9.1 Problem 9.1

9.1.1 Part a

The general solution for the vector potential in Lorentz gauge with source $\vec{J}(\vec{x},t)$ is:

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \int dt' \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\delta\left(t' + \frac{|\vec{x}-\vec{x}'|}{x} - t\right)$$

Then if we look at a Fourier-transformed (in time) field,

$$\begin{aligned}
\vec{A}(\vec{x},\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \vec{A}(\vec{x},t)e^{i\omega t} \\
&= \frac{1}{2\pi}\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} dt e^{i\omega t}\int d^3x' \int dt' \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\delta\left(t' + \frac{|\vec{x}-\vec{x}'|}{c} - t\right) \\
&= \frac{\mu_0}{8\pi^2}\int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\int dt' e^{i\omega t'}\vec{J}(\vec{x}',t') \\
&= \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\vec{J}(\vec{x}',\omega)
\end{aligned}$$

where $k = \omega/c$ and we have made no assumptions about the source.

9.1.2 Part b

If we have a charge q rotating in a circle of radius R about the z axis, it is easiest to use cylindrical polar coordinates, with the charge at $\vec{x}_q$ with coordinates $\rho = R$, $\varphi = \omega_0 t$, $z = 0$. The current density is

$$\vec{J}(\vec{x}',\omega) = \frac{1}{2\pi}qR\omega_0\frac{\delta(\rho - R)}{\rho}\delta(z)\int_{-\infty}^{\infty} dt\hat{e}_\varphi\delta(\varphi - \omega_0 t)e^{i\omega t}$$

A little care is needed in converting $\delta(\varphi - \omega_0 t)$ to a delta function in t, because it requires its argument to be zero **modulo** 2π, so

$$\delta(\varphi - \omega_0 t) = \frac{1}{\omega_0}\sum_{n\in\mathcal{Z}}\delta\left(\frac{t - (\varphi + 2\pi n)}{\omega_0}\right)$$

Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} dt\hat{e}_\varphi\delta(\varphi - \omega_0 t)e^{i\omega t} &= \frac{1}{\omega_0}\hat{e}_\varphi e^{i\omega\varphi/\omega_0}\sum_{n\in\mathcal{Z}} e^{2\pi i n\omega/\omega_0} \\
&= 2\pi\hat{e}_\varphi e^{i\omega\varphi/\omega_0}\sum_{m\in\mathcal{Z}}\delta(\omega - m\omega_0)
\end{aligned}$$

Now from the expansion of the Green's function we have

$$\vec{A}(\vec{x},\omega) = i\mu_0 q\omega_0 k \sum_{\ell m'} \int_0^\infty r'^2 dr' \delta(r'-R) j_\ell(kr_<) h_\ell^{(1)}(kr_>) Y_{\ell m'}(\theta,\varphi)$$
$$\cdot \sum_m \delta(\omega - m\omega_0) \int d\Omega' \delta(r'\cos\theta') \hat{e}'_\varphi e^{im\varphi'} Y^*_{\ell m'}(\theta',\varphi')$$

The angular integral restricts the values of ℓ and m' which can contribute. We have

$$\int d\Omega' \delta(r'\cos\theta') \hat{e}'_\varphi e^{im\varphi'} Y^*_{\ell m'}(\theta',\varphi') = \sqrt{\frac{(2\ell+1)}{4\pi r'} \frac{(\ell-m')!}{(\ell+m')!}}$$
$$\times \int_0^{2\pi} d\varphi' P_\ell^{m'}\left(\cos(\pi/2)\right) e^{-im'\varphi'} e^{im\varphi'} (-sin\varphi' \hat{e}_x + \cos\varphi' \hat{e}_y)$$
$$= \sqrt{\frac{(2\ell+1)\pi}{4} \frac{(\ell-m')!}{(\ell+m')!}} P_\ell^{m'}(0) \left(\hat{e}_x - i(m'-m)\hat{e}_y\right) \qquad (1)$$
$$\times \left[\delta(m'-m-1) + \delta(m'-m+1)\right]$$

Note that the φ' integral forces $m - m' = \pm 1$, so higher frequency modes, with $\omega = m\omega_0$, can only enter with $\ell \geq m-1$. Furthermore, as $Y_{\ell m'}(\pi/2, \varphi') = 0$ for $\ell - m'$ odd, even values of m have only odd values of ℓ, and vice-versa.

