Phys. 506 Electricity and Magnetism Winter 2004

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Problem Set 4

Total 40 Points

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1. Problem 9.10 **10 Points**

a). In the long-wavelength limit, in the source and its immediate vicinity electro- and magnetostatic equations apply. Thus, with Eq. 5.53 the magnetization density $\mathbf{M}$ is, using $\hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\sin\theta\hat{\phi} = -\sin\theta(-\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi)$, and $v_0 = \alpha c$

$$\begin{aligned}
\mathbf{M} &= \frac{1}{2}\mathbf{x} \times \mathbf{J}(\mathbf{x}) \\
&= \frac{1}{2}\left(\frac{-\mathrm{i}v_0}{2}\right) r\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}}\frac{1}{2} + \hat{\mathbf{z}}\frac{a_0}{z}\right) \\
&= \frac{1}{2}\left(\frac{-\mathrm{i}v_0}{2}\right) r\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}}\frac{1}{2} + \hat{\mathbf{z}}\frac{a_0}{z}\right)\rho(r,\theta) \\
&= \frac{1}{2}\left(\frac{\mathrm{i}v_0}{2}\right)\frac{ra_0}{z}\hat{\phi}\sin\theta\rho(r,\theta) \\
&= \frac{1}{2}\left(\frac{\mathrm{i}a_0v_0}{2}\right)\hat{\phi}\tan\theta\rho(r,\theta) \\
&= -(\hat{\mathbf{x}}\sin\phi - \hat{\mathbf{y}}\cos\phi)\left(\frac{\mathrm{i}a_0\alpha c}{4}\right)\tan\theta\rho(r,\theta) = \mathbf{M} \quad \text{, q.e.d.}
\end{aligned} \tag{1}$$

In the calculation of multipole moments at frequency ω_0, we may thus replace the current by the given effective magnetization density and set $\mathbf{J} = 0$. (Note that both $\mathbf{M}$ and $\mathbf{J}$ carry a time factor $\exp(-\mathrm{i}\omega_0 t)$, which is not shown.) Since $\mathbf{M}$ is of the form

$$\mathbf{M} = \hat{\phi}f(r,\theta)$$

with a function f that doesn't depend on ϕ, it is

$$\nabla\cdot\mathbf{M} = 0 \quad .$$

In the long-wavelength limit, for the multipole moments Eqs. 9.169 to 9.172 apply. Thus, with $\mathbf{J} = 0$ and $\nabla\cdot\mathbf{M} = 0$ both $M_{lm} = 0$ and $M'_{lm} = 0$. There are no magnetic multipoles.

From the orthogonality of the spherical harmonics, the only non-vanishing Q_{lm} is

$$Q_{10} = \int_\Omega\int_0^\infty r\left(\frac{2e}{\sqrt{6}a_0^4} r\exp(-\frac{3r}{2a_0})Y_{00}Y_{10}\right)Y_{10}^* r^2 dr d\Omega$$

$$\begin{aligned} &= \frac{2e}{\sqrt{24\pi}a_0^4}\int_0^\infty r^4\exp(-\frac{3r}{2a_0})dr \\ &= \frac{1}{\sqrt{6\pi}}\frac{256}{81}ea_0 \end{aligned} \tag{2}$$

To find the Q'_{lm}, we note

$$\begin{aligned} \nabla\cdot(\mathbf{r}\times\mathbf{M}) &= \frac{\mathrm{i}\alpha c a_0}{4}\nabla\cdot(\hat{\mathbf{r}}\times\hat{\phi})r\tan\theta\rho(r,\theta) \\ &= -\frac{\mathrm{i}\alpha c a_0}{4}\nabla\cdot(\hat{\theta}r\tan\theta\rho(r,\theta) \\ &= -\frac{\mathrm{i}\alpha c a_0}{4}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}r\sin\theta\tan\theta\rho(r,\theta) \end{aligned} \tag{3}$$

Since the angular dependence of ρ is $Y_{10}\propto\cos\theta$, this is

$$\nabla\cdot(\mathbf{r}\times\mathbf{M}) = -\frac{\mathrm{i}\alpha c a_0}{4}\frac{1}{\sin\theta}2\sin\theta\rho(r,\theta) = -\frac{\mathrm{i}\alpha c a_0}{2}\rho(r,\theta)$$

From Eq. 9.170 and the previous result on Q_{lm}, it is seen that the only non-vanishing Q'_{lm} is

$$Q'_{10} = -\frac{k\alpha a_0}{4}Q_{10} = -\frac{\pi}{2}\frac{a_0}{\lambda}\alpha Q_{10}$$

