Phys. 506 | Electricity and Magnetism | Winter 2004

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Problem Set 3

Total 40 Points

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1. Problem 9.1 **10 Points**

This problem deals with the implications of the fact that **negative frequencies are not allowed in the formalism of Chapter 9 of Jackson** (and other parts of the textbook that deal with harmonic time-dependence).

a): For a rigid charge distribution with a body frame with coordinates $(r, \theta, \tilde{\phi})$ rotating with an angular-velocity vector $\hat{\mathbf{z}}\omega$ in the laboratory frame with coordinates (r, θ, ϕ) it is

$$\rho(\mathbf{x}, t) = \rho(r, \theta, \tilde{\phi}) = \rho(r, \theta, \phi - \omega t)$$

The usual time-dependent multipole moments in the laboratory frame are

$$q_{lm}(t) = \int r^l Y^*_{lm}(\theta, \phi)\rho(r, \theta, \phi - \omega t) d^3x = \left\{ \int r^l Y^*_{lm}(\theta, \tilde{\phi})\rho(r, \theta, \tilde{\phi}) d^3x \right\} \exp(-\mathrm{i}m\omega t) = \tilde{q}_{lm} \exp(-\mathrm{i}m\omega t)$$

where $\tilde{q}_{lm}$ is a fixed multipole moment in the body frame ($\tilde{q}_{lm}$ can be thought of as the usual time-dependent multipole moment evaluated at $t = 0$, i.e. $\tilde{q}_{lm} = q_{lm}(t = 0)$). Since ρ is real and $Y_{l,-m} = (-1)^m Y^*_{l,m}$, it is $\tilde{q}_{l,-m} = (-1)^m \tilde{q}^*_{l,m}$.

The positive and negative values of m correspond to positive and negative frequencies. **Negative frequencies are not allowed in the formalism of Chapter 9 of Jackson** (and other parts of the textbook that deal with harmonic time-dependence). We must therefore re-write equations such that only positive frequencies occur, and deduce suitable multipole moments.

Consider, for instance, the (real-valued) electrostatic potential in the near-field limit at an observation point (r_o, θ_o, ϕ_o):

$$\begin{aligned}
\Phi(\mathbf{x}_o, t) &= \frac{1}{4\pi\epsilon_0} \sum_{l=0,m=-l}^{l=\infty,m=l} \left(\frac{4\pi}{2l+1}\right) \frac{1}{r_o^{l+1}} Y_{lm}(\theta_o, \phi_o) q_{lm}(t) \\
&= \frac{1}{4\pi\epsilon_0} \sum_{l,m\geq 0} \left(\frac{4\pi}{2l+1}\right) \frac{1}{r_o^{l+1}} \left[Y_{lm}(\theta_o, \phi_o) q_{lm}(t) + Y_{l,-m}(\theta_o, \phi_o) q_{l,-m}(t)\right] \\
&= \frac{1}{4\pi\epsilon_0} \sum_{l,m\geq 0} \left(\frac{4\pi}{2l+1}\right) \frac{1}{r_o^{l+1}} \left[Y_{lm}(\theta_o, \phi_o)\tilde{q}_{lm} \exp(-\mathrm{i}m\omega t) + Y_{l,-m}(\theta_o, \phi_o)\tilde{q}_{l,-m} \exp(+\mathrm{i}m\omega t)\right] \\
&= \frac{1}{4\pi\epsilon_0} \sum_{l,m\geq 0} \left(\frac{4\pi}{2l+1}\right) \frac{1}{r_o^{l+1}} \left[Y_{lm}(\theta_o, \phi_o)\tilde{q}_{lm} \exp(-\mathrm{i}m\omega t) + Y^*_{l,m}(\theta_o, \phi_o)\tilde{q}^*_{l,m} \exp(+\mathrm{i}m\omega t)\right] \\
&= \frac{1}{4\pi\epsilon_0} \sum_{l,m\geq 0} \left(\frac{4\pi}{2l+1}\right) \frac{1}{r_o^{l+1}} \mathrm{Re}\left\{Y_{lm}(\theta_o, \phi_o)\, 2\tilde{q}_{lm}\, \exp(-\mathrm{i}m\omega t)\right\}
\end{aligned}$$

We imply that for $m = 0$ the factor 2 in front of $\tilde{q}_{lm}$ is dropped. In the complex quantity of the last line only positive-frequency components $m\omega$ with $m \geq 0$ occur, as required. By inspection of the result we see that the multipole moments Q_{lm} suited for Chapter 9 are

$$Q_{lm} = \begin{cases} 2\tilde{q}_{lm} & , \quad m > 0 \\ \tilde{q}_{l,0} & , \quad m = 0 \\ 0 & , \quad m < 0 \end{cases}$$

with corresponding frequencies $m\omega$. Since static distributions don't radiate, the case $m = 0$ is quite irrelevant.

