Phys. 506 | Electricity and Magnetism | Winter 2004

Prof. G. Raithel

Problem Set 1

Total 30 Points

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1. Jackson 8.2 **10 Points**

a): The electric field is the same as in the 2-dimensional electrostatic problem of two concentric cylinders, *i.e.*

$$\mathbf{E}(\rho) = \hat{\rho} E_0 \frac{a}{\rho}$$

where E_0 denotes the field on the surface of the inner conductor, which has radius a. Since for TEM waves propagating in the $+z$-direction it is $\mathbf{H} = \frac{1}{\mu c}\hat{\mathbf{z}} \times \mathbf{E} = \frac{1}{Z}\hat{\mathbf{z}} \times \mathbf{E}$ with the plane-wave impedance $Z = \sqrt{\frac{\mu}{\epsilon}}$, it is

$$\mathbf{H}(\rho) = -\hat{\phi} E_0 \frac{a}{Z\rho} = -\hat{\phi} H_0 \frac{a}{\rho}$$

with $H_0 = \frac{E_0}{Z}$ being the magnetic field on the surface of the inner conductor. The Poynting vector

$$\mathbf{S}(\rho) = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \hat{\mathbf{z}} \frac{1}{2Z} |E_0|^2 \frac{a^2}{\rho^2} = \hat{\mathbf{z}} \frac{Z}{2} |H_0|^2 \frac{a^2}{\rho^2} \quad ,$$

and the power is

$$P = \int_a^b \mathrm{Re}\left[\hat{\mathbf{z}} \cdot \mathbf{S}\right] 2\pi\rho d\rho = Z\pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right) \quad \text{q.e.d.}$$

b): On the inner surface,

$$\left.\frac{dP}{dz}\right|_{inner} = -\frac{1}{2\sigma\delta} \int_{\rho=a} |\mathbf{H}(\rho = a)| dl = -\frac{\pi a}{\sigma\delta} |H_0|^2$$

and on the outer surface

$$\left.\frac{dP}{dz}\right|_{outer} = -\frac{1}{2\sigma\delta} \int_{\rho=b} |\mathbf{H}(\rho = b)| dl = -\frac{\pi b}{\sigma\delta} |H_0|^2 \frac{a^2}{b^2} = -\frac{\pi a^2}{b\sigma\delta} |H_0|^2$$

The power loss is the sum of the two,

$$\frac{dP}{dz} = -\frac{\pi}{a^2\sigma\delta} |H_0|^2 \left(\frac{1}{a} + \frac{1}{b}\right)$$

Then, the attenuation constant $\gamma = -\frac{1}{2P}\frac{dP}{dz}$,

$$\gamma = \frac{1}{2\sigma\delta Z}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)} \quad \text{q.e.d.}$$

c): Voltage: $V = \left|\int \mathbf{E}\cdot d\mathbf{l}\right| = \int_a^b \hat{\rho} E_0 \frac{a}{\rho}\cdot\hat{\rho} d\rho = H_0 Z a \ln\left(\frac{b}{a}\right)$.

Current: For TEM modes, the surface currents are in z-direction, *i.e.* on the inner conductor $K_z = H_0$. The total current $I = 2\pi a H_0$. Thus, the characteristic impedance Z_0 is

$$Z_0 = \frac{V}{I} = \frac{Z}{2\pi}\ln\left(\frac{b}{a}\right) \quad \text{q.e.d.}$$

Note that Z is the plane-wave impedance $Z = \sqrt{\frac{\mu}{\epsilon}}$.

d): The series resistance R satisfies $\frac{1}{2}RI^2 = \left|\frac{dP}{dz}\right|$, where the factor $\frac{1}{2}$ occurs, as usual, due to the use of complex quantities. Thus, with the above I and $\frac{dP}{dz}$

$$R = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right) \quad \text{q.e.d.}$$

