1 Problem 13.9

Using Jackson's equation 13.50 and the fact that $n = \sqrt{\varepsilon(\omega)}$ yields:

$$\cos\theta_c = \frac{1}{n\beta} \tag{1}$$

Now, we know that $K = (\gamma - 1)mc^2 = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)mc^2$. Solving this expression for β yields:

$$\beta = \frac{\sqrt{K^2 + 2Kmc^2}}{K + mc^2} \tag{2}$$

Plugging this into equation (1) yields:

$$\boxed{\cos\theta_c = \frac{K + mc^2}{n\sqrt{K^2 + 2Kmc^2}}}$$

Jackson's equation 13.48 gives us:

$$\frac{dE}{dx} = \int \frac{z^2e^2}{c^2}\omega\left(1 - \frac{1}{n^2\beta^2}\right)d\omega$$

A single energy quantum (i.e., a photon) radiated will have energy $\hbar\omega$. Thus, the above equation can be rewritten to express the number of quanta emitted:

$$\begin{aligned}
\frac{dN}{dx} &= \int \frac{z^2e^2}{\hbar c^2}\left(1 - \frac{1}{n^2\beta^2}\right)d\omega \\
&= \frac{z^2e^2}{\hbar c^2}\left(1 - \frac{1}{n^2\beta^2}\right)[\omega_{\text{max}} - \omega_{\text{min}}] \\
&= \frac{z^2e^2}{\hbar c^2}\left(1 - \frac{1}{n^2\beta^2}\right)\left[\frac{2\pi c}{n\lambda_{\text{min}}} - \frac{2\pi c}{n\lambda_{\text{max}}}\right] \\
&= \frac{2\pi z^2e^2}{n\hbar c}\left(1 - \frac{1}{n^2\beta^2}\right)\left[\frac{1}{\lambda_{\text{min}}} - \frac{1}{\lambda_{\text{max}}}\right]
\end{aligned}$$

For $z = 1$ (since we're dealing with isolated particles), $n = 1.5$, and $\lambda_{\text{min}} = 4000$ Å, this equation becomes:

$$\frac{dN}{dx} = 283\left(1 - \frac{1}{1.5^2\beta^2}\right)\ \text{cm}^{-1} \tag{3}$$

Plugging $K = 1$ MeV and $mc^2 = 0.511$ MeV into equation (2) yields $\beta = 0.941$. Plugging this into equation (3) yields $dN/dx = 149$ photons per cm.

Plugging $K = 500$ MeV and $mc^2 = 938$ MeV into equation (2) yields $\beta = 0.758$. Plugging this into equation (3) yields $dN/dx = 64$ photons per cm.

Plugging $K = 5000$ MeV and $mc^2 = 938$ MeV into equation (2) yields $\beta = 0.987$. Plugging this into equation (3) yields $dN/dx = 154$ photons per cm.

2 Problem 14.4

2.1 Part a

$$\begin{aligned} \vec{z} &= a\cos(\omega_0 t)\,\hat{z} \\ \vec{v} &= -a\omega_0 \sin(\omega_0 t)\,\hat{z} \\ \implies \vec{\beta} &= -\frac{a\omega_0}{c}\sin(\omega_0 t)\,\hat{z} \\ \dot{\vec{\beta}} &= -\frac{a\omega_0^2}{c}\cos(\omega_0 t)\,\hat{z} \end{aligned}$$

