1 Problem 12.7

1.1 Part a

We know that the $\vec{E}$ field from the particle is:

$$\vec{E} = q\frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3}$$

If we let $\vec{r'} = \vec{r} - \vec{r}_0$, this becomes:

$$\vec{E} = q\frac{\vec{r'}}{r'^3} = q\frac{x'\hat{x} + y'\hat{y} + z'\hat{z}}{(x'^2 + y'^2 + z'^2)^{3/2}}$$

Plugging this and $\vec{B} = -B\hat{z}$ into Jackson's equation 12.106:

$$\begin{aligned} P_{\text{field}} &= \frac{1}{4\pi c}\int \left(\vec{E} \times \vec{B}\right) d^3x \\ &= \frac{1}{4\pi c}\int_{x=-x_0}^{a-x_0}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty} \frac{q}{(x'^2 + y'^2 + z'^2)^{3/2}} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x' & y' & z' \\ 0 & 0 & -B \end{vmatrix} d^3x \\ &= -\frac{qB}{4\pi c}\int_{x=0-x_0}^{a-x_0}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty} \frac{y'\hat{x} - x'\hat{y}}{(x'^2 + y'^2 + z'^2)^{3/2}} d^3x \end{aligned}$$

By antisymmetry in y, we see that the x-component must be zero. Expresssing the y-component in polar coordinats in the y'-z' plane– that is, $\rho = \sqrt{y'^2 + z'^2}$:

$$P_{\text{field}} = \frac{qB}{4\pi c}\hat{y}\int_{x=-x_0}^{a-x_0}\int_{\rho=0}^{\infty}\int_{\theta=0}^{2\pi} \frac{x'dx'\rho d\rho d\theta}{(x'^2 + \rho^2)^{3/2}}$$

Evaluating this integral in Maple for the three regions yields:

$$P_{\text{field}} = \frac{qB}{c}\hat{y}\begin{cases} \frac{a}{2} & x_0 < 0 \\ \frac{a}{2} - x_0 & 0 < x_0 < a \\ -\frac{a}{2} & x_0 > a \end{cases}$$

The conjugate momentum of the particle comes from Jackson's equation 12.14:

$$\vec{P}_{\text{particle}} = \gamma\vec{m}\vec{v} + \frac{q}{c}\vec{A}$$

Because $\nabla \times \vec{A} = \vec{B}$ and $\vec{B} = -B\hat{z}$, $\vec{A} = -xB\hat{y}$. At $x = x_0$, $\vec{P}_{\text{particle}}$ becomes:

$$\vec{P}_{\text{particle}} = \gamma\vec{m}\vec{v} - \frac{qx_0B}{c}\hat{y} \tag{1}$$

Summing $\vec{P}_{\text{field}}$ and $\vec{P}_{\text{particle}}$ gives $\vec{G}$:

$$\boxed{\vec{G} = \gamma \vec{m}\vec{v} - \frac{qx_0 B}{c}\hat{y} + \frac{qB}{c}\hat{y}\begin{cases} \frac{a}{2} & x_0 < 0 \\ \frac{a}{2} - x_0 & 0 < x_0 < a \\ -\frac{a}{2} & x_0 > a \end{cases}}$$

Note that our answer is not gauge invariant. The $\vec{A}$ we chose is not the only choice in gauge which yields $\vec{B} = -B\hat{z}$. Because our answer depends on this choice in gauge, it is not gauge invariant.

1.2 Part b

$$F = \frac{\gamma m v^2}{r} = \frac{q}{c}\left|\vec{v} \times \vec{B}\right| = \frac{q}{c} vB$$
$$\implies \frac{\gamma m v^2}{r} = \frac{q}{c} vB$$

Solving this for r and plugging it into the y-component of equation (1) yields:

$$\begin{aligned} \left[P_{\text{particle}}\right]_y &= \gamma m v \sin\theta - \frac{qx_0 B}{c} \\ &= \gamma m v \left(\frac{a}{r}\right) - \frac{qx_0 B}{c} \\ &= \gamma m v a \left(\frac{qB}{\gamma c m v}\right) - \frac{qx_0 B}{c} \\ &= \frac{qaB}{c} - \frac{q(a)B}{c} = 0 \end{aligned}$$

where we've used the fact that $\sin\theta = a/r$, where θ is the deflection angle. We've also used the fact that $\vec{A}$ is constant in the region $x > a$, which is why we're able to make the substitution $x_0 = a$ in the last step.

