find $tt^* = 1 - \frac{3\delta}{a}$ and $t + t^* = 2\mathrm{Re}(t) = 2 - \frac{3\delta}{a}$. Thus,

$$
\frac{d\sigma}{d\Omega} = \frac{(ka)^6}{8k^2}\left[(5 - \frac{3\delta}{a})(1+\cos^2\theta) - 4(2 - \frac{3\delta}{a})\cos\theta\right]
$$

As a quick test we note that the result agrees with Eq. 10.72 in the limit $\delta \to 0$.

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d): According to Eq. 10.61, in the limit that only $l = 1$ is important, as in the given case, the total absorption cross section is

$$
\sigma_{abs} = \frac{3\pi}{2k^2}(2 - \alpha\alpha^* - \beta\beta^* - 2\mathrm{Re}(\alpha+\beta) - 2)
$$

The terms $\propto \alpha\alpha^*$ and $\propto \beta\beta^*$ are of order $(ka)^6$. The term $\propto \mathrm{Re}(\beta) = 0$, and the term $\propto \mathrm{Re}(\alpha)$ is of order $(ka)^3$. Thus, the only term of importance is $\mathrm{Re}(\alpha)$,

$$
\begin{aligned}
\mathrm{Re}(\alpha) &= \frac{2(ka)^3}{3}\left(\frac{-\frac{\delta}{a}(1+\frac{\delta}{2a}) - \frac{\delta}{2a}(1-\frac{\delta}{a})}{(1+\frac{\delta}{2a})^2 + \frac{\delta^2}{4a^2}}\right) \\
&\approx -\frac{2(ka)^3}{3}\left(\frac{3\delta}{2a}\right) = -(ka)^3\frac{\delta}{a} \qquad (3)
\end{aligned}
$$

where the last line is valid for $\delta \ll a$. Thus, in first order of δ it is

$$
\sigma_{abs} = \frac{3\pi}{2k^2}2(ka)^3\frac{\delta}{a} = 3\pi k\delta a^2 \quad \text{q.e.d.}
$$

For $\delta = a$ we use the first line of Eq. 3 to find $\mathrm{Re}(\alpha) = -\frac{2}{5}(ka)^3$, and

$$
\sigma_{abs}(\delta = a) = 3\pi k a^3 \times \frac{2}{5}
$$

This is only 40% of the result that would follow from the equation valid for $\delta \ll a$. Also, note that one cannot expect the underlying analysis of Chapter 8.1 to be very accurate for $\delta = a$.

4. Problem 10.9a **10 Points**

For ϵ_r close to 1, we can use the Born approximation. For the given case, the normalized polarization-resolved scattering amplitude in Born approximation is,

$$\frac{\epsilon^* \cdot \mathbf{A}_{sc}}{\mathbf{D}_0} = \frac{k^2}{4\pi}(\epsilon_r - 1)\epsilon^* \cdot \epsilon_0 \int_{r<a} \exp(i\mathbf{q} \cdot \mathbf{x}')d^3x'$$

The integral

$$\begin{aligned}
\int_{r<a} \exp(i\mathbf{q} \cdot \mathbf{x}')d^3x' &= \int_0^a r'^2 \left[\int \exp(iqr' \cos\theta')d\Omega'\right] dr' = 2\pi \int_0^a r'^2 \left[\int \exp(iqr' \cos\theta')d\cos\theta'\right] dr' \\
&= 4\pi \int_0^a r'^2 \frac{\sin(qr')}{qr'} dr' = \frac{4\pi}{q^3} \int_0^{qa} z \sin z dz \\
&= \frac{4\pi}{q^3}(\sin(qa) - qa\cos(qa)) = 4\pi a^3 \frac{j_1(qa)}{qa}
\end{aligned}$$