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b): At fixed location $\mathbf{x}$, we perform a discrete temporal Fourier transform,

$$\rho(\mathbf{x}, t) = \sum_{n=-\infty}^{\infty} c_n(\mathbf{x}) f_n(t)$$

with basis functions $f_n(t) = \frac{1}{\sqrt{T}} \exp(-in\omega t)$, where $T = \frac{2\pi}{\omega}$. Note the orthonormality $\int_{t=0}^{T} f_n^*(t) f_m(t) dt = \delta_{nm}$ and the closure $\sum_{n=-\infty}^{\infty} f_n^*(t') f_n(t) = \delta(t - t')$.

Then,

$$c_n(\mathbf{x}) = \int_{t=0}^{T} f_n^*(t) \rho(\mathbf{x}, t) dt = \frac{1}{\sqrt{T}} \int_{t=0}^{T} \exp(in\omega t) \rho(\mathbf{x}, t) dt$$

Noting $c_{-n}(\mathbf{x}) = c_n^*(\mathbf{x})$,

$$\begin{aligned} \rho(\mathbf{x}, t) &= c_0 f_0 + \sum_{n=1}^{\infty} \left[c_n(\mathbf{x}) f_n(t) + c_{-n}(\mathbf{x}) f_{-n}(t) \right] \\ &= c_0 f_0 + \sum_{n=1}^{\infty} \left[c_n(\mathbf{x}) f_n(t) + c_n^*(\mathbf{x}) f_n^*(t) \right] \\ &= c_0 f_0 + \sum_{n=1}^{\infty} \mathrm{Re} \left[2 c_n(\mathbf{x}) f_n(t) \right] \\ &= \frac{1}{T} \int_0^T \rho(\mathbf{x}, t) dt + \sum_{n=1}^{\infty} \mathrm{Re} \left[\left\{ \frac{2}{T} \int_0^T \rho(\mathbf{x}, t) \exp(in\omega t) dt \right\} \exp(-in\omega t) \right] \end{aligned} \tag{1}$$

Note that all frequencies are positive. By inspection we see that the charge densities to be used in Eq. 9.1 ff are

$$\rho(\mathbf{x}) = \begin{cases} \frac{2}{T} \int_0^T \rho(\mathbf{x}, t) \exp(in\omega t) dt & , \quad n > 0 \\ \frac{1}{T} \int_0^T \rho(\mathbf{x}, t) dt & , \quad n = 0 \end{cases} ,$$

where the frequencies $n\omega$ are all positive, as required. Since static distributions don't radiate, the case $n = 0$ is quite irrelevant. The multipole moments for frequency component $n\omega$ with $n > 0$ are

$$\begin{aligned}
Q_{lm} &= \frac{2}{T}\int d^3x \int_0^T dt\rho(r,\theta,\phi-\omega t)r^l Y_{lm}^*(\theta,\phi)\exp(\mathrm{i}n\omega t) \\
&= \frac{2}{T}\int d^3x \int_0^T dt\rho(r,\theta,\phi)r^l Y_{lm}^*(\theta,\phi+\omega t)\exp(\mathrm{i}n\omega t) \\
&= \frac{2}{T}\int d^3x \int_0^T dt\rho(r,\theta,\phi)r^l Y_{lm}^*(\theta,\phi)\exp(-\mathrm{i}m\omega t)\exp(\mathrm{i}n\omega t) \\
&= \delta_{mn}\left\{2\int \rho(r,\theta,\phi)r^l Y_{lm}^*(\theta,\phi)d^3x\right\}
\end{aligned}$$

For $n = 0$, drop the factor 2. This result is equivalent to that of part a).

