Phys. 506 | Electricity and Magnetism | Winter 2004

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Problem Set 2

Total 40 Points

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1. Problem 8.6 **10 Points**

a): TM_{mnp}:

$$\omega_{mnp} = \frac{1}{R\sqrt{\mu\epsilon}}\sqrt{x_{mn}^2 + p^2\pi^2 y^2} \quad \text{with} \quad y = \frac{R}{L}$$

where $m, p = 0, 1, 2..$ and $n = 1, 2, ..$ and x_{mn} is the n-th zero of $J_m(x)$. The frequencies of the TM_{mn0} do not depend on the cavity length L.

TE_{mnp}:

$$\omega_{mnp} = \frac{1}{R\sqrt{\mu\epsilon}}\sqrt{x_{mn}'^2 + p^2\pi^2 y^2} \quad \text{with} \quad y = \frac{R}{L}$$

where $m = 0, 1, 2..$ and $n, p = 1, 2, ..$ and x_{mn} is the n-th zero of $J_m(x)$. The frequencies of all TE-modes depend on both the cavity length and radius.

As the figure shows, the ground mode is either TE_{111} or TM_{010}, dependent on $\frac{R}{L}$.

At $y = 0.67$, the fundamental mode is the TM_{010}. Its fields are

$$\begin{aligned}
E_z &= \psi(\rho,\phi)\cos\left(\frac{p\pi z}{L}\right) = E_0 J_0\left(\frac{x_{01}\rho}{R}\right) \\
\mathbf{E}_t &= 0 \\
\mathbf{H}_t &= -\hat{\phi}\frac{\mathrm{i}\epsilon\omega}{\gamma^2}E_0\frac{x_{01}}{R}J_0'\left(\frac{x_{01}\rho}{R}\right) = \hat{\phi}\frac{\mathrm{i}\epsilon\omega}{\gamma^2}E_0\frac{x_{01}}{R}J_1\left(\frac{x_{01}\rho}{R}\right)
\end{aligned}$$

By Eq. 8.92 in Jackson, the intracavity energy is

$$U = E_0^2\frac{\pi L\epsilon}{2}\int_0^R \rho J_0^2\left(\frac{x_{01}\rho}{R}\right)d\rho = E_0^2\frac{\pi\epsilon L R^2}{4}J_1^2(x_{01})$$

The dissipation power due to Ohm-type heating is

$$\begin{aligned}
P &= \frac{1}{2\sigma\delta}\int_{surface}|\mathbf{H}|^2\,da \\
&= \frac{E_0^2}{2\sigma\delta}\frac{\epsilon^2\omega^2}{\gamma^4}\frac{x_{01}^2}{R^2}\left\{2\pi R L J_1^2(x_{01}) + 2\times 2\pi\int_0^R \rho J_1^2\left(\frac{x_{01}\rho}{R}\right)d\rho\right\}
\end{aligned}$$