Since the factor on the rhs is of order $\frac{1}{1000}\frac{1}{137}$ and the radiated power behaves as the square of the multipole moments, we can safely assume

$$Q'_{10} = 0$$

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b: Using Eq. 9.169, it is $a_E(1,0) = \frac{ck^3}{3\mathrm{i}}\sqrt{2}Q_{10}$. Also, the radiated power $P = \frac{Z_0}{2k^2}\left|a_E\right|^2$. Inserting the results of part a), it is

$$P = \frac{256^2}{81^2\cdot 9\cdot 6\pi}Z_0c^2k^4a_0^2e^2 \approx 1.02\times 10^{-9}\mathrm{W}$$

This can be expressed in the required unit, yielding

$$P = \left(\frac{2}{3}\right)^8\hbar\omega_0\left(\frac{\alpha^4 c}{a_0}\right)$$

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c: The transition rate is

$$\Gamma = \frac{P}{\hbar\omega_0} = \left(\frac{2}{3}\right)^8 \left(\frac{\alpha^4 c}{a_0}\right)$$

Numerically,

$$\Gamma = 6.27 \times 10^8 s^{-1} = (1.59ns)^{-1}$$

This equals the quantum mechanical decay rate of the hydrogen 2P level.

Note. The only non-zero multipole moment found in the classical calculation conforms with quantum mechanical selection rules explained in Chapter 9.8. First, in a transition from an upper 2P level into a lower 1S level the atomic angular momentum changes from 1 to 0 (with spin neglected). Thus, only $l = 1$ radiation can occur. Further, the transition from the 2P level into 1S reverses the parity of the atomic state, requiring an emission field mode with odd parity (that is, odd magnetic field). This only leaves electric $l = 1$ decay modes. Finally, in the given example both the upper and lower states have zero $z-$angular momentum. Thus, the emitted field cannot carry any $z-$angular momentum. In summary, the only multipole field allowed by selection rules is the $a_E(l = 1, m = 0)$, as found above.

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d: According to an earlier homework problem, for an elementary charge orbiting in the xy-plane at a radius $2a_0$, the only radiation multipole moment for dipole radiation is

$$Q_{11} = 2\tilde{Q}_{11}$$

where $\tilde{Q}_{11}$ is a "usual" spherical multipole evaluated in the rotating frame. Here, $\tilde{Q}_{11} = 2ea_0\sqrt{\frac{3}{8\pi}}\exp(-\mathrm{i}\phi_0)$ with a phase ϕ_0 that we may set to zero. Thus,

$$Q_{11} = 4\sqrt{\frac{3}{8\pi}}ea_0$$

leading to a radiated power of

$$P_{cl} = \left(\frac{3^2}{2^6}\right)\hbar\omega_0\left(\frac{\alpha^4 c}{a_0}\right)$$

The ratio of this classical power and the "quantum" power of part b) is

$$\frac{P_{cl}}{P_{qm}} = \frac{3^{10}}{2^{14}} = 3.60$$

2. Problem 9.16 **10 Points**

a): In this problem, a calculation in cartesian coordinates is the most straightforward. The current density is

$$\mathbf{J}(\mathbf{x}) = \hat{\mathbf{z}} I \delta(x)\delta(y)\sin(kz)$$

for $|z| \leq \lambda/2$. The radiation pattern is only relevant in the radiation zone. Thus, we calculate

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\frac{\exp(\mathrm{i}kr)}{r}\int_{source}\exp(-\mathrm{i}k\hat{\mathbf{n}}'\cdot\mathbf{x}')\mathbf{J}(\mathbf{x}')dxdydz \\
&= \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{I}{2\mathrm{i}}\frac{\exp(\mathrm{i}kr)}{r}\int_{z=-\lambda/2}^{z=+\lambda/2}\exp(-\mathrm{i}kz\cos\theta)\left(\exp(\mathrm{i}kz)-\exp(-\mathrm{i}kz)\right)dz \\
&= \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{I}{2\mathrm{i}}\frac{\exp(\mathrm{i}kr)}{r}\left[\frac{1}{\mathrm{i}k(1-\cos\theta)}\exp(\mathrm{i}kz(1-\cos\theta))+\frac{1}{\mathrm{i}k(1+\cos\theta)}\exp(-\mathrm{i}kz(1+\cos\theta))\right]_{z=-\lambda/2}^{z=+\lambda/2} \\
&= \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{I}{2k}\frac{2\mathrm{i}\exp(\mathrm{i}kr)}{r}\left[\frac{\sin(\pi(1+\cos\theta))}{(1+\cos\theta)}-\frac{\sin(\pi(1-\cos\theta))}{(1-\cos\theta)}\right] \\
&= \hat{\mathbf{z}}\frac{\mu_0 I}{2\pi}\frac{\exp(\mathrm{i}kr)}{\mathrm{i}kr}\left(\frac{\sin(\pi\cos\theta)}{\sin^2\theta}\right)
\end{aligned}$$