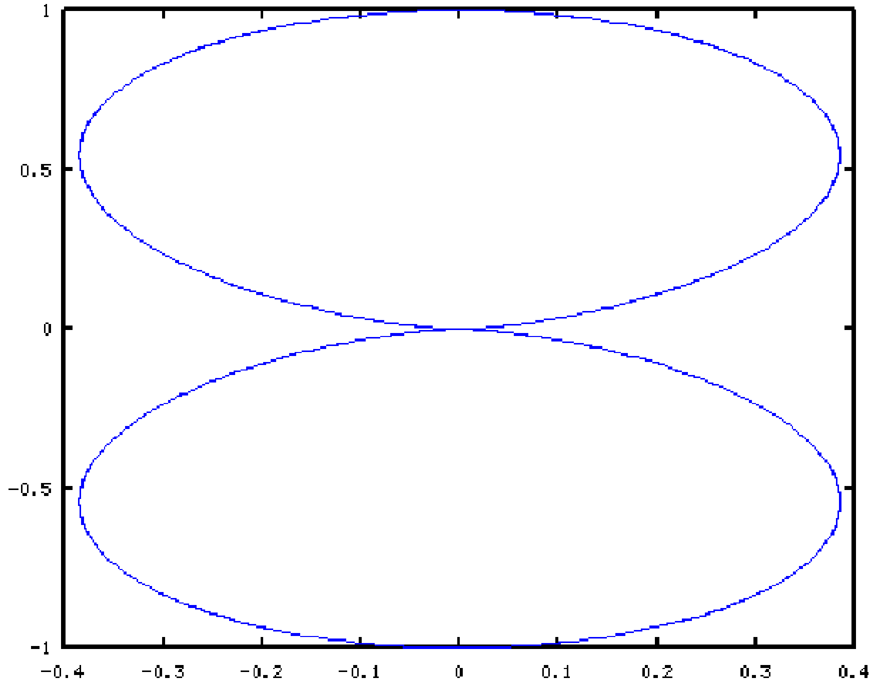
The observer is located at the zenith angle θ from the z-axis. Thus, the angle between $\hat{n}$ and $\dot{\vec{\beta}}$ is θ. Equation 14.20 becomes:

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{e^2}{4\pi c}\left|\hat{n}\times\left(\hat{n}\times\dot{\vec{\beta}}\right)\right|^2 \\ &= \frac{e^2}{4\pi c}\left|\hat{n}\times\dot{\vec{\beta}}\right|^2 \\ &= \frac{e^2}{4\pi c}\left|\dot{\vec{\beta}}\right|^2 \sin^2\theta \\ &= \frac{e^2 a^2 \omega_0^4}{4\pi c^3}\cos^2(\omega_0 t)\sin^2\theta \end{aligned}$$

The time average of $\cos^2(\omega_0 t)$ is $\frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\cos^2(\omega_0 t) = \frac{1}{2}$. Thus, the time average of $dP/d\Omega$ is:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2 a^2 \omega_0^4}{8\pi c^3}\sin^2\theta$$

The polar plot of this is shown below, where $\frac{e^2 a^2 \omega_0^4}{8\pi c^3}$ has been set to unity:



The total time-averaged power radiated can be determined by integrating the above expression over solid angle:

$$\begin{aligned}\langle P\rangle &= \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{e^2a^2\omega_0^4}{8\pi c^3}\sin^2\theta\sin\theta d\theta d\varphi \\ &= \frac{e^2a^2\omega_0^4}{3c^3}\end{aligned}$$

2.2 Part b

$$\begin{aligned}\vec{x} &= R\cos(\omega_0 t)\,\hat{x} + R\sin(\omega_0 t)\,\hat{y} \\ \vec{v} &= R\omega_0\sin(\omega_0 t)\,\hat{x} - R\omega_0\cos(\omega_0 t)\,\hat{y} \\ \Longrightarrow\quad \vec{\beta} &= \frac{R\omega_0}{c}\sin(\omega_0 t)\,\hat{x} - \frac{R\omega_0}{c}\cos(\omega_0 t)\,\hat{y} \\ \dot{\vec{\beta}} &= -\frac{R\omega_0^2}{c}\cos(\omega_0 t)\,\hat{x} - \frac{R\omega_0^2}{c}\sin(\omega_0 t)\,\hat{y}\end{aligned}$$