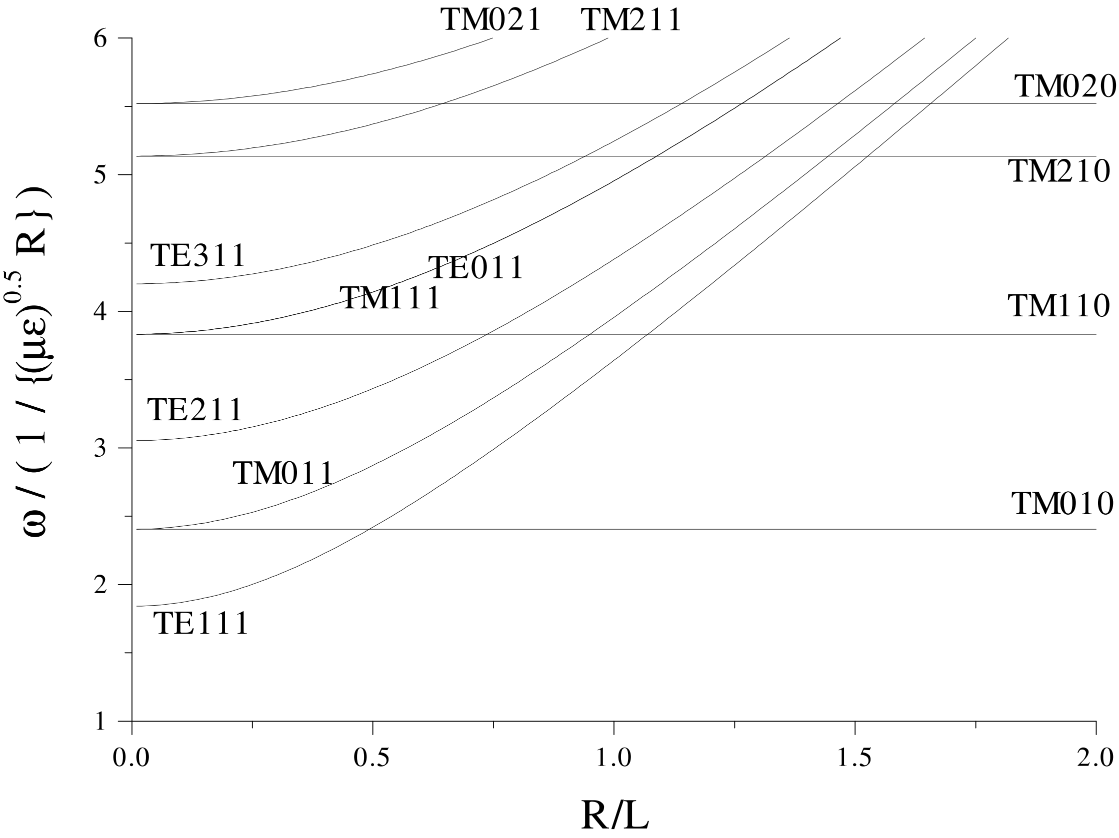

Figure 1: Frequencies of the lowest cavity modes

$$= \frac{E_0^2}{2\sigma\delta}\frac{2\pi\epsilon^2\omega^2}{\gamma^4}\frac{x_{01}^2}{R^2}\left\{RLJ_1^2(x_{01}) + R^2\left[J_1'^2(x_{01}) + \left(1 - \frac{1}{x_{01}^2}\right)J_1^2(x_{01})\right]\right\}$$

where the first term is from the mantle and the second from the end caps. The Q-factor then is, using $\gamma = \frac{x_{01}}{R}$,

$$\begin{aligned} Q &= \omega\frac{U}{P} \\ &= \frac{2L\sigma\delta x_{01}^2}{8\epsilon\omega R\left\{L + R\left[\frac{J_1'^2(x_{01})}{J_1^2(x_{01})} + \left(1 - \frac{1}{x_{01}^2}\right)\right]\right\}} \end{aligned}$$

Using the identity $xJ_\nu'(x) + \nu j_\nu(x) = xJ_{\nu-1}(x)$ with $\nu = 1$, it is $\frac{J_1'(x_{01})}{J_1(x_{01})} = -\frac{1}{x_{01}}$, and the result simplifies to

$$\begin{aligned} Q &= \frac{L\sigma\delta x_{01}^2}{4\epsilon\omega R\{L + R\}} \\ &= \omega\frac{\sigma\delta\mu}{4}\frac{RL}{L+R} \end{aligned} \tag{1}$$

where in the second line we have used that for the TM_{010}-mode $\omega^2 = \frac{1}{\mu\epsilon R^2} x_{01}^2$. Since also the skin depth $\delta = \sqrt{\frac{2}{\sigma\omega\mu_c}}$ and $c = \frac{1}{\sqrt{\mu\epsilon}}$ and $\mu = \mu_c$, this can be expressed as

$$Q = \frac{L\sqrt{R}}{R+L}\sqrt{\frac{\sigma\mu c x_{01}}{2}}$$

For a numerical result, use $\mu = \mu_0$, $x_{01} = 2.405$, $R = 2cm$, $L = 3cm$, $\sigma = \frac{1}{1.7\times10^{-8}\Omega m}$. Then one obtains a quite typical value,

$$Q = 13850$$

2. Problem 8.12 **10 Points**

TM-modes. We use the convention that the normal vector $\hat{\mathbf{n}}$ is pointing inward. The unperturbed solution, $\psi_0 = E_{z,0}$, is such that it vanishes on the unperturbed surface S_0. The surface perturbation function $\delta(x, y)$ equals the distance between perturbed and unperturbed surface in the direction of $\hat{\mathbf{n}}$, where according to our choice of $\hat{\mathbf{n}}$ an **inward deformation counts positive**.