In the radiation zone, $\mathbf{H} = \frac{\mathrm{i}k}{\mu_0}\hat{\mathbf{n}}\times\mathbf{A}$, and with $\hat{\mathbf{r}}\times\hat{\mathbf{z}} = -\sin\theta\hat{\phi}$

$$\mathbf{H} = -\hat{\phi}\frac{I}{2\pi}\frac{\exp(\mathrm{i}kr)}{r}\left(\frac{\sin(\pi\cos\theta)}{\sin\theta}\right)$$

The radiation pattern is $\frac{dP}{d\Omega} = r^2\frac{1}{2Z_0}\mathbf{E}\cdot\mathbf{E}^* = r^2\frac{Z_0}{2}\mathbf{H}\cdot\mathbf{H}^*$, yielding

$$\frac{dP}{d\Omega} = \frac{|I|^2 Z_0}{8\pi^2}\left(\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}\right)$$

The result is exact in the radiation-zone limit, $kr \gg 1$. For the plot, see Problem 9.17.

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b): The radiated power P

$$\begin{aligned}
P &= \frac{|I|^2 Z_0}{8\pi^2}\int_{-1}^{1}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}2\pi d\cos\theta \\
&= \frac{|I|^2 Z_0}{4\pi}\int_{-1}^{1}\frac{1-\cos^2(\pi x)}{1-x^2}dx \\
&= \frac{|I|^2 Z_0}{4\pi}\times 1.55718
\end{aligned}$$

Sine the radiation resistance is defined via $P = \frac{1}{2}R_{rad}|I|^2$, it is

$$R_{rad} = \frac{Z_0}{2\pi} \times 1.55718 = 93.36\Omega$$

3. Problem 9.17 **10 Points**

a): We use Eqs. 9.167 and 9.168 to obtain multipole moments that are NOT in the small-source approximation. Since Eqs. 9.167f are processed most efficiently in spherical coordinates, we use

$$\mathbf{J}(\mathbf{x}) = \hat{\mathbf{r}}\frac{I(r)}{2\pi r^2}\left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right]$$

and

$$\rho(\mathbf{x}) = \frac{1}{2\mathrm{i}\omega\pi r^2}\frac{dI(r)}{dr}\left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right]$$

with $I(r) = I\sin(kr)$ for $0 < r < \lambda/2$ and zero otherwise. It is easily verified that the continuity equation, $\nabla \cdot \mathbf{J} = \mathrm{i}\omega\rho$, holds. Since there is no intrinsic magnetization $\mathbf{M}$ and since at all locations $\mathbf{r}$ where there is current flowing it is $\mathbf{r}\times\mathbf{J} = 0$, the magnetic moments all vanish. From Eq. 9.167 we find the electric-multipole amplitudes

$$\begin{aligned}
a_E(l,m) &= \frac{k^2}{\mathrm{i}\sqrt{l(l+1)}}\int Y^*_{l,m}\left\{c\rho\left(\frac{d}{dr}rj_l(kr)\right) + \mathrm{i}k(\mathbf{r}\cdot\mathbf{J})j_l(kr)\right\}r^2 dr d\Omega \\
&= \frac{k^2}{\mathrm{i}\sqrt{l(l+1)}}\frac{1}{2\pi}\left\{\int Y^*_{l,m}\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]d\Omega\right\} \\
&\quad\times\left\{\int \frac{c}{\mathrm{i}\omega}\left(\frac{dI}{dr}\right)\left(\frac{d}{dr}rj_l(kr)\right) + \mathrm{i}krI(r)j_l(kr)dr\right\} \\
&= -\frac{k^2}{\sqrt{l(l+1)}}\left\{Y_{l,0}(\theta=0)+Y_{l,0}(\theta=\pi)\right\}\delta_{m,0}\left\{\int \frac{1}{k}\left(\frac{dI}{dr}\right)\left(\frac{d}{dr}rj_l(kr)\right) - krI(r)j_l(kr)dr\right\} \\
&= -\delta_{m,0}\delta_{l,even}\frac{2k}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}}\left\{\int\left(\frac{dI}{dr}\right)\left(\frac{d}{dr}rj_l(kr)\right) - k^2rI(r)j_l(kr)dr\right\} \\
&= -\delta_{m,0}\delta_{l,even}\frac{2k}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}}\left\{\int\left(\frac{d}{dr}\left(rj_l(kr)\frac{dI}{dr}\right)\right) - rj_l(kr)\left(\frac{d^2I}{dr^2}\right) - k^2rI(r)j_l(kr)dr\right\} \\
&= -\delta_{m,0}\delta_{l,even}\frac{2k}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}}\left\{\left[rj_l(kr)\frac{dI}{dr}\right]_0^L - \int_0^L rj_l(kr)\left(\frac{d^2I}{dr^2}+k^2I(r)\right)dr\right\}
\end{aligned}$$