Because this system has azimuthal symmetry (when averaged over a full period, which is what we will do in the next step), we can rotate the coordinate system such that the observer lies in the x-z plane. Thus,

$$\hat{n} = \cos\theta\hat{x} + \sin\theta\hat{z}$$

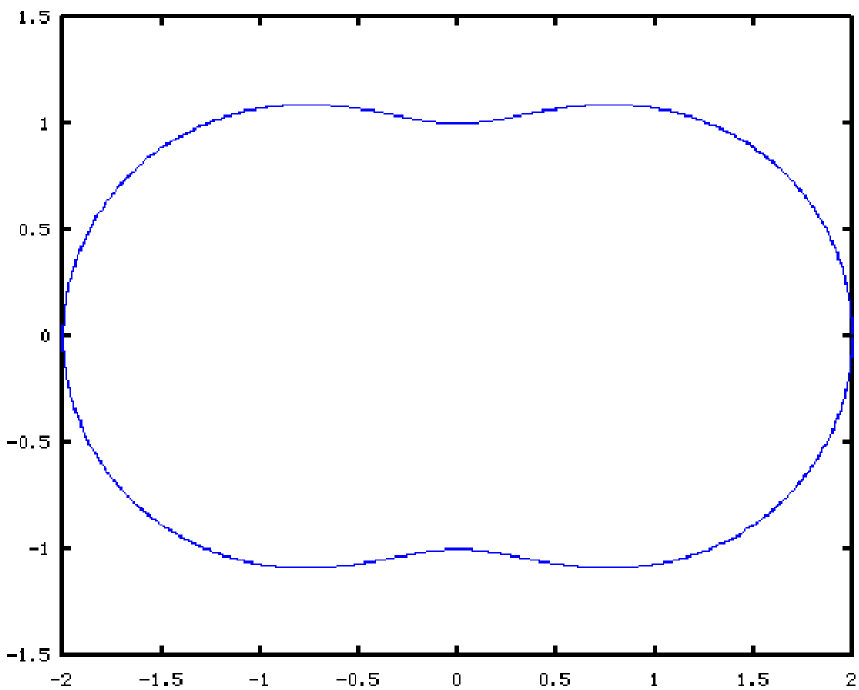
Note: here, θ is *not* the zenith angle, but the angle between the observer's position and the x-y plane. Equation 14.20 becomes:

$$
\begin{aligned}
\frac{dP}{d\Omega} &= \frac{e^2}{4\pi c}\left|\hat{n}\times\dot{\vec{\beta}}\right|^2 \\
&= \frac{e^2}{4\pi c}\left|\det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\theta & 0 & \sin\theta \\ -\frac{R\omega_0^2}{c}\cos(\omega_0 t) & -\frac{R\omega_0^2}{c}\sin(\omega_0 t) & 0 \end{bmatrix}\right|^2 \\
&= \frac{e^2}{4\pi c}\left|\frac{R\omega_0^2}{c}\sin(\omega_0 t)\cos\theta\hat{x} - \frac{R\omega_0^2}{c}\cos(\omega_0 t)\cos\theta\hat{y} - \frac{R\omega_0^2}{c}\sin(\omega_0 t)\sin\theta\right|^2 \\
&= \frac{R^2e^2\omega_0^4}{4\pi c^3}\left[\sin^2(\omega_0 t)\left(\cos^2\theta + \sin^2\theta\right) + \cos^2(\omega_0 t)\cos^2\theta\right] \\
&= \frac{R^2e^2\omega_0^4}{4\pi c^3}\left[\sin^2(\omega_0 t) + \cos^2(\omega_0 t)\cos^2\theta\right]
\end{aligned}
$$

The time average of $\cos^2(\omega_0 t)$ is $\frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\cos^2(\omega_0 t) = \frac{1}{2}$ while the time average of $\sin^2(\omega_0 t)$ is $\frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\sin^2(\omega_0 t) = \frac{1}{2}$. Thus, the time average of $dP/d\Omega$ is:

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{R^2e^2\omega_0^4}{8\pi c^3}\left(1+\cos^2\theta\right)$$

The polar plot of this is shown below, where $\frac{R^2e^2\omega_0^4}{8\pi c^3}$ has been set to unity:



The total time-averaged power radiated can be determined by integrating the above expression over solid angle:

$$\begin{aligned}
\langle P\rangle &= \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{R^2e^2\omega_0^4}{8\pi c^3}\left(1+\cos^2\theta\right)\sin\theta d\theta d\varphi \\
&= \frac{R^2e^2\omega_0^4}{4c^3}\left(2+\frac{2}{3}\right) \\
&= \frac{2R^2e^2\omega_0^4}{3c^3}
\end{aligned}$$