For $\ell = 0$ we have only $\omega = \omega_0$, and $P_0^0 = 1$, so (for $r > R$)

$$\vec{A}^{(0)}(\vec{x},\omega) = i\mu_0 q\omega_0 k \delta(\omega \pm \omega_0) R j_0(kR) h^{(1)}(kr) Y_{00} \frac{\pi}{4} P_0^0(0)(\hat{e}_x \mp i\hat{e}_y)$$
$$= i\mu_0 q\omega_0 k \delta(\omega \pm \omega_0) R \frac{\sin(kR)}{kR} \frac{e^{ikr}}{ikr} \frac{1}{\sqrt{4\pi}} \frac{\pi}{4} \cdot 1 \left(\hat{e}_x \mp i\hat{e}_y\right)$$
$$= \mu_0 qc \delta(\omega \pm \omega_0) \sin(\omega_0 R/c) \frac{e^{i\omega_0 r/c}}{r} \frac{\sqrt{\pi}}{8} \left(\hat{e}_x \mp i\hat{e}_y\right)$$

For $\ell = 1$, we have no contribution at w, as the magnetic dipole moment $\vec{m} = \frac{1}{2}\int \vec{x} \times \vec{J} = q\omega_0 R^2 \hat{e}_z$ is a constant, and the quadrupole term (and M_E),

$$\int x_\alpha x_\beta \rho(\vec{x}) = \begin{bmatrix} \cos^2(\omega_0 t) & \cos(\omega_0 t)\sin(\omega_0 t) \\ \sin(\omega_0 t)\cos(\omega_0 t) & \sin^2(\omega_0 t) \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 1+\cos(2\omega_0 t) & \sin(2\omega_0 t) \\ \sin(2\omega_0 t) & 1+\cos(2\omega_0 t) \end{bmatrix}$$

which has only components at $\omega = 2\omega_0$. From (1), for $m = 2$, we get a factor of

$$\frac{1}{r'} \sqrt{\frac{3\pi}{4} \frac{0!}{2!}} P_1^1(0)(\hat{e}_x i\hat{e}_y)$$

We have $j_1(kR) = \frac{\sin(kR)}{k^2R^2} = \frac{\cos(kR)}{kR}$, $Y_{11}(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$, $P_1^1(0) = -1$, $h_1^{(1)}(kr) = -\frac{e^{ikr}}{kr}\left(1+\frac{i}{kr}\right)$, and $k = 2\omega_0/c$. So,

$$\begin{aligned}\vec{A}^{(1)}(\vec{x},2\omega_0) &= i\mu_0 q\omega_0 kR^2\left(\frac{\sin(kR)}{k^2R^2} - \frac{\cos(kR)}{kR}\right)\frac{e^{ikr}}{kr}\left(1+\frac{i}{kr}\right)\\ &\times\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}\frac{1}{R}\sqrt{\frac{3\pi}{4}\frac{0!}{2!}}(\hat{e}_x + i\hat{e}_y)\\ &= \frac{3i\mu_0 cq}{16R}\left[\sin(kR) - 2\omega_0 R\cos(kR)\right]\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\\ &\times\sin\theta e^{i\varphi}\left(\hat{e}_x + i\hat{e}_y\right)\end{aligned}$$

9.7 Problem 9.7

To get the instantaneous power, we need to evaluate

$$\frac{dP}{d\Omega}(t) = r^2\hat{r}\cdot\left(\vec{E}(t)\times\vec{H}(t)\right)$$

and not just the average power. As the actual fields $\vec{E}(t)$ and $\vec{H}(t)$ are real, we need the Fourier transforms for negative ω, which are necessarily $\vec{E}(\vec{x},-\omega) = \left(\vec{E}(\vec{x},\omega)\right)^*$, and similarly for $\vec{H}$. Then,

$$\frac{dP}{d\Omega}(t) = r^2\hat{r}\cdot\left[\left(\int_{-\infty}^{\infty}\vec{E}(\omega)e^{-i\omega t}d\omega\right)\times\left(\int_{-\infty}^{\infty}\vec{H}(\omega')e^{-i\omega' t}d\omega'\right)\right]$$

The Fourier transformed vector potentials are, to leading order in $1/r$, all of the form

$$\vec{A}(\vec{x},\omega) = -i\frac{\mu_0\omega}{4\pi}\frac{e^{i\omega r/c}}{r}\vec{V}(\omega)$$

where $\vec{V} = \vec{p}$ for an electric dipole, $\vec{V} = c^1\hat{r}\times\vec{m}$ for a magnetic dipole, and $\vec{V} = \frac{i\omega}{6c}\hat{r}\cdot Q$ for an electric quadripole. Notice that all of these values of $\vec{V}(\omega)$, and of $\vec{A}$, satisfy the condition that $\omega\to-\omega$ is equivalent to complex conjugation. Again to leading order in $1/r$, we have

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{\omega^2}{4c\pi}\frac{e^{i\omega r/c}}{r}\hat{r}\times\vec{V}$$

and

$$\vec{E} = \frac{\mu_0\omega^2}{4\pi}\frac{e^{i\omega r/c}}{r}\hat{r}\times\left(\vec{V}\times\hat{r}\right)$$