The inductance per length is defined via the magnetic-field energy per length, $u_m = \frac{1}{4}\int \mathbf{B}\cdot\mathbf{H}^* da = \frac{2\pi}{4}\int \mathbf{B}\cdot\mathbf{H}^*\rho d\rho = \frac{1}{4}L|I|^2$, where the factor $\frac{1}{4}$ (in place of the magneto-static value $\frac{1}{2}$) is to the use of complex quantities. The given result obviously accounts for the field energies inside the guide, $u_{m,guide}$, and in the inner and outer skin regions, $u_{m,inner}$ and $u_{m,outer}$. In the guide,

$$u_{m,guide} = \frac{\pi a^2}{2}\mu\left|H_0\right|^2\ln\left(\frac{b}{a}\right)$$

In the inner guide wall it is $H = H_0\exp(-\xi/\delta)$, where ξ measures the depth from the guide surface. Assuming $a \gg \delta$, which is a good assumption except in pathetic cases,

$$u_{m,inner} = \frac{2\pi\mu_c}{4}\int_0^\infty \left|H_0\right|^2\exp(-2\xi/\delta)a d\xi = \frac{\pi\mu_c a\delta}{4}\left|H_0\right|^2$$

where μ_c is the permeability of the conductor, and similarly

$$u_{m,inner} = \frac{2\pi\mu_c}{4}\int_0^\infty \left|H_0\right|^2\left(\frac{a}{b}\right)^2\exp(-2\xi/\delta)b d\xi = \frac{\pi\mu_c a^2\delta}{4b}\left|H_0\right|^2$$

Summing over the linear magnetic-field energy densities, it is

$$u_m = \frac{\pi a^2}{2}\left|H_0\right|^2\left[\mu\ln\left(\frac{b}{a}\right) + \frac{\delta\mu_c}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\right]$$

and

$$L = \frac{4u_m}{|I|^2} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\delta\mu_c}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right) \quad \text{q.e.d.}$$

Note that normally $\delta \ll a < b$ and therefore the second term is much smaller than the first.

2. Jackson 8.4 **10 Points**

a): Following the analysis on page 369f with the replacement $\frac{p\pi}{d} \to k$ it is seen that the cutoff frequencies are

$\omega_{M,mn} = \frac{x_{mn}c}{R}$ for mode TM_{mn} with $m = 0, 1, ..$ and $n = 1, 2, ..$ and

$\omega_{E,mn} = \frac{x'_{mn}c}{R}$ for mode TE_{mn} with $m = 0, 1, ..$ and $n = 1, 2, ..$

There, x_{mn} is the n-th zero of the Bessel function $J_m(x)$ and x'_{mn} is the n-th zero $\frac{d}{dx}J_m(x)$. The fundamental mode is

TE_{11} with $\omega_{E,11} = \frac{1.841c}{R} =: \omega_0$.

with $c = \frac{1}{\sqrt{\mu\epsilon}}$. The next four higher modes are:

TM_{01} with $\omega_{M,01}/\omega_0 = 1.306$

TE_{21} with $\omega_{E,21}/\omega_0 = 1.659$

TE_{01} with $\omega_{E,01}/\omega_0 = 2.081$

TM_{11} with $\omega_{M,11}/\omega_0 = 2.081$

b): $\underline{TE_{11}}$: We require ψ to calculate the power P from Eq. 8.51 in Jackson, and all magnetic fields to calculate

$$dP/dz = -\frac{1}{2\sigma\delta}\int |\mathbf{H}|^2 dl \quad .$$

Here, it is

$$\begin{aligned} H_z &= \psi = H_0 J_1(\frac{x'_{11}}{R}\rho)\exp(\mathrm{i}\phi) \\ \mathbf{H}_t &= \frac{\mathrm{i}k}{\gamma^2}\nabla_t\psi = \frac{\mathrm{i}kH_0}{\gamma^2}\exp(\mathrm{i}\phi)\left[\hat{\rho}\frac{x'_{11}}{R}J'_1(\frac{x'_{11}}{R}\rho) + \hat{\phi}\frac{\mathrm{i}}{\rho}J_1(\frac{x'_{11}}{R}\rho)\right] \end{aligned}$$

with $\gamma^2 = \mu\epsilon\omega_{E,11}^2$ and $k^2 = \mu\epsilon(\omega^2 - \omega_{E,11}^2)$ and $\delta = \sqrt{\frac{2}{\mu_c\sigma\omega}}$.