3 Problem 14.10

3.1 Part a

Suppose the velocity is in the z-direction. Then, $\vec{\beta}=\beta\hat{z}$ and $\dot{\vec{\beta}}=\dot{\beta}\hat{z}$, where $\dot{\beta}$ is defined as follows:

$$\dot{\beta}=\begin{cases} 0 & t<0 \\ -\frac{\beta_{\text{init}}}{\Delta t} & 0\le t\le \Delta t \\ 0 & t>\Delta t \end{cases}$$

The observer is located at the zenith angle θ from the z-axis. Thus, the angle between $\hat{n}$ and $\dot{\vec{\beta}}$ is θ. Equation 14.38 (we use this equation since we're not given that the motion is nonrelativistic) becomes:

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{e^2}{4\pi c}\frac{\left|\hat{n}\times\left[\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right|^2}{\left(1-\hat{n}\cdot\vec{\beta}\right)^5} \\
&= \frac{e^2}{4\pi c}\frac{\left|\hat{n}\times\left(\hat{n}\times\dot{\vec{\beta}}-\vec{\beta}\times\dot{\vec{\beta}}\right)\right|^2}{\left(1-\hat{n}\cdot\vec{\beta}\right)^5} \\
&= \frac{e^2}{4\pi c}\frac{\left|\hat{n}\times\dot{\vec{\beta}}\right|^2}{\left(1-\hat{n}\cdot\vec{\beta}\right)^5} \\
&= \frac{e^2}{4\pi c}\frac{\dot{\beta}^2\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}
\end{aligned}$$

Integrating the above expression with respect to time will yield $dE/d\Omega$:

$$\frac{dE}{d\Omega}=\frac{e^2}{4\pi c}\sin^2\theta\int_{-\infty}^{\infty}\frac{\dot{\beta}\dot{\beta}}{\left(1-\beta\cos\theta\right)^5}dt$$

Letting $u = \beta$, $du = \dot{\beta} dt$ and setting the remaining $\dot{\beta}$ term to its piecewise definition yields:

$$\begin{aligned}\frac{dE}{d\Omega} &= \frac{e^2}{4\pi c}\sin^2\theta \int_{u=\beta_{\text{init}}}^{0} \frac{(-\beta_{\text{init}}/\Delta t)}{(1-u\cos\theta)^5} du \\ &= \frac{e^2\beta_{\text{init}}^2}{16\pi c\Delta t}\frac{(2-\beta_{\text{init}}\cos\theta)\left[1+(1-\beta_{\text{init}})^2\right]\sin^2\theta}{(1-\beta_{\text{init}}\cos\theta)^4}\end{aligned}$$

3.2 Part b

For $\gamma \gg 1$, we only need to consider small values of θ. We note that $1-\beta\cos\theta$ occurs quite frequently in our expression for $dE/d\Omega$; this can be approximated as:

$$\begin{aligned}1-\beta\cos\theta &= 1 - \underbrace{\left(1-\frac{1}{\gamma^2}\right)^{1/2}}_{\approx 1-\frac{1}{2\gamma^2}} \underbrace{\cos\theta}_{\approx 1-\frac{1}{2}\theta^2} \\ &\approx 1 - \left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{1}{2}\theta^2\right) \\ &= 1 - \left(1 - \frac{1}{2\gamma^2} - \frac{1}{2}\theta^2 + \frac{1}{4}\underbrace{\frac{1}{\gamma^2}\theta^2}_{\approx 0}\right) \\ &\approx \frac{1}{2\gamma^2} + \frac{1}{2}\theta^2\end{aligned}$$

Substituting this into the result from part a yields:

$$\frac{dE}{d\Omega} \approx \frac{e^2\beta_{\text{init}}^2}{16\pi c\Delta t}\frac{\left(1+\frac{1}{2\gamma^2}+\frac{1}{2}\theta^2\right)\left[1+\left(\frac{1}{2\gamma^2}+\frac{1}{2}\theta^2\right)^2\right]\sin\theta}{\left(\frac{1}{2\gamma^2}+\frac{1}{2}\theta^2\right)^4}$$

