1 Problem 11.6

1.1 Part a

We begin by differentiating the velocity addition formula (we will assume the space ship is traveling parallel to the Earth):

$$\begin{aligned}
u &= \frac{u' + v}{1 + \frac{u'v}{c^2}} \\
\frac{du}{dt} &= \frac{\left(1 + \frac{u'v}{c^2}\right)\left(\frac{du'}{dt} + \cancelto{0}{\frac{dv}{dt}}\right) - (u' + v)\frac{1}{c^2}\left(\frac{du'}{dt}v + u'\cancelto{0}{\frac{dv}{dt}}\right)}{\left(1 + \frac{u'v}{c^2}\right)^2} \\
&= \frac{1 - \frac{v^2}{c^2}}{\left(1 + \frac{u'v}{c^2}\right)^2}\frac{du'}{dt'}\underbrace{\frac{dt'}{dt}}_{1/\gamma}
\end{aligned}$$

Letting $u' = 0$ because the ship is, by definition, at rest in its own instantaneous reference frame:

$$\frac{du}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{3/2}\frac{du'}{dt'}$$

Letting $u = v$ because the space ship's velocity (according to Earth) is equal to the velocity of the space ship's reference frame relative to Earth's:

$$\begin{aligned}
\frac{dv}{dt} &= \left(1 - \frac{v^2}{c^2}\right)^{3/2}\frac{dv'}{dt'} \\
\int \frac{dv}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} &= \int \frac{dv'}{dt'}dt
\end{aligned}$$

Given that $dv'/dt' = g =$ constant,

$$\frac{v}{\sqrt{1 - \frac{v^2}{v^2}}} = gt$$

Solving for v yields:

$$v = \frac{gt}{\sqrt{1 + \frac{g^2t^2}{c^2}}} \tag{1}$$

Now, we integrate both sides of the equation for time dilation:

$$\begin{aligned}
\int dt' &= \int \frac{dt}{\gamma} \\
t' &= \int \sqrt{1 - \frac{v^2}{c^2}}dt
\end{aligned}$$

and substitute in equation (1):

$$t' = \int \sqrt{1 - \frac{g^2t^2}{c^2\left(1+\frac{g^2t^2}{c^2}\right)}}dt$$

$$= \int \left(1+\frac{g^2t^2}{c^2}\right)^{-1/2} dt$$

Looking up this integral in a table, we find that it is equal to:

$$t' = \frac{c}{g}\operatorname{arcsinh}\left(\frac{gt}{c}\right)$$

Solving for t in terms of t' yields:

$$t = \frac{c}{g}\sinh\left(\frac{gt'}{c}\right)$$

For the first leg of the journey, $t' = 5$ years. In addition, $g = 9.86$ m/s^2, and $c = 3 \times 10^8$ m/s:

$$t = \frac{3\times 10^8}{9.86}\sinh\left(\frac{(9.86)(5\times 3.16\times 10^7)}{3\times 10^8}\right) \times \frac{1}{3.16\times 10^7}$$

$$= 86 \text{ years}$$

The total journey is 4×70 years $= 344$ years. Hence, it is the year $2100 + 344 = \boxed{2444}$.

1.2 Part b

In the first two legs, the rocket ship traveled:

$$d = 2\int_0^T v dt$$

$$= 2\int_0^{5\times 3.16\times 10^7} \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}}dt$$

$$= 7.83 \times 10^{16} \text{ m}$$

2 Problem 11.11

Letting $A_1 = e^{\lambda L}$ and $A_2 = e^{\lambda(L+\delta L)}$,

$$A(\lambda) = A_2 A_1^{-1} = e^{\lambda(L+\delta L)}e^{-\lambda L} \quad (2)$$

We want to prove that to first order in δL,

$$A(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda}{n!} \Omega_n(L, \delta L) \tag{3}$$

where $\Omega_0 = I$, $\Omega_1(L, \delta L) = \delta L$ and $\Omega_n(L, \delta L) = [L, \Omega_{n-1}(L, \delta L)]$ for $n \geq 2$. Replacing the left hand side of equation (3) with the Taylor series of $A(\lambda)$ yields:

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} A^{(n)}(0) = \sum_{n=0}^{\infty} \frac{\lambda}{n!} \Omega_n(L, \delta L)$$

Hence, it is sufficient to prove:

$$A^{(n)}(0) = \Omega_n(L, \delta L)$$

We will prove this by induction.