Using Eq. 8.51,

$$P = \frac{\pi |H_0|^2 \omega^2}{\omega_{E,11}^2}\sqrt{\frac{\mu}{\epsilon}}\sqrt{1 - \frac{\omega_{E,11}^2}{\omega^2}}\int_0^R \rho J_1^2(\frac{x'_{11}}{R}\rho)d\rho$$

where the integral equals $\frac{R^2}{2}(1 - \frac{1}{x'^2_{11}})J_1^2(x'_{11})$.

Also, on the surface $|H_z| = |H_0|J_1(x'_{11})$ and $|\mathbf{H}_t| = \frac{k|H_0|}{\gamma^2 R}J_1(x'_{11})$, and thus

$$\left|\frac{dP}{dz}\right| = \frac{1}{2\sigma\delta}2\pi R(|H_z|^2 + |\mathbf{H}_t|^2) = \frac{\pi R |H_0|^2}{\sigma\delta}J_1^2(x'_{11})(1 + \frac{k^2}{\gamma^4 R^2})$$

The attenuation constant for a hollow brass guide, for which $\mu = \mu_c = \mu_0$ and $\epsilon = \epsilon_0$ is then found to be

$$\beta_{E,11}(\omega) = \frac{1}{2P}\left|\frac{dP}{dz}\right| = \frac{1}{R}\sqrt{\frac{\epsilon_0}{2\sigma}}\frac{\mu_0\epsilon_0\omega_{E,11}^4 R^2 + \omega^2 - \omega_{E,11}^2}{\sqrt{\omega}\sqrt{\omega^2 - \omega_{E,11}^2}(\mu_0\epsilon_0\omega_{E,11}^2 R^2 - 1)}$$

$\underline{TM_{01}}$: The required fields are, using $\gamma = \frac{x_{01}}{R}$ and $Z = \frac{k}{\epsilon\omega}$,

$$\begin{aligned}
E_z &= \psi = E_0 J_0(\frac{x_{01}}{R}\rho) \\
\mathbf{E}_t &= \frac{\mathrm{i}k}{\gamma^2}\nabla_t\psi = \frac{\mathrm{i}kE_0}{\gamma^2}\left[\hat{\rho}\frac{x_{01}}{R}J_0'(\frac{x_{01}}{R}\rho)\right] = \hat{\rho}\frac{\mathrm{i}kE_0 R}{x_{01}}J_0'(\frac{x_{01}}{R}\rho) \\
\mathbf{H}_t &= \frac{1}{Z}\hat{\mathbf{z}}\times\mathbf{E}_t = -\hat{\phi}\frac{\mathrm{i}kE_0 R}{Zx_{01}}J_0'(\frac{x_{01}}{R}\rho)
\end{aligned}$$

Using Eq. 8.51, calculating

$$\left|\frac{dP}{dz}\right| = \frac{1}{2\sigma\delta}2\pi R|\mathbf{H}_t|^2 \quad ,$$

and evaluating $\beta = \frac{1}{2P}\left|\frac{dP}{dz}\right|$ for a hollow brass guide ($\mu = \mu_c = \mu_0$ and $\epsilon = \epsilon_0$), it is found

$$\beta_{M,01}(\omega) = \frac{1}{R}\sqrt{\frac{\epsilon_0\omega^3}{2\sigma(\omega^2 - \omega_{M,01}^2)}}$$