The perturbation will not significantly alter the mode function. We may assume that the effect of the wall deformation is that the mode function ψ_0 **shifts together with the wall**, thereby maintaining the proper boundary condition $\psi = 0$ on S. The leading dependence of ψ_0 near the unperturbed surface S_0 is $\psi(\xi) = \xi \frac{\partial}{\partial n}\psi_0$, where ξ measures the normal distance from S_0 ($\xi > 0$ inside the guide). Shifting ψ such that its zero moves from S_0 to S then is equivalent to assuming a **perturbed boundary condition**

$$\psi = -\delta(x, y)\frac{\partial \psi_0}{\partial n} \quad \text{on} \quad S_0$$

The equations and boundary conditions for ψ_0 and ψ and the respective eigenvalues γ_0^2 and γ^2 are

$$\begin{aligned} (\nabla_t^2 + \gamma_0^2)\psi_0 = 0 \qquad & \text{and} \qquad \psi_0 = 0 \quad \text{on} \quad S_0 \\ (\nabla_t^2 + \gamma^2)\psi = 0 \qquad & \text{and} \qquad \psi = -\delta(x, y)\frac{\partial \psi_0}{\partial n} \quad \text{on} \quad S_0 \end{aligned}$$

Using Green's 2nd identity in two dimensions with $\hat{\mathbf{n}}$ pointing inward for ψ and ψ_0^* then yields

$$\int (\psi \nabla_t^2 \psi_0^* - \psi_0^* \nabla_t^2 \psi) da = \oint_{S_0} (\psi_0^* \frac{\partial \psi}{\partial n} - \psi \frac{\partial \psi_0^*}{\partial n}) dl$$

Inserting the above eigenvalue equations and boundary values on S_0 it is

$$(\gamma^2 - \gamma_0^2) \int \psi \psi_0^* da = \oint \delta(x, y) \left| \frac{\partial \psi}{\partial n} \right|^2 dl$$

where we have also used that the eigenvalues are real. Since in the area integral the difference between ψ and ψ_0 will only produce higher-order corrections, we may set $\psi = \psi_0$ in the area integral and get

$$(\gamma^2 - \gamma_0^2) = \frac{\oint_{S_0} \delta(x, y) \left| \frac{\partial \psi}{\partial n} \right|^2 dl}{\int |\psi_0|^2 da}$$

and for the wavenumbers, $k^2 = \mu\epsilon\omega^2 - \gamma^2$,

$$(k^2 - k_0^2) = -\frac{\oint_{S_0} \delta(x, y) \left| \frac{\partial \psi}{\partial n} \right|^2 dl}{\int |\psi_0|^2 da}$$

The **sign of the result** must be such that a reduction of the guide cross section, corresponding to positive δ, must result in a reduction of k (*i.e.* an increase of the wavelength in the guide). Our result satisfies this requirement, but it differs from the result stated in Jackson by a minus sign. The result stated in Jackson obviously corresponds to a choice of the normal vector $\hat{\mathbf{n}}$ pointing outward (which is opposite to the convention used in the corresponding portion of the text in Ch. 8.6).

TE-modes. We use the convention that the normal vector $\hat{\mathbf{n}}$ is pointing inward. The unperturbed solution, $\psi_0 = H_{z,0}$, is such that its normal derivative $\frac{\partial}{\partial n}\psi_0$ vanishes on the unperturbed surface S_0.

The effect of the wall deformation is that the mode function ψ_0 **shifts together with the wall**, thereby maintaining the proper boundary condition $\frac{\partial}{\partial n}\psi_0 = 0$ on S. The leading dependence of the normal derivative of ψ_0 near the unperturbed surface S_0 is $\frac{\partial}{\partial n}\psi_0(\xi) = \xi\,\frac{\partial^2}{\partial n^2}\psi_0\Big|_{\xi=0}$, where ξ measures the normal distance from S_0 ($\xi > 0$ inside the guide). Shifting ψ_0 such that the zero of its normal derivative moves from S_0 to S then is equivalent to assuming a **perturbed boundary condition**

$$\frac{\partial}{\partial n}\psi = -\delta(x,y)\frac{\partial^2}{\partial n^2}\psi_0 \quad \text{on} \quad S_0$$

The equations and boundary conditions for ψ_0 and ψ and the respective eigenvalues γ_0^2 and γ^2 are

$$(\nabla_t^2 + \gamma_0^2)\psi_0 = 0 \qquad \text{and} \qquad \frac{\partial\psi_0}{\partial n} = 0 \quad \text{on} \quad S_0$$
$$(\nabla_t^2 + \gamma^2)\psi = 0 \qquad \text{and} \qquad \frac{\partial\psi}{\partial n} = -\delta(x,y)\frac{\partial^2\psi_0}{\partial n^2} \quad \text{on} \quad S_0$$

Using Green's 2nd identity in two dimensions with $\hat{\mathbf{n}}$ pointing inward for ψ and ψ_0^* and inserting the above eigenvalue equations then yields

$$(\gamma^2 - \gamma_0^2)\int \psi\psi_0^* da = \oint_{S_0}(\psi_0^*\frac{\partial\psi}{\partial n} - \psi\frac{\partial\psi_0^*}{\partial n})dl$$