where the antenna half-length $L = \lambda/2$. We also use the definition $\delta_{l,even} = 1$ for even l and $\delta_{l,even} = 0$ for odd l. For the given $I(r) = I\sin(kr)$ it is $\frac{d^2I}{dr^2} + k^2I(r) = 0$, and

$$\begin{aligned}
a_E(l,m) &= -\delta_{m,0}\delta_{l,even}\frac{2Ik}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}}\left[rj_l(kr)k\cos(kr)\right]_0^{\lambda/2} \\
&= \delta_{m,0}\delta_{l,even}\frac{2Ik}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}}\frac{\lambda}{2}j_l(\pi)k \\
&= \delta_{m,0}\delta_{l,even}Ikj_l(\pi)\sqrt{\frac{\pi(2l+1)}{l(l+1)}}
\end{aligned} \tag{4}$$

In the long-wavelength approximation, we use Eq. 9.169-9.172. We already note that the long-wavelength approximation cannot be expected to be tremendously accurate in the given case, because the antenna length is not small compared with the wavelength.

As before, all moments vanish except the $Q_{l,0}$ with even l. It is

$$\begin{aligned} Q_{l,m} &= \int Y^*_{l,m} r^l \rho d^3x \\ &= \frac{1}{2\pi \mathrm{i}\omega} \left\{ \int r^l \left(\frac{dI(r)}{dr} \right) dr \right\} \left\{ \int Y^*_{l,m} \left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1) \right] d\Omega \right\} \\ &= \frac{Ik}{2\pi \mathrm{i}\omega} \left\{ \int_0^L r^l \cos(kr) dr \right\} \left\{ 4\pi \sqrt{\frac{2l+1}{4\pi}} \delta_{m,0} \delta_{l,even} \right\} \\ &= \delta_{m,0}\delta_{l,even} \frac{2Ik}{\mathrm{i}\omega} \sqrt{\frac{2l+1}{4\pi}} \left\{ \int_0^L r^l \cos(kr) dr \right\} \end{aligned}$$

With $a_E(l,m) = \frac{ck^{l+2}}{\mathrm{i}(2l+1)!!} \sqrt{\frac{l+1}{l}} Q_{lm}$, the electric-multipole amplitudes are, in the long-wavelength limit,

$$a_E(l,m) = -\delta_{m,0}\delta_{l,even} \frac{2Ik^{l+2}}{(2l+1)!!} \sqrt{\frac{l+1}{l}} \sqrt{\frac{2l+1}{4\pi}} \left\{ \int_0^L r^l \cos(kr) dr \right\}$$

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b): The exact lowest non-vanishing amplitude is

$$a_E(2,0) = Ikj_2(\pi) \sqrt{\frac{5\pi}{6}}$$

Using only this moment, the radiated power is

$$P = \frac{Z_0}{2k^2} \left| a_E(2,0) \right|^2 = \frac{1}{2} \left[\frac{Z_0 5\pi}{6} |j_2(\pi)|^2 \right] |I|^2 = \frac{1}{2} \left[\frac{Z_0 5\pi}{6} \frac{9}{\pi^4} \right] |I|^2 = \frac{1}{2} \left[\frac{15 Z_0}{2\pi^3} \right] |I|^2$$

The numerical value for the radiation resistance (term in rectangular brackets) is

$$R_{rad} = \left[\frac{15 Z_0}{2\pi^3} \right] = 91.12\Omega$$

The radiation pattern follows from Eq. 9.151 and Table 9.1,

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \left| a_E(2,0) \right|^2 \left| \mathbf{X}_{2,0} \right|^2 = \frac{1}{2} R_{rad} |I|^2 \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

The lowest non-vanishing amplitude in the long-wavelength approximation is

$$\begin{aligned} a_E(2,0) &= -\frac{2Ik^4}{15}\sqrt{\frac{15}{8\pi}}\left\{\int_0^{\lambda/2} r^2\cos(kr)dr\right\} \\ &= -\frac{Ik^4}{\sqrt{30\pi}}\left\{\frac{1}{k^3}\int_0^{\pi} x^2\cos(x)dx\right\} \\ &= \frac{2\pi Ik}{\sqrt{30\pi}} = Ik\sqrt{\frac{2\pi}{15}} \end{aligned}$$