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c): We have already generally shown that both methods a) and b) lead to identical multipole moments that are simply related to the multipole moments in the body frame. For a charge q located at $(R, \theta = \pi/2, \phi = \phi_0)$ rotating about the z-axis with frequency ω_0, the body-frame charge density in spherical and cylindrical coordinates is

$$\rho(\mathbf{x}) = q\frac{\delta(r-R)}{R^2}\delta(\cos\theta)\delta(\phi-\phi_0) = q\frac{\delta(r-R)}{R}\delta(z)\delta(\phi-\phi_0)$$

and it is

Monopole moment:

Spherical: $Q_{00} = \frac{q}{\sqrt{4\pi}}$. Cartesian: $Q = q$. Since $m = 0$, the frequency of the monopole moment is zero. This is generally the case, and explains why monopole moments do not occur in radiation problems.

Dipole moment:

Spherical:

$$\begin{aligned}
Q_{11} &= 2\int rY_{11}^*\rho(\mathbf{x})d^3x = -2qR\sqrt{\frac{3}{8\pi}}\exp(-\mathrm{i}\phi_0) \\
Q_{10} &= 0 \\
Q_{1-1} &= 0
\end{aligned} \tag{2}$$

The frequency of Q_{11} is $m\omega_0 = \omega_0$.

Cartesian: Use Eq. 4.5 in Jackson to find

$$\begin{aligned}
p_x &= \frac{Q_{11} - Q_{1,-1}}{-2\sqrt{\frac{3}{8\pi}}} = qR\exp(-\mathrm{i}\phi_0) \\
p_y &= \frac{Q_{11} + Q_{1,-1}}{2\mathrm{i}\sqrt{\frac{3}{8\pi}}} = \mathrm{i}qR\exp(-\mathrm{i}\phi_0)
\end{aligned}$$

$$p_z = \frac{Q_{10}}{\sqrt{\frac{3}{4\pi}}} = 0$$

Thus, $\mathbf{p} = qR\exp(-\mathrm{i}\phi_0)(\hat{\mathbf{x}}+\mathrm{i}\hat{\mathbf{y}})$. The frequency is ω_0, and the dipole moment with explicitly displayed time dependence is

$$\mathbf{p} = qR\exp(-\mathrm{i}\phi_0)(\hat{\mathbf{x}}+\mathrm{i}\hat{\mathbf{y}})\exp(-\mathrm{i}\omega_0 t)$$

Note. One may choose the time origin such that the global phase term $\exp(-\mathrm{i}\phi_0)$ becomes 1.

Note. It is still instructive to obtain the cartesian moments by first calculating the harmonic charge densities and then their moments. With

$$\rho(\mathbf{x},t) = q\frac{\delta(r-R)}{R}\delta(z)\delta(\phi-\phi_0-\omega_0 t)$$

Frequency zero:

$$\rho_0(\mathbf{x}) = \frac{1}{T}\int_0^T q\frac{\delta(r-R)}{R}\delta(z)\delta(\phi-\phi_0-\omega_0 t)dt = \frac{q}{T\omega_0}\frac{\delta(r-R)}{R}\delta(z) = \frac{q}{2\pi}\frac{\delta(r-R)}{R}\delta(z)$$

which has a zero-frequency cartesian monopole moment, $Q = q$.

n-th harmonic frequency:

$$\rho(\mathbf{x}) = \frac{2}{T}\int_0^T q\frac{\delta(r-R)}{R}\delta(z)\delta(\phi-\phi_0-\omega_0 t)\exp(\mathrm{i}n\omega_0 t)dt = \frac{q}{\pi}\frac{\delta(r-R)}{R}\delta(z)\exp(\mathrm{i}n(\phi-\phi_0))$$

from which we can see, for instance, that the electric-dipole components are radiating at the fundamental (as is generally the case),

$$\begin{aligned} p_x &= \frac{q}{\pi}\exp(-\mathrm{i}n\phi_0)\int r\cos\phi\frac{\delta(r-R)}{R}\delta(z)\exp(\mathrm{i}n\phi)rdrdzd\phi = qR\exp(-\mathrm{i}\phi_0)\delta_{n,1} \\ p_y &= \frac{q}{\pi}\exp(-\mathrm{i}n\phi_0)\int r\sin\phi\frac{\delta(r-R)}{R}\delta(z)\exp(\mathrm{i}n\phi)rdrdzd\phi = \mathrm{i}qR\exp(-\mathrm{i}\phi_0)\delta_{n,1} \\ p_z &= 0 \end{aligned} \tag{3}$$