We begin by showing that this holds for $n = 0$ and $n = 1$:

$$\begin{aligned} A^{(0)}(0) &= I \quad \checkmark \\ A^{(1)}(0) &= (L + \delta L) - L = \delta L \quad \checkmark \end{aligned}$$

Now, we assume that to first order, $A^{(n)}(0) = \Omega_n(L, \delta L)$ in order to prove that to first order, $A^{(n+1)}(0) = \Omega_{n+1}(L, \delta L)$. Now, we assume $f^{(n)}(0) = \Omega_n(A, B)$ in order to prove that $f^{(n+1)}(0) = \Omega_{n+1}(A, B)$ (induction).

$$\begin{aligned} & A^{(n)}(0) = \Omega_n(A, B) = [L, \Omega_{n-1}(L, \delta L)] \\ \implies & A^{(n)}(0) = \left[L, A^{(n-1)}(0)\right] = LA^{(n-1)}(0) - A^{(n-1)}L \end{aligned}$$

Taking the derivative of both sides,

$$\begin{aligned} A^{(n+1)}(0) &= LA^{(n)}(0) - A^{(n)}L = \left[L, A^{(n)}(0)\right] \\ &= [L, \Omega_n(L, \delta L)] \\ &= \Omega_{n+1}(L, \delta L) \end{aligned}$$

Hence, we have proved equation (3) by induction.

$$A^{(n)}(0) = \Omega_n(L, \delta L)$$

Letting $\lambda = 1$ in equation (3) and expanding the summation yields:

$$A = I + \delta L + \frac{1}{2!} [L + \delta L] + \frac{1}{3!} [L + [L + \delta L]] + \dots$$

3 Problem 11.14

3.1 Part a

3.1.1 $F^{\alpha\beta}F_{\alpha\beta}$

We solve for the Lorentz scalar $F^{\alpha\beta}F_{\alpha\beta}$:

$$\begin{aligned} F^{\alpha\beta}F_{\alpha\beta} &= -F^{\alpha\beta}F_{\beta\alpha} \\ &= -F^{\alpha\beta}F_{\beta\gamma}\delta^{\alpha}_{\gamma} \end{aligned}$$

We know that $F^{\alpha\beta}F_{\beta\gamma}$ is index notation for matrix multiplication and δ^{α}_{γ} takes the trace of the resulting multiplication. Hence:

$$F^{\alpha\beta}F_{\alpha\beta} = -\text{trace}\left(\boldsymbol{F^{\alpha\beta}}\boldsymbol{F_{\beta\alpha}}\right) \tag{4}$$

where $\boldsymbol{F^{\alpha\beta}}$ is defined in Jackson's equation 11.137:

$$\boldsymbol{F^{\alpha\beta}} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

To find $\boldsymbol{F_{\alpha\beta}}$, we invert the signs on only the first row and first column (or, just using Jackson's equation 11.138 also works):

$$\boldsymbol{F_{\alpha\beta}} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

Plugging these matrices into equation (4) yields:

$$\begin{aligned} F^{\alpha\beta}F_{\alpha\beta} &= -\text{trace}\left(\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}\begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}\right) \\ &= 2B_x^2 + 2B_y^2 + 2B_z^2 - 2E_x^2 - 2E_y^2 - 2E_z^2 \\ &= 2\left(|\mathbf{B}|^2 - |\mathbf{E}|^2\right) \end{aligned}$$

3.1.2 $\mathcal{F}^{\alpha\beta}F_{\alpha\beta}$

Next, we solve for the Lorentz scalar $\mathcal{F}^{\alpha\beta}F_{\alpha\beta}$:

$$\begin{aligned} \mathcal{F}^{\alpha\beta}F_{\alpha\beta} &= -\mathcal{F}^{\alpha\beta}F_{\beta\alpha} \\ &= -\mathcal{F}^{\alpha\beta}F_{\beta\gamma}\delta^{\alpha}_{\gamma} \end{aligned}$$

$$\mathcal{F}^{\alpha\beta}F_{\alpha\beta} = -\text{trace}\left(\boldsymbol{\mathcal{F}^{\alpha\beta}F_{\beta\alpha}}\right) \tag{5}$$

where $\boldsymbol{\mathcal{F}^{\alpha\beta}}$ is defined in equation 11.140:

$$\boldsymbol{\mathcal{F}^{\alpha\beta}} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