Thus, $\frac{\epsilon^* \cdot \mathbf{A}_{sc}}{\mathbf{D}_0} = k^2(\epsilon_r - 1)\epsilon^* \cdot \epsilon_0 a^3 \frac{j_1(qa)}{qa}$, and

$$\frac{d\sigma}{d\Omega}(\epsilon, \epsilon_0) = \left|\frac{\epsilon^* \cdot \mathbf{A}_{sc}}{\mathbf{D}_0}\right|^2 = k^4 a^6 (\epsilon_r - 1)^2 \left|\frac{j_1(qa)}{qa}\right|^2 |\epsilon^* \cdot \epsilon_0|^2$$

Averaging over the incident and summing over the exit polarizations leads to

$$\frac{d\sigma}{d\Omega} = k^4 a^6 (\epsilon_r - 1)^2 \left|\frac{j_1(qa)}{qa}\right|^2 \frac{1}{2}(1 + \cos^2\theta)$$

(see Chapter 10.1 and lecture). We note that by the law of cosines it is $q = k\sqrt{2(1 - \cos\theta)}$, where θ is the scattering angle.

Trends. For $ka \gg 1$, $qa = ka\sqrt{2(1 - \cos\theta)}$ also tends to be $\gg 1$. This limit applies in all cases except for θ less than a critical value $\sim \frac{1}{ka}$. The critical angle $\theta_c = \frac{1}{ka}$ corresponds to a small forward-scattering cone with solid angle $\frac{\pi}{k^2a^2}$, in which the limit $qa \gg 1$ does not apply. However, for $ka \gg 1$ the solid angle $\frac{\pi}{k^2a^2}$ will be negligibly small, and can be ignored. For an estimate, we may thus assume $qa \gg 1$ for all θ.

Since for large $qa = x$ it is $j_1(x) \approx \frac{1}{x}\sin(x - \pi/2)$, the following approximate scaling applies:

$$\left|\frac{j_1(qa)}{qa}\right|^2 \propto \left|\frac{1}{(qa)^2}\right|^2 = \frac{1}{(qa)^4}$$

Thus, the scattering cross section approximately scales as

$$\frac{d\sigma}{d\Omega} \propto a^2 \left(\frac{k}{q}\right)^4$$

and is clearly peaked at small q, corresponding to scattering in forward directions. Consequently, in the integral for the total scattering cross section we may set $\cos\theta = 1$ and get

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = k^4 a^6 (\epsilon_r - 1)^2 \int \left| \frac{j_1(qa)}{qa} \right|^2 \frac{1}{2}(1 + \cos^2\theta) d\Omega \approx k^4 a^6 (\epsilon_r - 1)^2 \int \left| \frac{j_1(qa)}{qa} \right|^2 d\Omega$$

Also, with $q = k\sqrt{2(1 - \cos\theta)}$ it is $\frac{dq}{d\cos\theta} = -\frac{k^2}{q}$ and

$$\sigma \approx -2\pi k^4 a^6 (\epsilon_r - 1)^2 \int_{2k}^{0} \left| \frac{j_1(qa)}{qa} \right|^2 \frac{q}{k^2} dq = -2\pi k^2 a^4 (\epsilon_r - 1)^2 \int_{2k}^{0} \frac{j_1^2(qa)}{q} dq = 2\pi k^2 a^4 (\epsilon_r - 1)^2 \int_{0}^{2ka} \frac{j_1^2(x)}{x} d(x)$$

Since for large x the scaling of $\frac{j_1^2(x)}{x}$ is $\sim \frac{1}{x^3}$, for the purpose of an estimate we may extend the integration range to infinity,

$$\begin{aligned} \sigma &\approx 2\pi k^2 a^4 (\epsilon_r - 1)^2 \int_0^\infty \frac{j_1^2(x)}{x} d(x) = 2\pi k^2 a^4 (\epsilon_r - 1)^2 \times \frac{1}{4} \\ \sigma &\approx \frac{\pi}{2} k^2 a^4 (\epsilon_r - 1)^2 \quad \text{q.e.d} \end{aligned}$$