Higher moments. As explained above, only moments with $m > 0$ exist. For $m > 0$, the spherical moments have oscillation frequencies $m\omega_0$ and magnitudes

$$Q_{lm} = 2q\int r^l\frac{\delta(r-R)}{R^2}\delta(\cos\theta)\delta(\phi-\phi_0)Y^*_{lm}(\theta,\phi)r^2drd\cos\theta d\phi$$

$$
\begin{aligned}
&= 2qR^l Y_{lm}^*(\pi/2, \phi_0) \\
&= 2qR^l \exp(-\mathrm{i}m\phi_0)\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)
\end{aligned}
$$

Thereby, for even $l-m$ it is $P_l^m(0) = (-1)^{\frac{l+m}{2}} \frac{(l+m)!}{2^l\left(\frac{l-m}{2}\right)!\left(\frac{l+m}{2}\right)!}$, while for odd $l-m$ it is $P_l^m(0) = 0$. Thus, radiation occurs at all harmonic frequencies $m\omega_0$. The lowest-order non-zero multipole at frequency $m\omega_0$ is $Q_{l=m,m}$. Non-zero higher-order multipoles at frequency $m\omega_0$ are $Q_{l=m+2,m}$, $Q_{l=m+4,m}$ etc.

2. Problem 9.2 **10 Points**

According to Problem 9.1, it is

$$Q_{lm} = \begin{cases} 2\tilde{q}_{lm} & , \quad m > 0 \\ \tilde{q}_{l,0} & , \quad m = 0 \\ 0 & , \quad m < 0 \end{cases}$$

where the frequencies are $m\omega$ and

$$\tilde{q}_{lm} = \int r^l Y^*_{lm}(\theta,\phi)\rho(r,\theta,\phi(t=0))d^3x$$

Here,

$$\rho(\mathbf{x}, t=0) = \frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\left[\delta(\phi) + \delta(\phi+\pi) - \delta(\phi+\pi/2) - \delta(\phi+3\pi/2)\right]$$

and

$$\begin{aligned} Q_{lm} &= 2\int r^{l+2}\frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\left[\delta(\phi)+\delta(\phi+\pi)-\delta(\phi+\pi/2)-\delta(\phi+3\pi/2)\right]Y^*_{lm}(\theta,\phi)dr d\cos\theta d\phi \\ &= 2qR^l\left[Y^*_{lm}(0,0)+Y^*_{lm}(0,\pi)-Y^*_{lm}(0,\pi/2)-Y^*_{lm}(0,3\pi/2)\right] \\ &= 2qR^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P^m_l(0)\left[1+\exp(im\pi)-\exp(im\pi/2)-\exp(im3\pi/2)\right] \\ &= 2qR^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\left[(-1)^{\frac{l+m}{2}}\frac{(l+m)!}{2^l\left(\frac{l-m}{2}\right)!\left(\frac{l+m}{2}\right)!}\right]\times 4 \end{aligned} \tag{4}$$

where for the result to be different from zero it must be both $l-m$ even and $m = 2+4p$ with integer $p = 1, 2, 3...$ Thus, the lowest non-zero moments are Q_{22}, Q_{42}, Q_{62}, ... and Q_{66}, Q_{86}, ... Also, there is no magnetic dipole moment, because the net circular current is zero. Thus, in the long wavelength limit the leading radiation term comes from Q_{22}, which radiates at a frequency 2ω and has a value, following the above formula, of

$$Q_{22} = qR^2\sqrt{\frac{30}{\pi}}$$

Since the sidelength $a = R\sqrt{2}$, it also is

$$Q_{22} = qa^2\sqrt{\frac{15}{2\pi}}$$

There are no other non-zero spherical quadrupole moments.

The quadrupole radiation field in the far field is given by Eqs. 9.169 and 9.149 (applied in that order),

$$\begin{aligned} \mathbf{H} &= \exp(\mathrm{i}kr - 2\mathrm{i}\omega t)\frac{-\mathrm{i}^3 ck^4}{\mathrm{i}kr3 \times 5}\sqrt{\frac{3}{2}}qa^2\sqrt{\frac{15}{2\pi}}\mathbf{X}_{22} = \exp(\mathrm{i}kr - 2\mathrm{i}\omega t)\frac{qa^2ck^3}{r}\sqrt{\frac{1}{20\pi}}\mathbf{X}_{22} \\ \mathbf{E} &= Z_0\mathbf{H} \times \mathbf{n} \end{aligned}$$