Setting $\psi = \psi_0$ in the area integral and inserting the boundary values on S_0 it is

$$(\gamma^2 - \gamma_0^2) = -\frac{\oint_{S_0}\delta(x,y)\psi_0^*\frac{\partial^2\psi_0}{\partial n^2}dl}{\int|\psi_0|^2\,da}$$

and for the wavenumbers, $k^2 = \mu\epsilon\omega^2 - \gamma^2$,

$$(k^2 - k_0^2) = \frac{\oint_{S_0}\delta(x,y)\psi_0^*\frac{\partial^2\psi_0}{\partial n^2}dl}{\int|\psi_0|^2\,da}$$

Concerning the **sign of the result**, note that δ changes sign upon reversal on $\hat{\mathbf{n}}$, while $\frac{\partial^2\psi_0}{\partial n^2}$ does not. Our result differs from the result stated in Jackson by a minus sign, again reflecting the fact the result stated in Jackson assumes a normal vector $\hat{\mathbf{n}}$ pointing outward.

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b): The depicted deformation $\delta(y) = \delta\frac{y}{b}$ along the vertical sides and $\delta = 0$ along the horizontal sides. The depicted case corresponds to positive δ. The line integral only needs to be evaluated along the vertical sides. We can use unnormalized mode functions.

<u>TM_{11}.</u>

$$\psi_0 = E_z = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

The normal derivative $\frac{\partial\psi_0}{\partial n}$ on the vertical sides $x = 0$ and $x = a$ is

$$\left|\frac{\partial\psi_0}{\partial n}\right| = \frac{\pi}{a}\sin\left(\frac{\pi y}{b}\right)$$

The line integral

$$\oint_{S_0} \delta(x,y)\left|\frac{\partial\psi}{\partial n}\right|^2 dl = 2\frac{\pi^2\delta}{a^2 b}\int y\sin^2\left(\frac{\pi y}{b}\right)dy = \frac{\pi^2\delta b}{2a^2}$$

The area integral

$$\int |\psi_0|^2\, da = \frac{1}{4}ab$$

and

$$\gamma^2 - \gamma_0^2 = k_0^2 - k^2 = \frac{\oint_{S_0}\delta(x,y)\left|\frac{\partial\psi}{\partial n}\right|^2 dl}{\int|\psi_0|^2\, da} = \frac{2\pi^2\delta}{a^3}$$

Since $k_0^2 = \mu\epsilon\omega^2 - \pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$, the perturbed value of k^2 is

$$k^2 = k_0^2 - \frac{\pi^2}{a^2}\frac{2\delta}{a} = \mu\epsilon\omega^2 - \frac{\pi^2}{a^2}\left(1+\frac{2\delta}{a}\right) - \frac{\pi^2}{b^2}$$

<u>TE_{10}.</u>

$$\psi_0 = H_z = \cos\left(\frac{\pi x}{a}\right)$$

On the surfaces $x = 0$ and $x = a$, it is $\psi_0^* = \pm 1$ and $\frac{\partial^2\psi_0}{\partial n^2} = \mp\frac{\pi^2}{a^2}$ (upper signs for $x = 0$, lower for $x = a$).

Thus,

$$\oint_{S_0}\delta(x,y)\psi_0^*\frac{\partial^2\psi_0}{\partial n^2}dl = -2\frac{\pi^2}{a^2}\frac{\delta}{b}\int_0^b y\,dy = -\frac{\pi^2 b\delta}{a^2}$$

The area integral

$$\int |\psi_0|^2 \, da = \frac{1}{2} ab$$

and

$$\gamma^2 - \gamma_0^2 = k_0^2 - k^2 = -\frac{\oint_{S_0} \delta(x,y) \psi_0^* \frac{\partial^2 \psi_0}{\partial n^2} \, dl}{\int |\psi_0|^2 \, da} = \frac{2\pi^2 \delta}{a^3}$$

Since $k_0^2 = \mu\epsilon\omega^2 - \frac{\pi^2}{a^2}$, the perturbed value of k^2 is

$$k^2 = k_0^2 - \frac{\pi^2}{a^2} \frac{2\delta}{a} = \mu\epsilon\omega^2 - \frac{\pi^2}{a^2} \left(1 + \frac{2\delta}{a} \right)$$

3. Problem 8.13 **10 Points**

a):

Damping-induced mixing of TM-modes..