So evaluating the power per sterradian:

$$\frac{dP}{d\Omega}(t) = r^2 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{-i(\omega+\omega')t} \frac{\mu_0 \omega^2}{4\pi} \frac{\omega'^2}{4c\pi} \frac{e^{i(\omega+\omega')}}{r^2}$$
$$\times \hat{r} \cdot \left(\left[\hat{r} \times \left(\vec{V}(\omega) \times \hat{r} \right) \right] \times \left[\hat{r} \times \vec{V}(\omega') \right] \right)$$
$$= \frac{\mu_0}{16c\pi^2} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{-i(\omega+\omega')(t-r/c)} \omega^2 \omega'^2$$
$$\times \left(\vec{V}(\omega) \cdot \vec{V}(\omega') - \hat{r} \cdot \vec{V}(\omega) \hat{r} \cdot \vec{V}(\omega') \right)$$
$$= \frac{\mu_0}{16c\pi^2} \left\{ \int_{-\infty}^{\infty} d\omega e^{-i\omega t'} \omega^2 \left(\vec{V}(\omega) \right)_j \right\}$$
$$\left\{ \int_{-\infty}^{\infty} d\omega' e^{-i\omega' t} \omega'^2 \left(\vec{V}(\omega') \right)_k \right\} (\delta_{jk} - \hat{r}_j \hat{k})$$

where $t' = t - r/c$.

For the electric dipole,

$$\vec{V}(\omega) = \vec{p}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt'' \vec{p}(t'') e^{i\omega t''}$$

so, $\omega^2 \vec{V}(\omega) = \vec{p}(\omega) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \frac{d^2}{d\tau^2} \vec{p}(\tau) e^{i\omega\tau}$ and

$$\int_{-\infty}^{\infty} d\omega e^{-i\omega t'} \omega^2 \left(\vec{V}(\omega) \right)_j = - \frac{d^2}{d\tau^2} \vec{p}(\tau) \Big|_{\tau=t'}$$

Nothing that $\left| \left(\hat{r} \times \vec{A} \right) \times \hat{r} \right|^2 = \vec{A}^2 - \left(\hat{r} \cdot \vec{A} \right)^2$, we have, all together,

$$\frac{dP}{d\Omega}(t) = \frac{Z_0}{16\pi^2 c^2} \left| \left(\hat{r} \times \frac{d^2}{dt'^2} \vec{p}(t') \right) \times \hat{r} \right|^2$$

For the magnetic dipole, $\vec{p}$ is replaced by $\hat{r} \times \vec{m}$, and $(\hat{r} \times \vec{A})^2 - (\hat{r} \cdot (\hat{r} \times \vec{A}))^2 = (\hat{r} \times \vec{A})$, so for the magnetic dipole

$$\frac{dP}{d\Omega}(t) = \frac{Z_0}{16\pi^2 c^4} \left| \hat{r} \times \frac{d^2}{dt'^2} \vec{m}(t') \right|^2$$

Finally, for the electric quadripole, we need

$$\omega^2 \vec{V}(\omega) = \frac{i}{6c} \omega^3 \hat{r} \cdot \mathbf{Q}(\omega) = \frac{-1}{12\pi c} \int_{-\infty}^{\infty} d\tau \hat{r} \cdot \mathbf{Q}(\tau) \frac{d^3}{d\tau^3} e^{i\omega\tau}$$
$$= \frac{1}{12\pi c} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \hat{r} \cdot \frac{d^3}{d\tau^3} \mathbf{Q}$$

$$\int_{-\infty}^{\infty} d\omega e^{-i\omega t'} e^{-i\omega t'} \omega^2 \vec{V}(\omega) = \frac{1}{6c} \frac{d^3}{d\tau^3} \mathbf{Q}(\tau) \Big|_{\tau=t'} = \frac{1}{6c} \frac{d^3}{dt'^3} \vec{\mathbf{Q}}(\hat{r}, t')$$

So, just as for the electric dipole, we have:

$$\begin{aligned}\frac{dP}{d\Omega}(t) &= \frac{Z_0}{16\pi^2 c^2}\frac{1}{6^2 c^2}\left|\left(\hat{r}\times\frac{d^3}{dt'^3}\vec{\mathbf{Q}}(\hat{r},t')\right)\times\hat{r}\right| \\ &= \frac{Z_0}{576\pi^2 c^4}\left|\left(\hat{r}\times\frac{d^3}{dt'^3}\vec{\mathbf{Q}}(\hat{r},t')\right)\times\hat{r}\right|^2\end{aligned}$$