Expressing the results in a normalized frequency,

$$x := \frac{\omega}{\omega_{E,11}}$$

and the damping constants in units of $\frac{1}{R}\sqrt{\frac{\epsilon_0\omega_{E,11}}{2\sigma}}$,

$$\tilde{\beta} = \beta / \left(\frac{1}{R}\sqrt{\frac{\epsilon_0\omega_{E,11}}{2\sigma}}\right) \quad ,$$

it is

$$\begin{aligned}
\tilde{\beta}_{E,11}(x) &= \frac{1.841^2 + x^2 - 1}{\sqrt{x}\sqrt{x^2-1}(1.841^2 - 1)} \\
\tilde{\beta}_{M,01}(x) &= \sqrt{\frac{x^3}{x^2 - 1.306^2}}
\end{aligned}$$

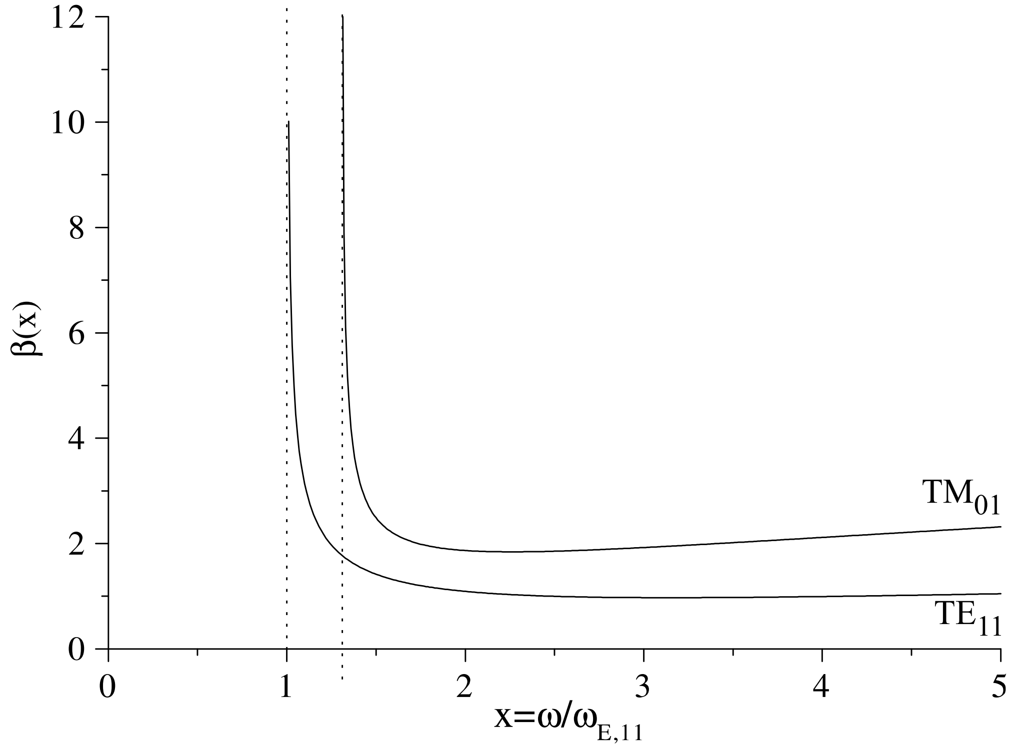

Figure 1: Damping constants in scaled units

3. Jackson 8.5 **10 Points**

a):

The problem has a discrete $\pi/2$ rotation symmetry. Thus, any mode TX_{mn} is degenerate with TX_{nm}, where $X = E$ or $X = M$. For $m \neq n$, we can use the superposition principle to form linear superpositions of degenerate modes that satisfy the applicable boundary conditions on the diagonal (in addition to the sides that form a right angle).

TM-modes:

$$E_z(x,y) = E_0 \left[\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) - \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\right]$$

for $n, m = 1, 2, 3..$ and $n \neq m$. It is $E_z(x,y) = 0$ for $x = y$. The corresponding cutoff frequencies are

$$\omega_{M,mn} = \frac{\pi}{a\sqrt{\mu\epsilon}}\sqrt{m^2+n^2}$$

The lowest TM-mode has $m = 1$ and $n = 2$.