Note that we set up the system such that the canonical momentum of the particle in the y-direction was initially zero. Thus, the canonical momentum in the y-direction is conserved. However, the mechanical momentum is not conserved– initially, the mechanical momentum was zero, but it grows to qaB/c after exiting the magnetic field.

The momentum of the electromagnetic field will not be conserved, either, by conservation of $\vec{G}$.

1.3 Part c

In this case, the particle is trapped in the region where the field exists because the particle will continuously revolve around within the field since it doesn't have enough momentum to leave. So, we should pick a symmetric gauge. We note that a symmetric $\vec{A}$ exists which gives $\vec{B} = -B\hat{z}$

$$\vec{A} = -\frac{B}{2}\left(x\hat{y} + y\hat{x}\right)$$

Following the same arguments we did in the previous part with this new value of $\vec{A}$ yields:

$$\begin{aligned}
\left[P_{\text{particle}}\right]_y &= \gamma m v \sin\theta - \frac{q x_0 B}{2c} \\
&= \gamma m v \left(\frac{a/2}{r}\right) - \frac{q x_0 B}{2c} \\
&= \gamma m v a \left(\frac{qB}{\gamma 2 c m v}\right) - \frac{q x_0 B}{2c} \\
&= \frac{qaB}{2c} - \frac{q(a)B}{2c} = 0
\end{aligned}$$

By conservation of canonical momentum, canonical momentum of the particle in the y-direction was initially zero (as was the case in the previous part). Because our choice in $\vec{A}$ is now symmetric, we aren't seeing the mechanical momentum growing, but it's now conserved as well.

Finally, the momentum of the electromagnetic field is also conserved by conservation of $\vec{G}$.

2 Problem 12.9 (part a only)

Substituting $\hat{n} = \hat{r}$ into equation 5.56 (remembering to convert to Jackson's new choice of units for this half of the book):

$$\begin{aligned}
\vec{B}(\vec{r}) &= -\frac{1}{r^3}\left[3\hat{r}\left(\hat{r}\cdot\vec{M}\right) - \vec{M}\right] \\
&= -\frac{1}{r^3}\left[3M\cos\theta\hat{r} - M\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)\right] \\
&= -\frac{M}{r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right] \qquad (2)
\end{aligned}$$

We know that the line element $d\vec{s}$ is:

$$d\vec{s} = dr\hat{r} + r d\theta\hat{\theta} + r\sin\varphi d\varphi\hat{\varphi}$$

Along a line of differential force, $d\vec{s}$ is parallel to $\vec{B}$. Hence,

$$\begin{aligned}
\frac{dr}{r d\theta} &= \frac{B_r}{B_\theta} \\
&= \frac{2\cos\theta}{\sin\theta}
\end{aligned}$$

$$\frac{dr}{r} = 2\frac{\cos\theta}{\sin\theta}d\theta$$
$$= 2\frac{d(\sin\theta)}{\sin\theta}$$
$$\int\frac{dr}{r} = 2\int\frac{du}{u}$$

where $u = \sin\theta$.

$$\implies \ln(r) = 2\ln(\sin\theta) + C$$
$$\boxed{r = r_0\sin^2\theta}$$

Plugging this into equation (2) yields:

$$\vec{B}(\vec{r}) = -\frac{M}{r_0^3\sin^6\theta}\left[2\cos\theta\hat{r} - \sin\theta\hat{\theta}\right]$$
$$\left|\vec{B}(\vec{r})\right| = \frac{M}{r_0^3\sin^6\theta}\sqrt{\left[4\cos^2\theta + \sin^2\theta\right]}$$
$$= \boxed{\frac{M}{r_0^3\sin^6\theta}\sqrt{3\cos^2\theta + 1}} \tag{3}$$

3 Problem 12.10

Substituting equation (3) into Jackson's equation 12.72 yields (at time $t = 0$):

$$v_{||}^2 = v_0^2 - v_{\perp,0}^2\frac{B}{B_0}$$
$$= \left(v_{||,0}^2 + v_{\perp,0}^2\right) - v_{\perp,0}^2\frac{M\sqrt{3\cos^2\theta + 1}}{B_0 r_0^3\sin^6\theta}$$