Using only this moment, the radiated power is

$$P = \frac{Z_0}{2k^2}\left|a_E(2,0)\right|^2 = \frac{1}{2}\left[\frac{2Z_0\pi}{15}\right]|I|^2$$

The numerical value for the radiation resistance (term in rectangular brackets) is

$$R_{rad} = \left[\frac{2Z_0\pi}{15}\right] = 157.8\Omega$$

The radiation pattern follows from Eq. 9.151 and Table 9.1,

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|a_E(2,0)\right|^2\left|\mathbf{X}_{2,0}\right|^2 = \frac{1}{2}R_{rad}\left|I\right|^2\frac{15}{8\pi}\sin^2\theta\cos^2\theta$$

Discussion of 9.16 and 9.17.

The radiation resistances found are

$$\begin{aligned} R_1 &= R_{rad,exact} = 93.36\Omega \\ R_2 &= R_{rad,a20,exact} = 91.12\Omega \\ R_3 &= R_{rad,a20,approx} = 157.8\Omega \end{aligned}$$

It is $R_2 < R_1$. This is to be expected, because the total radiated powers of multipoles add incoherently. Thus, by neglecting higher exact multipoles we will slightly underestimate the radiated power, which is equivalent to underestimating the radiation resistance. In the given case, from R_1 and R_2 it follows that by neglecting higher-order exact multipoles we underestimate the radiated power by 2.4% (this is not so bad).

It is $R_3 >> R_1$. This is not unexpected, because by making the small-source approximation we essentially neglect destructive interference of radiation originating from different portions of the source. The destructive interference reduces the radiation efficiency of sources that are not much smaller than the wavelength. In the case of large sources, neglecting this destructive interference can lead to gross overestimates of the radiated power, as in our case.