Expanding and replacing $\gamma^2\theta^2$ with ξ yields:

$$\frac{dE}{d\Omega} \approx \frac{e^2\beta_{\text{init}}^2}{16\pi c\Delta t}\frac{16\gamma^8+4\gamma^4+8\xi\gamma^4+4\xi^2\gamma^4+8\gamma^6+2\gamma^2+6\xi\gamma^2+6\xi^2\gamma^2+8\xi\gamma^6+2\xi^3\gamma^2}{(1+\xi)^4}\underbrace{\sin^2\theta}_{\approx\theta^2}$$

Note that the γ^8 term will dominate. Hence,

$$\frac{dE}{d\Omega} \approx \frac{e^2\beta_{\text{init}}^2}{16\pi c\Delta t}\frac{16\gamma^8}{(1+\xi)^4}\theta^2$$

Letting $\theta^2 = \xi\gamma^{-2}$:

$$\frac{dE}{d\Omega} \approx \frac{e^2\beta_{\text{init}}^2}{\pi c\Delta t}\frac{\gamma^6}{(1+\xi)^4}\xi$$

We note that $d\Omega = 2\pi \sin\theta d\theta \approx 2\pi\theta d\theta$. Letting $\theta = \sqrt{\xi}\gamma^{-1}$ and $d\theta = \frac{1}{2}\xi^{-1/2}\gamma^{-1}$, we find that $d\Omega = \pi\gamma^{-2}$ Substituting this for $d\Omega$ in the above expression yields:

$$\boxed{\frac{dE}{d\xi} \approx \frac{e^2\beta_{\text{init}}^2\gamma^4}{c\Delta t}\frac{\xi}{(1+\xi)^4}}$$

$$\begin{aligned}\sqrt{\langle\theta^2\rangle} &= \sqrt{\langle\xi/\gamma^2\rangle} \\ &= \sqrt{\langle\xi\rangle}/\gamma \end{aligned} \tag{4}$$

$$\langle\xi\rangle = \frac{\int_0^\infty \xi\frac{dE}{d\xi}}{\int_0^\infty \frac{dE}{d\xi}} = \frac{1/3}{1/6} = 2$$

Plugging this into equation (4) yields:

$$\boxed{\sqrt{\langle\theta^2\rangle} = \sqrt{2}/\gamma}$$

Integrating our expression for $dE/d\xi$ with respect to ξ yields an expression for E:

$$\begin{aligned}E &= \frac{e^2\beta^2\gamma^4}{c\Delta t}\int_0^\infty \frac{\xi}{(1+\xi)^4} \\ &= \frac{e^2\beta^2\gamma^4}{c\Delta t}\left(\frac{1}{6}\right) \\ &= \frac{e^2\beta^2}{6c\Delta t}\left(1-\beta^2\right)^{-2}\end{aligned}$$

And differentiating this with respect to time gives the power:

$$\begin{aligned}P &= \frac{e^2\left(2\beta\dot{\beta}\gamma^4 - 2\beta^2\left(1-\beta^2\right)^{-3}2\beta\dot{\beta}\right)}{6c\Delta t} \\ &= \frac{e^2\left(2\beta\dot{\beta}\gamma^4 - 4\beta^3\dot{\beta}\gamma^6\right)}{6c\Delta t} \\ &= \frac{e^2\left(2\left(-\Delta t\dot{\beta}\right)\dot{\beta}\gamma^4 - 4\left(-\Delta t\dot{\beta}\right)^3\dot{\beta}\gamma^6\right)}{6c\Delta t} \\ &= \frac{e^2\left(-\dot{\beta}^2\gamma^4 + 2\Delta t^2\dot{\beta}^4\gamma^6\right)}{3c} \\ &= \frac{2e^2\dot{\beta}^2\gamma^6}{3c}\left(-\frac{1}{2\gamma^2} + \Delta t^2\dot{\beta}^2\right) \\ &\approx \frac{2e^2\dot{\beta}^2\gamma^6}{3c}\end{aligned}$$

which agrees with equation 14.43.