When the particle reaches latitude λ, it will be at a turning point– hence, $v_{||} = 0$:

$$\frac{v_{||,0}^2}{v_{\perp,0}^2} = \frac{M\sqrt{3\sin^2\alpha + 1}}{B_0 r_0^3\cos^6\lambda} - 1$$
$$\tan^2\alpha = \frac{M\sqrt{3\sin^2\lambda + 1}}{B_0 r_0^3\cos^6\lambda} - 1$$

where we've used the fact that $\theta = \frac{\pi}{2} - \alpha$.

Finally, we note that equation (2) yields $B = M/r_0^3$ at $\theta = \pi/2$. Substituting this value for B_0 into the above equation and solving for α yields:

$$\alpha = \arctan\left[\left(\frac{\sqrt{3\sin^2\lambda + 1}}{\cos^6\lambda} - 1\right)^{1/2}\right]$$

From problem 9, we know that $r = r_0 \sin^2\theta$. In terms of R, R_0, and λ, this is $R_0 = R\cos^2\alpha$. Hence,

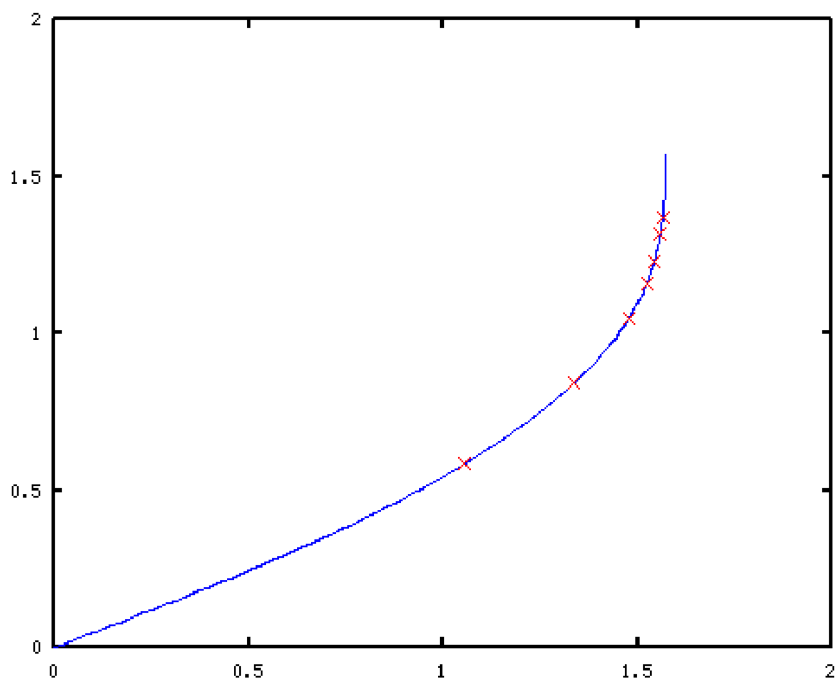
$$\lambda = \arccos\left(\sqrt{\frac{R_0}{R}}\right)$$

A plot of λ vs. a with the values of α marked for $R/R_0 = 1.2, 1.5, 2.0, 2.5, 3, 4, 5$ was generated using the following Matlab code:

```
lambda = 0:0.01:pi/2;
alpha = atan(sqrt(sqrt(3*sin(lambda).^2 + 1)./cos(lambda).^6 - 1));
RR0 = [1.2, 1.5, 2.0, 2.5, 3, 4, 5];
lambda_RR0 = asec(RR0);
alpha_RR0 = atan(sqrt(sqrt(3*sin(lambda_RR0).^2 + 1)./cos(lambda_RR0).^6 - 1));

plot(alpha,lambda,'b',alpha_RR0,lambda_RR0,'rx');
```

The resulting lot is shown below.