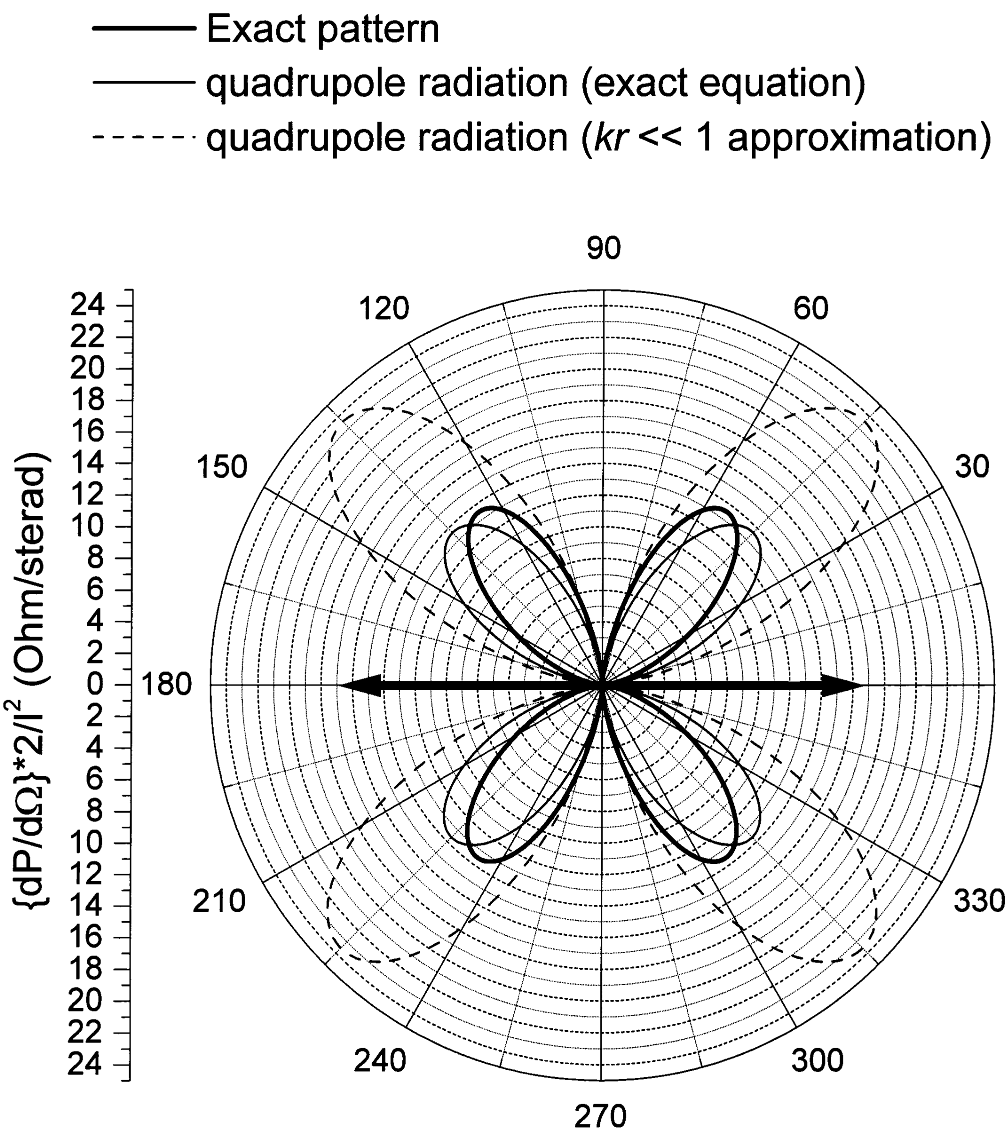

Figure 1: Radiation patterns for the indicated cases. **Bold and solid**: exact calculation. Solid: Lowest exact multipole term (this term is due to $a_E(2,0)$). Dashed: Same multipole term in the long-wavelength approximation.

4. Problem 9.22 **10 Points**

a): Electric-multipole modes = TM modes. We use Eq. 9.122 as starting point. Since the fields must be regular at $r = 0$, we choose $j_l(kr)$ for all radial functions. The generic form of the field of a TM_{lm*}-mode, with amplitude $a_E(l,m)$ set to 1, then is

$$\begin{aligned}\mathbf{H} &= j_l(kr)\mathbf{X}_{lm} \\ \mathbf{E} &= \frac{\mathrm{i}Z_0}{k}\nabla\times\mathbf{H} = \frac{\mathrm{i}Z_0}{k}\nabla\times j_l(kr)\mathbf{X}_{lm}\end{aligned}$$

The boundary conditions are that at $r = a$ the electric field must only have a radial component and the magnetic field must be transverse. The second condition is automatically satisfied because of the transversality of the $\mathbf{X}_{lm}$. To match the first, we use

$$\mathbf{X}_{l,m} = \frac{1}{\sqrt{l(l+1)}}\hat{\mathbf{L}}Y_{l,m} = \frac{1}{\sqrt{l(l+1)}}\frac{1}{\mathrm{i}}\left(\hat{\phi}\partial_\theta - \hat{\theta}\frac{1}{\sin\theta}\partial_\phi\right)Y_{l,m}$$

to first write out the $\mathbf{H}-$field components,

$$\begin{aligned}H_r &= 0 \\ H_\theta &= -\frac{m}{\sqrt{l(l+1)}}j_l(kr)\frac{1}{\sin\theta}Y_{l,m} \\ H_\phi &= \frac{1}{\mathrm{i}\sqrt{l(l+1)}}j_l(kr)\partial_\theta Y_{l,m}\end{aligned}$$

Then, the electric-field components follow from $\mathbf{E} = \frac{\mathrm{i}Z_0}{k}\nabla\times\left(\hat{\theta}H_\theta + \hat{\phi}H_\phi\right)$,

$$\begin{aligned}E_r &= \frac{\mathrm{i}Z_0}{k}\frac{1}{r\sin\theta}\left[\partial_\theta\sin\theta H_\phi - \partial_\phi H_\theta\right] \\ &= \frac{\mathrm{i}Z_0}{k}\frac{1}{\mathrm{i}\sqrt{l(l+1)}}\frac{j_l(kr)}{r\sin\theta}\left[\partial_\theta\sin\theta\partial_\theta + \mathrm{i}m\partial_\phi\frac{1}{\sin\theta}\right]Y_{l,m} \\ &= \frac{Z_0}{k}\frac{1}{\sqrt{l(l+1)}}\frac{j_l(kr)}{r}\left[\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta - \frac{m^2}{\sin^2\theta}\right]Y_{l,m} \\ &= -Z_0\sqrt{l(l+1)}\frac{j_l(kr)}{kr}Y_{l,m} \\ E_\theta &= -\frac{\mathrm{i}Z_0}{k}\frac{1}{r}\partial_r rH_\phi \\ &= -\frac{Z_0}{\sqrt{l(l+1)}}\frac{1}{kr}\left[\frac{d}{dr}rj_l(kr)\right]\left[\partial_\theta Y_{l,m}\right] \\ E_\phi &= \frac{\mathrm{i}Z_0}{k}\frac{1}{r}\partial_r rH_\theta \\ &= -\frac{\mathrm{i}Z_0 m}{\sqrt{l(l+1)}}\frac{1}{kr}\left[\frac{d}{dr}rj_l(kr)\right]\left[\frac{1}{\sin\theta}Y_{l,m}\right]\end{aligned}$$