The radiation pattern is, following Eqn. 9.151,

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|a_E(2,2)\right|^2\left|\mathbf{X}_{22}\right|^2$$

where $a_E(2,2) = \frac{ck^4}{\mathrm{i}\,15}\sqrt{\frac{3}{2}}qa^2\sqrt{\frac{15}{2\pi}}$. Using further that $k = 2\omega/c$ and the table 9.1 one finds

$$\frac{dP}{d\Omega} = \frac{Z_0\omega^6q^2a^4}{2\pi^2}(1 - \cos^4\theta)$$

This result can also be obtained directly from the fields, because

$$\frac{dP}{d\Omega} = \frac{r^2}{2}\mathrm{Re}\left[\hat{\mathbf{n}} \cdot (\mathbf{E} \times \mathbf{H}^*)\right]$$

This may be integrated, or one may use Eq. 9.154, to find

$$P = \int_{4\pi}\frac{dP}{d\Omega}d\Omega = \frac{Z_0}{2k^2}\left|a_E(2,2)\right|^2 = \frac{8Z_0\omega^6q^2a^4}{5\pi c^4}$$

Note. From Q_{22} and Eqns. 4.6 one may also derive the cartesian quadrupole tensor

$$Q = 3qa^2\begin{pmatrix} 1 & \mathrm{i} & 0 \\ \mathrm{i} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ,$$

and then use Eqs. 9.45 and 9.49 to arrive at the same result. This method is less elegant, however.

3. Problem 9.3 **10 Points**

We show that there is a non-zero electric-dipole moment. From that it follows that the leading radiation term in the long-wavelength approximation is the electric-dipole radiation.

We also show that the magnetic-dipole moment is zero. This step is not really necessary, because electric-dipole radiation dominates magnetic-dipole radiation of the same order by a factor of order $(kd)^{-2}$.

From Eq. 3.38 in Jackson we see that in the near field the scalar potential produced by the hemispheres is

$$\Phi(r,\theta,t) = \frac{V(t)}{\sqrt{\pi}} \sum_{j=1}^{\infty} (-1)^{j+1} \frac{(2j-1/2)\Gamma(j-1/2)}{j!} \left(\frac{a}{r}\right)^{2j} P_{2j-1}(\cos\theta)$$

The electric-dipole term of that corresponds to $j = 1$, i.e.

$$\Phi_{E1}(r,\theta,t) = \frac{V}{\sqrt{\pi}}(3/2)\Gamma(1/2)\left(\frac{R}{r}\right)^2 (\cos\theta)\cos(\omega t) = \mathrm{Re}\left\{\frac{3VR^2}{2r^2}\cos\theta\exp(-\mathrm{i}\omega t)\right\} = \mathrm{Re}\left\{\frac{p}{4\pi\epsilon_0 r^2}\cos\theta\exp(-\mathrm{i}\omega t)\right\}$$

We thus see by comparison that the complex dipole moment vector at frequency ω is

$$\mathbf{p} = 6VR^2\pi\epsilon_0\hat{\mathbf{z}}$$

In the long-wavelength = small-source approximation, a non-vanishing electric-dipole moment produces the dominant radiation. Eqns. 9.19, 9.23 and 9.24 then yield, in the radiation zone,

$$\begin{aligned}
\mathbf{H} &= -\frac{3V}{2Z_0}(kR)^2\frac{\exp(\mathrm{i}kr)}{r}\sin\theta\hat{\phi} \\
\mathbf{E} &= -\frac{3V}{2}(kR)^2\frac{\exp(\mathrm{i}kr)}{r}\sin\theta\hat{\theta} \\
\frac{dP}{d\Omega} &= \frac{9V^2}{8Z_0}(kR)^4\sin^2\theta \\
P &= 3\pi(kR)^4\frac{V^2}{Z_0}
\end{aligned}$$

where $k = \frac{\omega}{c}$.