Ideal degenerate guide modes without damping and common eigenvalue γ_0^2:

$$(\nabla_t^2 + \gamma_0^2)\psi_0^{(i)} = 0 \quad \text{and} \quad \psi_0^{(i)} = 0 \quad \text{on} \quad S_0$$

with $i = 1, 2, ..N$. For small damping, the mixed modes are well-defined linear superpositions of the degenerate ideal modes,

$$\psi = \sum_{i=1}^{N} a_i \psi_0^{(i)}$$

The objective is to find the coefficients a_i. Since there are as many mixed modes as there are unperturbed ones, there will be N independent mixed modes (characterized by independent sets of a_i).

As shown in Sec. 8.6 of Jackson, the mixed modes are slightly altered due to the surface conductivity such that they satisfy an equation with a perturbed boundary condition

$$(\nabla_t^2 + \gamma^2)\psi = 0 \quad \text{and} \quad \psi = f\frac{\partial \psi}{\partial n} = f\sum_i a_i \frac{\partial \psi_0^{(i)}}{\partial n} \quad \text{on} \quad S_0$$

where $f = (1 + \mathrm{i})\frac{\mu_c \delta}{2\mu}\frac{\omega^2}{\omega_0^2}$ with the unperturbed cutoff frequency $\omega_0 = \frac{\gamma_0}{\sqrt{\epsilon\mu}}$ and the skin depth $\delta = \sqrt{\frac{2}{\mu_c \sigma \omega}}$.

Using Green's 2nd identity in two dimensions with $\hat{\mathbf{n}}$ pointing inward for ψ and $\psi_0^{(j)*}$ then yields

$$\int (\psi \nabla_t^2 \psi_0^{(j)*} - \psi_0^{(j)*} \nabla_t^2 \psi) da = \oint_{S_0} (\psi_0^{(j)*} \frac{\partial \psi}{\partial n} - \psi \frac{\partial \psi_0^{(j)*}}{\partial n}) dl$$

Inserting the above eigenvalue equations, the sum expression for ψ, the boundary values on S_0, and using the reality of the eigenvalues, it is

$$(\gamma^2 - \gamma_0^2) \sum_{i=1}^{N} a_i \int \psi_0^{(i)} \psi_0^{(j)*} da = -f \sum_{i=1}^{N} a_i \oint \frac{\partial \psi_0^{(i)}}{\partial n} \frac{\partial \psi_0^{(j)*}}{\partial n} dl$$

Since by assumption an orthogonality condition

$$\int \psi_0^{(i)} \psi_0^{(j)*} da = N_i \delta_{ji}$$

with norms N_i applies, the equation can be resorted into

$$\sum_{i=1}^{N} \left[N_i(\gamma^2 - \gamma_0^2)\delta_{ji} + \Delta_{ji} \right] a_i = 0$$

with $\Delta_{ji} = f \oint_{S_0} \frac{\partial \psi_0^{(i)}}{\partial n} \frac{\partial \psi_0^{(j)*}}{\partial n} dl$. **Q.e.d.**

Mixing of TM-modes due to surface deformation.

The equations and perturbed boundary conditions for the mixed modes ψ are, as seen in Problem 8.12,

$$(\nabla_t^2 + \gamma^2)\psi = 0 \quad \text{and} \quad \psi = -\delta(x,y)\frac{\partial \psi}{\partial n} = -\delta(x,y)\sum_i a_i \frac{\partial \psi_0^{(i)}}{\partial n} \quad \text{on} \quad S_0$$

($\hat{\mathbf{n}}$ pointing inward). Using Green's 2nd identity in two dimensions with $\hat{\mathbf{n}}$ pointing inward for ψ and $\psi_0^{(j)*}$ then yields

$$\sum_{i=1}^{N} \left[N_i(\gamma^2 - \gamma_0^2)\delta_{ji} + \Delta_{ji} \right] a_i = 0$$

with $\Delta_{ji} = -\oint_{S_0} \delta(x,y) \frac{\partial \psi_0^{(i)}}{\partial n} \frac{\partial \psi_0^{(j)*}}{\partial n} dl$. **Q.e.d.**

Note. Having $\hat{\mathbf{n}}$ point outward reverses the sign of δ; this produces the result given in the textbook.

Mixing of TE-modes due to surface deformation.