The cavity frequencies follow from the requirement $E_\theta = E_\phi = 0$ at $r = a$. The frequencies can be obtained from the transcendental equation

$$\left[\frac{d}{dr} r j_l(kr)\right]_{r=a} = \left[\frac{d}{dx} x j_l(x)\right]_{x=ka} = 0$$

Denoting the n-th root of $\frac{d}{dx}(x j_l(x))$ with x'_{ln}, it is $ka = \omega_{lmn}\frac{a}{c} = x'_{ln}$. The resonance frequencies thus are

$$\omega_{lmn} = \frac{x'_{ln} c}{a}$$

Note that $l = 0$ does not exist, and that the frequencies are degenerate in m, i.e. for given l and n there are $2l+1$ TM-modes with the same frequency.

Magnetic-multipole modes = TE modes. We use Eq. 9.122 as starting point. The generic form of the field of a TE_{lm*}-mode, with amplitude $a_M(l,m)$ set to 1, then is

$$\begin{aligned}
\mathbf{H} &= \frac{-i}{k}\nabla \times j_l(kr)\mathbf{X}_{lm} \\
\mathbf{E} &= Z_0 j_l(kr)\mathbf{X}_{lm}
\end{aligned}$$

Comparison with the analogous equation for TM-modes shows that the fields of the TE-modes are obtained by replacing the former $\mathbf{H}$ with $\mathbf{E}/Z_0$ and the former $\mathbf{E}$ with $-Z_0\mathbf{H}$. Thus, for TE-modes it is

$$\begin{aligned}
E_r &= 0 \\
E_\theta &= -\frac{Z_0 m}{\sqrt{l(l+1)}} j_l(kr)\frac{1}{\sin\theta}Y_{l,m} \\
E_\phi &= \frac{Z_0}{\mathrm{i}\sqrt{l(l+1)}} j_l(kr)\partial_\theta Y_{l,m}
\end{aligned}$$

and

$$\begin{aligned}
H_r &= \sqrt{l(l+1)}\frac{j_l(kr)}{kr}Y_{l,m} \\
H_\theta &= \frac{1}{\sqrt{l(l+1)}}\frac{1}{kr}\left[\frac{d}{dr} r j_l(kr)\right]\left[\partial_\theta Y_{l,m}\right] \\
H_\phi &= \frac{\mathrm{i}m}{\sqrt{l(l+1)}}\frac{1}{kr}\left[\frac{d}{dr} r j_l(kr)\right]\left[\frac{1}{\sin\theta}Y_{l,m}\right]
\end{aligned}$$

The conditions of vanishing transverse electric and vanishing normal magnetic field at $r = a$ are satisfied via the transcendental equation

$$j_l(ka) = 0$$

Denoting the n-th root of $j_l(x)$ with x_{ln}, it is $ka = \omega_{lmn}\frac{a}{c} = x_{ln}$. The resonance frequencies thus are

$$\omega_{lmn} = \frac{x_{ln}c}{a}$$

Again, $l = 0$-modes don't exist, and for given l and n there are $2l+1$ TE-modes with the same frequency.

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b): (required for TE-modes only). From $\omega_{lmn} = \frac{2\pi c}{\lambda_{lmn}} = \frac{x_{ln}c}{a}$ we see that

$$\frac{\lambda_{lmn}}{a} = \frac{2\pi}{x_{ln}}$$

Numerically we find the lowest roots of spherical Bessel functions to be $x_{11} = 4.493$, $x_{21} = 5.763$, $x_{31} = 6.988$ and $x_{12} = 7.725$. The lowest four TE-modes therefore are:

l	n	$\frac{\lambda_{lmn}}{a}$
1	1	1.398
2	1	1.090
3	1	0.899
1	2	0.813

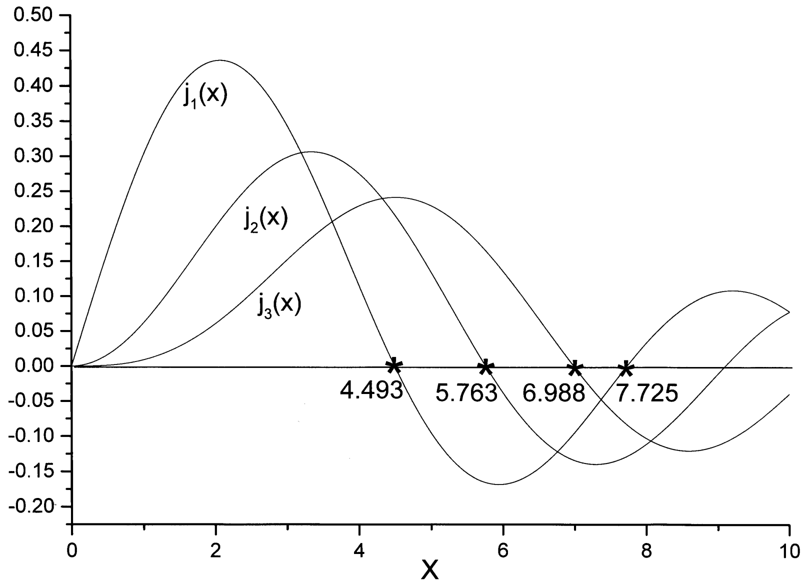

Figure 2: Lowest spherical Bessel functions and their roots.

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c):

The lowest TE-modes are the degenerate $TE_{l=1,m=-1,n=1}$, $TE_{l=1,m=0,n=1}$ and $TE_{l=1,m=1,n=1}$-modes. To obtain their fields, use the above general equations for the TE-fields to obtain:

$l = 1, m = 1$:

$$
\begin{aligned}
E_r &= 0 \\
E_\theta &= \frac{Z_0}{\sqrt{2}}\sqrt{\frac{3}{8\pi}} j_1(\frac{x_{11}}{a}r)\exp(\mathrm{i}\phi) \\
E_\phi &= -\frac{Z_0}{\mathrm{i}\sqrt{2}}\sqrt{\frac{3}{8\pi}} j_1(\frac{x_{11}}{a}r)\cos\theta\exp(\mathrm{i}\phi) \\
H_r &= -\sqrt{2}\sqrt{\frac{3}{8\pi}}\frac{j_1(\frac{x_{11}}{a}r)}{\frac{x_{11}}{a}r}\sin\theta\exp(\mathrm{i}\phi) \\
H_\theta &= -\frac{1}{\sqrt{2}}\sqrt{\frac{3}{8\pi}}\frac{a}{x_{11}r}\left[\frac{d}{dr}r j_1(\frac{x_{11}}{a}r)\right]\cos\theta\exp(\mathrm{i}\phi) \\
H_\phi &= -\frac{\mathrm{i}}{\sqrt{2}}\sqrt{\frac{3}{8\pi}}\frac{a}{x_{11}r}\left[\frac{d}{dr}r j_1(\frac{x_{11}}{a}r)\right]\exp(\mathrm{i}\phi)
\end{aligned}
$$

$l = 1, m = 0$:

$$
\begin{aligned}
E_r &= 0 \\
E_\theta &= 0 \\
E_\phi &= -\frac{Z_0}{\mathrm{i}\sqrt{2}}\sqrt{\frac{3}{4\pi}} j_1(\frac{x_{11}}{a}r)\sin\theta \\
H_r &= \sqrt{2}\sqrt{\frac{3}{4\pi}}\frac{j_1(\frac{x_{11}}{a}r)}{\frac{x_{11}}{a}r}\cos\theta \\
H_\theta &= -\frac{1}{\sqrt{2}}\sqrt{\frac{3}{4\pi}}\frac{a}{x_{11}r}\left[\frac{d}{dr}r j_1(\frac{x_{11}}{a}r)\right]\sin\theta \\
H_\phi &= 0
\end{aligned}
$$

$l = 1, m = -1$:

$$
\begin{aligned}
E_r &= 0 \\
E_\theta &= \frac{Z_0}{\sqrt{2}}\sqrt{\frac{3}{8\pi}} j_1(\frac{x_{11}}{a}r)\exp(-\mathrm{i}\phi) \\
E_\phi &= \frac{Z_0}{\mathrm{i}\sqrt{2}}\sqrt{\frac{3}{8\pi}} j_1(\frac{x_{11}}{a}r)\cos\theta\exp(-\mathrm{i}\phi) \\
H_r &= \sqrt{2}\sqrt{\frac{3}{8\pi}}\frac{j_1(\frac{x_{11}}{a}r)}{\frac{x_{11}}{a}r}\sin\theta\exp(-\mathrm{i}\phi) \\
H_\theta &= \frac{1}{\sqrt{2}}\sqrt{\frac{3}{8\pi}}\frac{a}{x_{11}r}\left[\frac{d}{dr}r j_1(\frac{x_{11}}{a}r)\right]\cos\theta\exp(-\mathrm{i}\phi) \\
H_\phi &= -\frac{\mathrm{i}}{\sqrt{2}}\sqrt{\frac{3}{8\pi}}\frac{a}{x_{11}r}\left[\frac{d}{dr}r j_1(\frac{x_{11}}{a}r)\right]\exp(-\mathrm{i}\phi)
\end{aligned}
$$