To show that the magnetic-dipole moment is zero, we note that for symmetry reasons the surface current has even spatial parity, i.e. $\mathbf{K}(\theta,\phi) = \mathbf{K}(\pi-\theta,\phi+\pi)$. Thus,

$$\begin{aligned}
\mathbf{m} &= \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\cos\theta=-1}^{1} \mathbf{x}\times\mathbf{K}(\theta,\phi)R^2 d\phi d\cos\theta \\
&= \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\cos\theta=0}^{1} \left[\mathbf{x}\times\mathbf{K}(\theta,\phi) + (-\mathbf{x})\times\mathbf{K}(\pi-\theta,\phi+\pi)\right] R^2 d\phi d\cos\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\cos\theta=0}^{1}\left[\mathbf{x}\times\mathbf{K}(\theta,\phi)-\mathbf{x}\times\mathbf{K}(\theta,\phi)\right]R^2 d\phi d\cos\theta \\
&= 0
\end{aligned}
$$

4. Problem 9.5 **10 Points**

a): For $\mathbf{A}(\mathbf{x})$, copy Eqns. 9.13-9.16 of Jackson. For $\Phi(\mathbf{x})$, write the analogue of Eq. 9.30 for $\Phi(\mathbf{x})$,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{\exp(\mathrm{i}kr)}{r}\left(\frac{1}{r} - \mathrm{i}k\right)\int \rho(\mathbf{x}')\hat{\mathbf{n}}\cdot\mathbf{x}'d^3x' = \frac{1}{4\pi\epsilon_0}\frac{\exp(\mathrm{i}kr)}{r}\left(\frac{1}{r} - \mathrm{i}k\right)\hat{\mathbf{n}}\cdot\mathbf{p}$$

b):

$$\begin{aligned}
\mathbf{B} &= \nabla\times\mathbf{A} = -\frac{\mathrm{i}\mu_0\omega}{4\pi}\nabla\times\left(\mathbf{p}\frac{\exp(\mathrm{i}kr)}{r}\right) \\
&= -\frac{\mathrm{i}\mu_0\omega}{4\pi}\left[\left(\nabla\frac{\exp(\mathrm{i}kr)}{r}\right)\times\mathbf{p} + \frac{\exp(\mathrm{i}kr)}{r}(\nabla\times\mathbf{p})\right] = \\
&= -\frac{\mathrm{i}\mu_0\omega}{4\pi}\left(\hat{\mathbf{n}}\frac{\exp(\mathrm{i}kr)}{r}\left[\mathrm{i}k - \frac{1}{r^2}\right]\right)\times\mathbf{p} \\
&= \frac{ck^2\mu_0}{4\pi}\frac{\exp(\mathrm{i}kr)}{r}\left[1 - \frac{1}{\mathrm{i}kr}\right](\hat{\mathbf{n}}\times\mathbf{p}) \qquad (5)
\end{aligned}$$

One way to obtain $\mathbf{E}$ is

$$\begin{aligned}
\mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{A} - \nabla\Phi \\
&= \frac{1}{4\pi\epsilon_0}\frac{\exp(\mathrm{i}kr)}{r}\left\{k^2\mathbf{p} + \hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot\mathbf{p}\left(\left(\frac{1}{r} - \mathrm{i}k\right)^2 + \frac{1}{r^2}\right)\right] - \hat{\theta}\left[\hat{\theta}\cdot\mathbf{p}\left(\frac{1}{r} - \mathrm{i}k\right)\left(\frac{1}{r}\right)\right] - \hat{\phi}\left[\hat{\phi}\cdot\mathbf{p}\left(\frac{1}{r} - \mathrm{i}k\right)\left(\frac{1}{r}\right)\right]\right\}
\end{aligned}$$

where we have used $\partial_\theta(\hat{\mathbf{r}}\cdot\mathbf{p}) = \mathbf{p}\cdot\hat{\theta}$ and $\partial_\phi(\hat{\mathbf{r}}\cdot\mathbf{p}) = (\mathbf{p}\cdot\hat{\phi})\sin\theta$. The result simplifies to

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\exp(\mathrm{i}kr)}{r}k^2\{\mathbf{p} - p_r\hat{\mathbf{r}}\} - \frac{1}{4\pi\epsilon_0}\frac{\exp(\mathrm{i}kr)}{r^2}\left(\frac{1}{r} - \mathrm{i}k\right)\left\{\hat{\theta}p_\theta + \hat{\phi}p_\phi - 2\hat{\mathbf{r}}p_r\right\}$$

Noting that $\hat{\mathbf{r}} = \hat{\mathbf{n}}$, the first curly bracket equals $(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}$ and the second $\mathbf{p} - 3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})$ we find the final result,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{\mathbf{n}}\times\mathbf{p})\frac{\exp(\mathrm{i}kr)}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p}) - \mathbf{p}]\left(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\right)\exp(\mathrm{i}kr)\right\} \qquad (6)$$