The equations and perturbed boundary conditions for the mixed modes ψ are, as seen in Problem 8.12,

$$(\nabla_t^2 + \gamma^2)\psi = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial n} = -\delta(x,y)\frac{\partial^2 \psi_0}{\partial n^2} = -\delta(x,y)\sum_i a_i \frac{\partial^2 \psi_0^{(i)}}{\partial n^2} \quad \text{on} \quad S_0$$

($\hat{\mathbf{n}}$ pointing inward). Using Green's 2nd identity in two dimensions with $\hat{\mathbf{n}}$ pointing inward for ψ and $\psi_0^{(j)*}$ then yields

$$\sum_{i=1}^{N} \left[N_i(\gamma^2 - \gamma_0^2)\delta_{ji} + \Delta_{ji} \right] a_i = 0$$

with $\Delta_{ji} = +\oint_{S_0} \delta(x,y) \frac{\partial^2 \psi_0^{(i)}}{\partial n^2} \psi_0^{(j)*} dl$. **Q.e.d.**

Note. Having $\hat{\mathbf{n}}$ point outward reverses the sign of δ; this produces the result given in the textbook.

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b): The given mode functions are orthogonal. We can set $B_0 = 1$. The norm values are then both equal to

$$\begin{aligned} N: \quad &= \quad N_+ = N_- = 2\pi \int_{\rho=0}^{R} \rho J_1^2\left(\frac{x'_{11}}{R}\rho\right) d\rho \\ &= \quad 2\pi \left[\frac{\rho^2}{2} J_1'^2\left(\frac{x'_{11}}{R}\rho\right) + \frac{\rho^2}{2}\left(1 - \frac{1}{\rho^2 \frac{x'^2_{11}}{R^2}}\right) J_1^2\left(\frac{x'_{11}}{R}\rho\right)\right]_0^R \\ &= \quad \pi R^2 J_1^2(x'_{11})\left(1 - \frac{1}{x'^2_{11}}\right) \end{aligned}$$

The surface deformation can be written as $\delta(\phi) = \Delta R \cos(2\phi)$. It is then seen that $\Delta_{++} = \Delta_{--} = 0$. Also,

$$\Delta_{+-} = R\Delta R \oint_{S_0} \cos(2\phi) \frac{\partial^2 \psi_0^{(-)}}{\partial n^2} \psi_0^{(+)*} d\phi$$

where $\frac{\partial^2 \psi_0^{(-)}}{\partial n^2} = \left(\frac{x'_{11}}{R}\right)^2 J_1''\left(\frac{x'_{11}}{R}\rho\right)_{\rho=R} \exp(-\mathrm{i}\phi) = \left(\frac{x'_{11}}{R}\right)^2 J_1''(x'_{11}) \exp(-\mathrm{i}\phi)$. Also, $\Delta_{+-} = \Delta_{-+}$. Thus,

$$\begin{aligned} \Delta: \quad &= \quad \Delta_{+-} = \Delta_{-+} = R\Delta R \left(\frac{x'_{11}}{R}\right)^2 J_1''(x'_{11}) J_1(x'_{11}) \oint_{S_0} \exp(-2\mathrm{i}\phi)\cos(2\phi) d\phi \\ &= \quad \pi \frac{\Delta R}{R} x'^2_{11} J_1(x'_{11}) J_1''(x'_{11}) \\ &= \quad \pi \frac{\Delta R}{R} x'^2_{11} J_1(x'_{11}) \left[J_1(x)\left(\frac{1}{x^2} - 1\right) - J_1'(x)\frac{1}{x}\right]_{x=x'_{11}} \\ &= \quad \pi \frac{\Delta R}{R} x'^2_{11} J_1^2(x'_{11}) \left(\frac{1}{x'^2_{11}} - 1\right) \end{aligned}$$

The new eigenvalues γ^2 are found by setting the determinant

$$\left| \begin{pmatrix} N(\gamma^2 - \gamma_0^2) & \Delta \\ \Delta & N(\gamma^2 - \gamma_0^2) \end{pmatrix} \right| = 0$$

yielding

$$\gamma^2 = \gamma_0^2 \pm \frac{|\Delta|}{|N|} = \gamma_0^2 \pm \frac{\Delta R}{R}\left(\frac{x'_{11}}{R}\right)^2 = \gamma_0^2 (1 \pm \frac{\Delta R}{R})$$

Therefore, the parameter λ asked for in the problem is 1.