TE-modes:

$$H_z(x,y) = H_0 \left[\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)\right]$$

for $n, m = 0, 1, 2, ..$ but not $n = m = 0$. As required, the normal derivative on the diagonal,

$$\frac{\partial}{\partial n}H_z = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)H_z$$

vanishes for $x = y$. The corresponding cutoff frequencies are also

$$\omega_{E,mn} = \frac{\pi}{a\sqrt{\mu\epsilon}}\sqrt{m^2+n^2}$$

The lowest TE-mode has $m = 0$ and $n = 1$.

Note. Since each pair of degenerate modes of the square guide gives only one mode of the triangular guide, there are about half as many modes in the triangular guides as there are in the square guide. This observation makes sense, because generally the mode density (=number of modes per frequency interval) at large frequencies is approximately proportional to the guide cross section.

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b): TE_{10}:

We need to calculate the power and the magnetic field on the surfaces. With an (unnormalized)

$$\psi = H_z = \cos\left(\frac{\pi x}{a}\right) + \cos\left(\frac{\pi y}{a}\right)$$

and, by symmetry,

$$\int_{triangle} \psi\psi^* dxdy = \frac{1}{2}\int_{square} \psi\psi^* dxdy = \frac{1}{2}\int_{square} \left[\cos^2\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{\pi y}{a}\right) + 2\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\right] dxdy = \frac{a^2}{2}$$

Magnetic-field amplitudes on the sides: Use $H_z = \psi$ and

$$\mathbf{H}_t = \frac{\mathrm{i}k}{\gamma^2}\nabla_t H_z = -\frac{\mathrm{i}k\pi}{\gamma^2 a}\left[\hat{\mathbf{x}}\sin\left(\frac{\pi x}{a}\right) + \hat{\mathbf{y}}\sin\left(\frac{\pi y}{a}\right)\right]$$

to see that on the side $x = 0$ it is $H_z(y) = 1 + \cos\left(\frac{\pi y}{a}\right)$ and $H_t(y) = \frac{k\pi}{\gamma^2 a}\sin\left(\frac{\pi y}{a}\right)$. Then, along that side it is

$$\int_{x=0,y=0}^{y=a} |\mathbf{H}|^2\, dy = a\left[\frac{3}{2} + \frac{1}{2}\frac{k^2\pi^2}{\gamma^4 a^2}\right]$$

The $y = 0$-side yields the same result. On the diagonal $x = y$ it is $H_z(y) = 2\cos\left(\frac{\pi y}{a}\right)$ and $H_t(y) = \sqrt{2}\frac{k\pi}{\gamma^2 a}\sin\left(\frac{\pi y}{a}\right)$. Then, along the diagonal it is

$$\int_{x=y,y=0}^{y=a} |\mathbf{H}|^2\, dl = \sqrt{2}\int_{x=y,y=0}^{y=a} |\mathbf{H}|^2\, dy = \sqrt{2}a\left[2 + \frac{k^2\pi^2}{\gamma^4 a^2}\right]$$

and the sum over all three sides,

$$\oint |\mathbf{H}|^2\, dl = a(3 + 2\sqrt{2}) + \frac{k^2\pi^2}{\gamma^4 a}(1 + \sqrt{2}) = a(3 + 2\sqrt{2}) + \frac{k^2\pi^2}{\gamma^4 a}(1 + \sqrt{2})$$

Use $k^2 = \mu\epsilon(\omega^2 - \omega_{E,01}^2)$ and $\gamma_{E,01} = \omega_{E,01}\sqrt{\mu\epsilon} = \frac{\pi}{a}$ to get

$$\oint |\mathbf{H}|^2\, dl = a(2 + 2\sqrt{2}) + a\frac{\omega^2}{\omega_{E,01}^2}(1 + \sqrt{2})$$

With Eq. 8.51 and $\left|\frac{dP}{dz}\right| = \frac{1}{2\sigma\delta}\oint |\mathbf{H}|^2\, dl$ and $\beta = \frac{1}{2P}\left|\frac{dP}{dz}\right|$

$$\beta_{E,01} = \frac{1}{\sigma\delta_{E,01}}\sqrt{\frac{\epsilon}{\mu}}\sqrt{\frac{\omega}{\omega_{E,01}}}\frac{1}{a}\frac{1}{\sqrt{1 - \frac{\omega_{E,01}^2}{\omega^2}}}\left[(2 + \sqrt{2})\frac{\omega_{E,01}^2}{\omega^2} + (1 + \sqrt{2})\right]$$

where $\delta_{E,01} = \sqrt{\frac{2}{\mu_c\sigma\omega_{E,01}}}$. The result has been written in a form analogous with Eq. 8.63 in Jackson.