To find $\boldsymbol{F_{\alpha\beta}}$, we invert the signs on only the first row and first column:

$$\mathbf{F}_{\alpha\beta} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix}$$

Plugging the matrices $\boldsymbol{\mathcal{F}^{\alpha\beta}}$ and $\boldsymbol{F_{\alpha\beta}}$ into equation (5) yields:

$$\begin{aligned} F^{\alpha\beta}F_{\alpha\beta} &= -\text{trace}\left(\begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}\begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}\right) \\ &= -4B_xE_x - 4B_yE_y - 4B_zE_z \\ &= -4\mathbf{E}\cdot\mathbf{B} \end{aligned}$$

3.1.3 $\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}$

Finally, we solve for the Lorentz scalar $\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}$:

$$\begin{aligned} \mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} &= -\mathcal{F}^{\alpha\beta}\mathcal{F}_{\beta\alpha} \\ &= -\mathcal{F}^{\alpha\beta}\mathcal{F}_{\beta\gamma}\delta^{\alpha}_{\gamma} \end{aligned}$$

$$\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} = -\text{trace}\left(\boldsymbol{\mathcal{F}^{\alpha\beta}\mathcal{F}_{\beta\alpha}}\right) \tag{6}$$

Plugging the matrices $\boldsymbol{\mathcal{F}^{\alpha\beta}}$ and $\boldsymbol{F_{\alpha\beta}}$ into equation (6) (defined above) yields:

$$\begin{aligned} F^{\alpha\beta}F_{\alpha\beta} &= -\text{trace}\left(\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}\begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix}\right) \\ &= 2E_x^2 + 2E_y^2 + 2E_z^2 - 2B_x^2 - 2B_y^2 - 2B_z^2 \\ &= 2\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) \end{aligned}$$

Because $|\mathbf{B}|^2 - |\mathbf{E}|^2$, $|\mathbf{E}|^2 - |\mathbf{B}|^2$, and $\mathbf{E}\cdot\mathbf{B}$ are the only three possible ways we can combine vectors $\mathbf{E}$ and $\mathbf{B}$ to produce scalars which are quadratic in $\mathbf{E}$ and $\mathbf{B}$, there are not any other invariants quadratic in the field strengths $\mathbf{E}$ and $\mathbf{B}$.

3.2 Part b

No. Proof by contradiction:

Assume that there exists one reference frame in which we see a purely electric field (i.e., $|\mathbf{B}| = 0$) and another frame where we see a purely magnetic field (i.e., $|\mathbf{E}| = 0$), disregarding the trivial case where $|\mathbf{E}| = |\mathbf{B}| = 0$.

We have just shown that $|\mathbf{E}|^2 - |\mathbf{B}|^2$ is an invariant– that is, it must remain constant across all inertial frames. This invariant is equal to $|\mathbf{E}|^2$ in the frame where $|\mathbf{B}| = 0$ and it's equal to $-|\mathbf{B}|^2$ in the frame where $|\mathbf{E}| = 0$. Hence:

$$|\mathbf{E}|^2 = -|\mathbf{B}|^2$$

which is impossible since the square of a magnitude of a vector cannot be negative. So, **no**, it is not possible to have an electromagnetic field which appears as a purely electric field in one inertial frame and appears as a purely magnetic field in another inertial frame.

Let S be a reference frame where there exists a nonzero electric field and let S' be a reference frame where the electric field vanishes. Equating the invariants between these two fields yields:

$$|\mathbf{B}|^2 - |\mathbf{E}|^2 = |\mathbf{B}'|^2 \implies |\mathbf{E}|^2 = |\mathbf{B}|^2 - |\mathbf{B}'|^2$$
$$\mathbf{E} \cdot \mathbf{B} = 0$$

$$\boxed{\implies \begin{cases} |\mathbf{E}|^2 < |\mathbf{B}|^2 \\ \mathbf{E} \cdot \mathbf{B} = 0 \end{cases}}$$