<u>Eigenfunctions</u>

From

$$\begin{pmatrix} N(\gamma^2 - \gamma_0^2) & \Delta \\ \Delta & N(\gamma^2 - \gamma_0^2) \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = 0$$

and $(\gamma^2 - \gamma_0^2) = \gamma_0^2 \frac{\Delta R}{R}$ it follows

$$a_- = \mp\gamma_0^2 \frac{\Delta R}{R} \frac{N}{\Delta} a_+$$

With $\frac{N}{\Delta} = -\gamma_0^{-2} \frac{R}{\Delta R}$ we then find

$$a_- = \pm a_+$$

where the upper sign corresponds to the larger perturbed value of γ^2. The solutions satisfying the boundary conditions on the deformed guide walls are:

$$\psi_1 = H_0 J_1 \left(\frac{x'_{11}}{R} \rho \right) \cos\phi \quad \text{with} \quad \gamma^2 = \gamma_0^2 (1 + \frac{\Delta R}{R})$$

and

$$\psi_2 = H_0 J_1 \left(\frac{x'_{11}}{R} \rho \right) \sin\phi \quad \text{with} \quad \gamma^2 = \gamma_0^2 (1 - \frac{\Delta R}{R})$$

Interpretation. The electric field is transverse and found to be

$$\mathbf{E}_{1,t} = -\frac{\mathrm{i}\mu\omega H_0}{\gamma^2} \left[\hat{\phi}\gamma_0 J_1' (\gamma_0\rho) \cos\phi + \hat{\rho}\frac{1}{\rho} J_1 (\gamma_0\rho) \sin\phi \right]$$

and

$$\mathbf{E}_{2,t} = -\frac{\mathrm{i}\mu\omega H_0}{\gamma^2} \left[\hat{\phi}\gamma_0 J_1' (\gamma_0\rho) \sin\phi - \hat{\rho}\frac{1}{\rho} J_1 (\gamma_0\rho) \cos\phi \right]$$

Noting that $\gamma^2 \approx \gamma_0^2$, near the axis, where $\rho \ll R$, the electric field reduces to

$$\mathbf{E}_{1,t} = -\hat{\mathbf{y}} \frac{\mathrm{i}\mu\omega H_0}{2\gamma_0}$$

$$\mathbf{E}_{2,t} = \hat{\mathbf{x}} \frac{\mathrm{i}\mu\omega H_0}{2\gamma_0}$$

Thus, the mixed modes have transverse electric (and magnetic) fields that are aligned with the major and minor axes of the ellipse present after deformation.

Note (unwarranted). For the adopted inward direction of the normal vector $\hat{\mathbf{n}}$ and for $\frac{\Delta R}{R} > 0$ the deformation is such that the guide becomes stretched in y- and squished in x-direction. Thus, the mode the eigenvalue γ^2 of which shifts upward (corresponding to downshifting k^2 and upshifting guide wavelength) has an electric field polarized parallel to the long axis of the deformation ellipse. The mode the eigenvalue

γ^2 of which shifts downward has an electric field polarized transverse to the long axis of the deformation ellipse.

Note. You recall that the direction of the normal vector $\hat{\mathbf{n}}$ played a role in the analysis. In the present case, reversing the choice of the direction of $\hat{\mathbf{n}}$ reverses the sign of $\frac{\Delta R}{R}$. As a result, the mixed modes 1 and 2 and the associated eigenvalues become swapped, ensuring that the physical results remain the same. The number values for the perturbed eigenvalues must be independent of the direction of $\hat{\mathbf{n}}$, and the association between the polarizations of the up- and downshifting modes and the orientation of the deformation ellipse must be independent of the direction of $\hat{\mathbf{n}}$.

4. Problem 8.20 **10 Points**

a): TM-modes. The origin is chosen in the lower left corner, and we use the normalized ode functions Eq. 8.135 and 8.136. For the current element, we use

$$\mathbf{J}(\mathbf{x})d^3x = I_0 d\mathrm{l} = I_0 R \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} d\phi$$

and

$$\mathbf{x} = \begin{pmatrix} R\cos\phi \\ h + R\sin\phi \end{pmatrix}$$

There, R is the loop radius, h the height of the loop center, and $-\pi/2 < \phi < \pi/2$. We assume, without loss of generality, $z = 0$. Then, the normalized-mode amplitudes are

$$A_i^{\pm} = -\frac{Z_i}{2} \int \mathbf{J}(\mathbf{x}) \cdot \mathbf{E}_i^{\mp} d^3x$$

which for the TM-modes of the given rectangular guide and current yields

$$\begin{aligned} A_{mn}^{\pm} &= -\frac{Z_{TM}}{2} \frac{2\pi I_0 R}{\gamma_{mn}\sqrt{ab}} \int_{-\pi/2}^{\pi/2} \left\{ \frac{m}{a}(-\sin\phi)\cos\left(\frac{m\pi R\cos\phi}{a}\right) \sin\left(\frac{n\pi(h+R\sin\phi)}{b}\right) \right. \\ &\quad \left. + \frac{n}{b}\cos\phi \sin\left(\frac{m\pi R\cos\phi}{a}\right) \cos\left(\frac{n\pi(h+R\sin\phi)}{b}\right) \right\} d\phi \end{aligned}$$