$\underline{TM_{12}}$: We use

$$\psi = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right) - \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)$$

which yields

$$\int_{triangle} \psi\psi^* dxdy = \frac{a^2}{4}$$

Also, $H_z = 0$, and the magnitude of the magnetic field is

$$\begin{aligned}
H &= H_t = \frac{1}{Z}E_t = \frac{\epsilon\omega}{\gamma^2}\left|\nabla_t\psi\right| \\
&= \frac{\epsilon\omega}{\gamma^2}\left|\hat{\mathbf{x}}\left[\frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right) - \frac{2\pi}{a}\cos\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\right]\right. \\
&\quad \left.\hat{\mathbf{y}}\left[\frac{2\pi}{a}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi y}{a}\right) - \frac{\pi}{a}\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\right]\right|
\end{aligned}$$

On the side $x = 0$ it is

$$\int_{x=0,y=0}^{y=a} \left|\mathbf{H}\right|^2 dy = \frac{5}{2}\frac{\epsilon^2\omega^2\pi^2}{\gamma^4 a}$$

The same applies on the side $y = 0$. On the diagonal $x = y$, it is

$$H_t = \sqrt{2}\frac{\epsilon\omega}{\gamma^2}\left|\frac{2\pi}{a}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right) - \frac{\pi}{a}\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)\right|$$

and

$$\int_{x=y,y=0}^{y=a} H_t^2 dl = \sqrt{2}\int_{x=y,y=0}^{y=a} H_t^2 dx = 5\frac{\epsilon^2\omega^2\pi^2}{\gamma^4 a}$$

The line integral over all three sides in

$$\oint \left|\mathbf{H}\right|^2 dl = 10\frac{\epsilon^2\omega^2\pi^2}{\gamma^4 a}$$

Use $\gamma_{M,12} = \omega_{M,12}\sqrt{\mu\epsilon} = \sqrt{5}\frac{\pi}{a}$ and Eq. 8.51 and $\left|\frac{dP}{dz}\right| = \frac{1}{2\sigma\delta}\oint\left|\mathbf{H}\right|^2 dl$ and $\beta = \frac{1}{2P}\left|\frac{dP}{dz}\right|$ to find

$$\beta_{M,12} = \frac{1}{\sigma\delta_{M,12}}\sqrt{\frac{\epsilon}{\mu}}\sqrt{\frac{\omega}{\omega_{M,12}}}\frac{4}{a}\frac{1}{\sqrt{1-\frac{\omega_{E,01}^2}{\omega^2}}}$$

For the corresponding modes in a square guide, double the area integrals $\int\psi\psi^* dxdy$, and for the line integrals double the results over the vertical and horizontal sides of the triangular guide and leave out the diagonals.

$\underline{TM_{E,01}}$: The area integral doubles and the line integral becomes

$$\oint\left|\mathbf{H}\right|^2 dl = 4a + 2a\frac{\omega^2}{\omega_{E,01}^2}$$

The resultant damping constant is

$$\beta_{E,01,square} = \beta_{E,01,triangle} \times \frac{1}{2} \frac{2 + 4\frac{\omega_{E,01}^2}{\omega^2}}{(1+\sqrt{2}) + (2+\sqrt{2})\frac{\omega_{E,01}^2}{\omega^2}}$$

$\underline{TM_{12}}$: The area integral doubles and the line integral remains unchanged. Thus,

$$\beta_{M,12,square} = \beta_{M,12,triangle} \times \frac{1}{2}$$