4 Problem 21

4.1 Part a

Using the Coulomb force law:

$$\begin{aligned} F &= \frac{kq_1q_2}{R^2} \\ m\frac{v^2}{R} &= \frac{kZe^2}{R^2} \\ \omega_0^2 = \frac{v^2}{R^2} &= \frac{kZe^2}{mR^3} \end{aligned}$$

Plugging this value for ω_0 into the result from problem 14.4.b yields:

$$\begin{aligned} P &= \frac{2R^2e^2\omega_0^4}{3c^3} \\ &= \frac{2R^2e^2}{3c^3}\left(\frac{kZe^2}{mR^3}\right)^2 \\ &= \frac{2k^2Z^2e^6}{3m^2R^4c^3} \end{aligned}$$

According to the problem statement, Bohr's correspondence principle states that $P = \hbar\omega_0/\tau \implies 1/\tau = P/\hbar\omega_0$:

$$\begin{aligned} \frac{1}{\tau} &= \frac{P}{\hbar\omega_0} \\ &= \frac{2k^2Z^2e^6}{3m^2R^4c^3\hbar\omega_0} \end{aligned} \tag{5}$$

Now, we use the Rydberg formula to find an expression for ω_0 in terms of n (in order to get our answer in the desired form):

$$\omega_0 = \frac{2\pi}{\lambda} = 2\pi R_{\mathrm{Ryd}}Z^2\left[-\Delta\left(\frac{1}{n^2}\right)\right]$$

Where $\Delta\left(\frac{1}{n^2}\right) = \frac{1}{n_2^2} - \frac{1}{n_1^2}$. But since n_1 and n_2 are close, $\Delta\left(\frac{1}{n^2}\right) \approx \frac{\partial}{\partial n}\left(\frac{1}{n^2}\right) = -\frac{2}{n^3}$. Hence:

$$\omega_0 = 4\pi R_{\mathrm{Ryd}}\frac{Z^2}{n^3}$$

where R_{Ryd} is the Rydberg constant: $R_{\mathrm{Ryd}} = \frac{me^4}{8\varepsilon_0^2h^3c}$. Substituting this into the above expression yields:

$$\omega_0 = 4\pi\frac{Z^2me^4}{8\varepsilon_0^2h^3cn^3}$$

Also, the allowed orbital radius is:

$$R = \frac{n^2\hbar^2}{Zke^2m}$$

Substituting the above two equations into equation (5) and simplifying yields:

$$\begin{aligned}
\frac{1}{\tau} &= 32k^6\pi^2\varepsilon_0^2\frac{Z^4e^{10}m}{3n^5\hbar^6c^2} \\
&= 16k^4\left(\frac{1}{4\pi\varepsilon_0}\right)^2\pi^2\varepsilon_0^2\frac{2}{3}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{\hbar c}\right)^4\frac{mc^2}{\hbar}\frac{1}{n^5} \\
&= \frac{2}{3}\frac{k^4e^2}{\hbar c}\left(\frac{Ze^2}{\hbar c}\right)^4\frac{mc^2}{\hbar}\frac{1}{n^5}
\end{aligned}$$

Converting to Gaussian units, we let $k = 1$. Moreover, the Rydberg constant does not have a c in the denominator, which means that we need to divide the above expression by an overall factor of c. Thus, $1/\tau$ becomes:

$$\frac{1}{\tau} = \frac{2}{3}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{\hbar c}\right)^4\frac{mc^2}{\hbar}\frac{1}{n^5}$$

4.2 Part b

Setting $Z = 1$ and substituting in the values of the physical constants, the result from part a becomes:

$$\begin{aligned}
&\frac{1}{\tau} \approx 1 \times 10^{10}\frac{1}{n^5} \\
\implies &\tau \approx 1 \times 10^{-10}n^5
\end{aligned}$$

	n	classical	quantum
$2p \to 1s$	2	3.2×10^{-9}	1.6×10^{-9}
$4f \to 3d$	4	1.0×10^{-7}	7.3×10^{-8}
$6h \to 5g$	6	7.8×10^{-7}	6.1×10^{-7}