Noting that the integrands are total differentials,

$$\begin{aligned} A_{mn}^{\pm} &= -\frac{Z_{TM}}{2} \frac{2\pi I_0 R}{\gamma_{mn}\sqrt{ab}} \int_{-\pi/2}^{\pi/2} \left\{ \left[\frac{d}{d\phi}\sin\left(\frac{m\pi R\cos\phi}{a}\right)\right] \sin\left(\frac{n\pi(h+R\sin\phi)}{b}\right) \right. \\ &\quad \left. + \left[\frac{d}{d\phi}\sin\left(\frac{n\pi(h+R\sin\phi)}{b}\right)\right] \sin\left(\frac{m\pi R\cos\phi}{a}\right) \right\} \frac{1}{\pi R} d\phi \\ &= -\frac{Z_{TM}}{2} \frac{2\pi I_0 R}{\gamma_{mn}\sqrt{ab}} \int_{-\pi/2}^{\pi/2} \frac{d}{d\phi}\left[\sin\left(\frac{m\pi R\cos\phi}{a}\right) \sin\left(\frac{n\pi(h+R\sin\phi)}{b}\right)\right] \frac{1}{\pi R} d\phi \\ &= -\frac{Z_{TM}}{2} \frac{2\pi I_0 R}{\gamma_{mn}\sqrt{ab}} \frac{1}{\pi R} \left[\sin\left(\frac{m\pi R\cos\phi}{a}\right) \sin\left(\frac{n\pi(h+R\sin\phi)}{b}\right)\right]_{-\pi/2}^{\pi/2} \\ &= 0 \end{aligned}$$

Interpretation. The wire loop couples to the longitudinal magnetic field, which is absent for TM-modes. Thus, for the given geometry the TM excitation amplitudes vanish.

Note. The integrand can be expanded for the case $R \ll a, b$. For TM-modes, the result also is $A_{mn}^{\pm} = 0$.

---

b): TE$_{10}$-mode.

From Eq. 8.136 in Jackson and the subsequent comment it follows that the normalized fields are

$$
\begin{aligned}
E_z &= 0 \\
E_x &= 0 \\
E_y &= \frac{\sqrt{2}\pi}{\gamma_{01} a\sqrt{ab}} \sin\left(\frac{\pi x}{a}\right)
\end{aligned}
$$

from which

$$A^{\pm}_{E10} = -\frac{Z_{TE}}{2}\frac{\sqrt{2}\pi I_0 R}{\gamma_{10} a\sqrt{ab}} \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} \sin\left(\frac{\pi R\cos\phi}{a}\right)\cos\phi d\phi$$

Use $Z_{TE} = \mu\omega/k$ and $k = \sqrt{\frac{\omega^2}{c^2} - \gamma^2}$ and $\gamma_{10} = \pi/a$ and $\int_{-\pi/2}^{\pi/2} \sin(x\cos\phi)\cos\phi d\phi = \pi J_1(x)$ to see

$$A^{\pm}_{E10} = -\frac{\mu\omega I_0 R\pi}{\sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}}\sqrt{2ab}} J_1\left(\frac{\pi R}{a}\right)$$

---

c): Power in TE$_{10}$-mode. It is, according to Eq. 8.136 and the subsequent comment,

$$\psi = H_z = A^{\pm}_{E10} H_{z,E01} = \frac{\mu\omega I_0 R\pi}{\sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}}\sqrt{2ab}} J_1\left(\frac{\pi R}{a}\right) \frac{\sqrt{2}\mathrm{i}\gamma_{01}}{kZ_{TE}\sqrt{ab}} \cos\left(\frac{\pi x}{a}\right)$$

Use $Z_{TE} = \mu\omega/k$ and $k = \sqrt{\frac{\omega^2}{c^2} - \gamma^2}$ and $\gamma_{10} = \pi/a$, and Eq. 8.51, $P = const \int |\psi|^2 \, da$, to find the power

$$P = \frac{I_0^2 R^2 Z\pi^2}{4ab} J_1^2\left(\frac{\pi R}{a}\right)$$

For small argument $J_1(x) = x/2$. Thus, for $R \ll a$ it is

$$P \approx \frac{I_0^2 Z}{16}\frac{a}{b}\left(\frac{\pi R}{a}\right)^4$$

Note. The result also follows from Eq. 8.133,

$$P = \frac{1}{2Z_i}|A_i|^2$$