4 Problem 12.14

4.1 Part a

Using Jackson's equation 11.73 ($x^\alpha = g_{\alpha\beta} x^\beta$):

$$\begin{aligned}
\partial_\alpha A_\beta \partial^\alpha A^\beta &= g_{\alpha\mu} g_{\beta\nu} \partial^\alpha A^\beta \partial^\mu A^\nu \\
\frac{\partial}{\partial(\partial^\alpha \partial^\beta)} \left(\partial_\alpha A_\beta \partial^\alpha A^\beta\right) &= g_{\alpha\mu} g_{\beta\nu} \left(\partial^\mu A^\nu + \partial^\alpha A^\beta \delta_{\alpha\mu} \delta_{\mu\nu}\right) \\
&= 2\partial^\alpha A^\beta
\end{aligned}$$

Thus, using the following for our Lagrangian density:

$$\mathcal{L} = -\frac{1}{8\pi} \partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha$$

We get:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial^\alpha A^\beta)} &= -\frac{1}{4\pi} \partial^\alpha A^\beta \\
\frac{\partial \mathcal{L}}{\partial A^\beta} &= -\frac{1}{c} J_\beta
\end{aligned}$$

Thus, the Euler-Lagrange equation becomes:

$$\begin{aligned}
\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial^\alpha A^\beta)} &= \frac{\partial \mathcal{L}}{\partial A^\alpha} \\
\partial_\alpha \left(-\frac{1}{4\pi} \partial^\alpha A^\beta\right) &= -\frac{1}{c} J_\beta \\
\partial_\alpha \partial^\alpha A^\beta &= \frac{4\pi}{c} J_\beta \\
\partial^\alpha \partial_\alpha A_\beta &= \frac{4\pi}{c} J_\beta \qquad (4)
\end{aligned}$$

Substituting equation 11.136 into equation 11.141 yields:

$$\begin{aligned}
\partial^\alpha F_{\alpha\beta} &= \frac{4\pi}{c} J_\beta \\
\partial^\alpha \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) &= \frac{4\pi}{c} J_\beta \\
\partial^\alpha \partial_\alpha A_\beta - \partial_\beta \partial^\alpha A_\alpha &= \frac{4\pi}{c} J_\beta
\end{aligned}$$

This reduces to equation (4) when $\boxed{\partial^\alpha A_\alpha = 0}$. This is the condition for which the above Euler-Lagrange equation reduces to the Maxwell equations.

4.2 Part b

Taking the difference between the Lagrangian density we were given in this problem and equation 12.85 yields:

$$\begin{aligned}\Delta\mathcal{L} &= \frac{1}{16\pi}\left(F_{\alpha\beta}F^{\alpha\beta} - \partial_\alpha A_\beta \partial^\alpha A^\beta\right)\\ &= \frac{1}{16\pi}\left[\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) - \partial_\alpha A_\beta \partial^\alpha A^\beta\right]\\ &= \frac{1}{16\pi}\left[\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\beta A_\alpha \partial^\alpha A_\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha + \partial_\beta A_\alpha \partial^\beta A^\alpha - \partial_\alpha A_\beta \partial^\alpha A^\beta\right]\\ &= -\frac{1}{16\pi}\partial_\beta A_\alpha \partial^\alpha A_\beta \qquad (5)\end{aligned}$$

Using the product rule, we know that:

$$\begin{aligned}\partial_\beta\left(A_\alpha \partial^\alpha A^\beta\right) &= \partial_\beta A_\alpha \partial^\alpha A^\beta + A_\alpha \partial_\beta \partial^\alpha A^\beta\\ &= \partial_\beta A_\alpha \partial^\alpha A^\beta + A_\beta \partial^\alpha \partial_\beta A^\beta\end{aligned}$$

Using the Lorentz condition that $\partial_\beta A^\beta$, $\partial_\beta\left(A_\alpha \partial^\alpha A^\beta\right) = \partial_\beta A_\alpha \partial^\alpha A^\beta$. Substituting this into equation (5) yields:

$$\Delta\mathcal{L} = -\frac{1}{16\pi}\partial_\beta\left(A_\alpha \partial^\alpha A_\beta\right)$$

Thus, the difference between the Lagrangian densities is equal to a 4-divergence.

From equation 12.84, we know that the action is:

$$A = \int \mathcal{L} d^4x$$

Thus, the difference in the actions between these two Lagrangian densities is:

$$\Delta A = \int \partial_\beta\left(A_\alpha \partial^\alpha A_\beta\right) d^4x$$

Because the integral over all space of a divergence is zero, the action is unaffected.