8.9 Problem 8.9

8.9.1

$$k^2 = \frac{\int_V \vec{E}^* \cdot \left[\nabla \times \nabla \times \vec{E}\right] d^3x}{\int_V \vec{E}^* \cdot \vec{E} d^3x}$$

$$\begin{aligned}
\delta k^2 =& \frac{\cancel{\int_V \vec{E}^* \cdot \vec{E} d^3x} \int_V \vec{E}^* \cdot \left[\nabla \times \nabla \times \left(\vec{E} + \delta\vec{E}\right)\right]}{\left(\int_V \vec{E}^* \cdot \vec{E} d^3x\right)^{\cancel{2}}} - \frac{\int_V \vec{E}^* \left[\nabla \times \nabla \times \vec{E}\right] d^3x \int \vec{E}^* \cdot \left(\vec{E} + \delta\vec{E} d^3x\right)}{\left(\int_V \vec{E}^* \cdot \vec{E} d^3x\right)^2} \\
=& \frac{\int_V \vec{E}^* \cdot \left[\nabla \times \nabla \times \vec{E} + \nabla \times \nabla \times \delta\vec{E}\right]}{\left(\int_V \vec{E}^* \cdot \vec{E} d^3x\right)} \\
& - \frac{\int_V \vec{E}^* \left[\nabla \times \nabla \times \vec{E}\right] d^3x \cancel{\int \vec{E}^* \cdot \vec{E} d^3x}}{\left(\int_V \vec{E}^* \cdot \vec{E} d^3x\right)^{\cancel{2}}} - \frac{\int_V \vec{E}^* \left[\nabla \times \nabla \times \vec{E}\right] d^3x \int \vec{E}^* \cdot \delta\vec{E} d^3x}{\left(\int_V \vec{E}^* \cdot \vec{E} d^3x\right)^2} \\
=& \frac{\int_V \vec{E}^* \cdot \left[\nabla \times \nabla \times \delta\vec{E}\right]}{\int_V \vec{E}^* \cdot \vec{E} d^3x} - \underbrace{\frac{\int_V \vec{E}^* \left[\nabla \times \nabla \times \vec{E}\right] d^3x}{\int_V \vec{E}^* \cdot \vec{E} d^3x}}_{k^2} \frac{\int_V \vec{E}^* \cdot \delta\vec{E} d^3x}{\int_V \vec{E}^* \cdot \vec{E} d^3x} \\
=& \frac{\int_V \vec{E}^* \cdot \left[\nabla \times \nabla \times \delta\vec{E}\right] d^3x - k^2 \int \vec{E}^* \cdot \delta\vec{E} d^3x}{\int_V \vec{E}^* \cdot \vec{E} d^3x} \qquad (1)
\end{aligned}$$

We now examine the integral $\int_V \vec{E}^* \cdot \left[\nabla \times \nabla \times \delta\vec{E}\right] d^3x$:

$$\begin{aligned}
\int_V \vec{E}^* \cdot \left[\nabla \times \nabla \times \delta\vec{E}\right] &= \int_V E_i^* \frac{\partial}{\partial x_\ell} \frac{\partial}{\partial x_j} \delta E_k \underbrace{\varepsilon_{mjk}\varepsilon_{mi\ell}}_{\delta_{ij}\delta_{k\ell} - \delta_{j\ell}\delta_{ik}} d^3x \\
&= \int_V E_i^* \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \delta E_k d^3x - \int_i \int_k \left(\int_i \underbrace{E_i^* \frac{\partial}{\partial x_j}}_{u} \underbrace{\frac{\partial}{\partial x_j} \delta E_i}_{dv} dx_j \right) dx_k dx_i \\
&= \int_V E_i^* \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} \delta E_k d^3x - \int_i \int_k \left(\cancelto{0}{E_i^* \frac{\partial}{\partial x_j} \delta E_i \Big|_V} - \int_i \underbrace{\frac{\partial}{\partial x_j} \delta E_i}_{dv} \underbrace{\frac{\partial}{\partial x_j} E_i^*}_{u} dx_j \right) dx_k dx_i
\end{aligned}$$

$$= \int_V E_i^* \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\delta E_k d^3x - \int_i\int_k \Big(- E_i^* \underbrace{\cancelto{0}{\frac{\partial}{\partial x_j}\delta E_i}\Big|_V} + \int_i \underbrace{\frac{\partial}{\partial x_j}\delta E_i}_{dv} \underbrace{\frac{\partial}{\partial x_j}E_i^*}_{u} dx_j \Big) dx_k dx_i$$

$$= \int_V E_i^* \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\delta E_k d^3x - \int_V \delta E_i \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} E_i^* dx^3$$

$$= \int_V \left[\vec{E}^* \cdot \nabla \left(\nabla \cdot \delta\vec{E}\right) - \delta\vec{E}\cdot\nabla^2\vec{E}^*\right] d^3x$$

Using the fact that $\vec{E}^* \cdot \nabla\cancelto{0}{\left(\nabla\cdot\delta\vec{E}\right)} = 0 = \delta\vec{E}\cdot\nabla\cancelto{0}{\left(\nabla\cdot\vec{E}^*\right)}$:

$$\int_V \vec{E}^* \cdot \left[\nabla\times\nabla\times\delta\vec{E}\right] = \int_V \delta\vec{E}\cdot \underbrace{\left[\nabla\left(\nabla\cdot\vec{E}^*\right) - \nabla^2\vec{E}^*\right]}_{\nabla\times\nabla\times\vec{E}^*} d^3x$$

$$= \int_V \delta\vec{E}\cdot\left[\nabla\times\nabla\times\vec{E}^*\right] d^3x$$

Plugging this into equation (1) yields:

$$\delta k^2 = \frac{\int_V \delta\vec{E}\cdot\left[\nabla\times\nabla\times\vec{E}^*\right] d^3x - k^2\int_V \vec{E}^*\cdot\delta\vec{E} d^3x}{\int_V \vec{E}^*\cdot\vec{E} d^3x}$$

$$= \frac{\int_V \delta\vec{E}\cdot\left(\left[\nabla\times\nabla\times\vec{E}^*\right] - k^2\vec{E}^*\right) d^3x}{\int_V \vec{E}^*\cdot\vec{E} d^3x}$$

$$= 0$$

where, in the last step, we've used the vector Helmholtz equation, $\nabla\times\nabla\times\vec{E} = k^2\vec{E}$.

Thus, there are no first-order changes in k^2, indicating that there are only second-order and higher changes in k^2.

8.9.2

Given that $\vec{E} = E_0\cos(\pi\rho/2R)\hat{z}$,

$$\nabla\times\left(\nabla\times\vec{E}\right) = \nabla\underbrace{\left(\nabla\cdot\vec{E}\right)}_{0} - \nabla^2\vec{E}$$

$$= -\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\vec{E}}{\partial\rho}\right) + \frac{1}{\rho^2}\cancelto{0}{\frac{\partial^2\vec{E}}{\partial\varphi^2}} + \cancelto{0}{\frac{\partial^2\vec{E}}{\partial z^2}}\right)$$

$$= E_0\left[\frac{\pi}{2R\rho}\sin\left(\frac{\pi\rho}{2R}\right) + \frac{\pi^2}{4R^2}\cos\left(\frac{\pi\rho}{2R}\right)\right]\hat{z}$$

Plugging this into the definition of k^2 gives:

$$k^2 = \frac{\int_0^R \left[\frac{\pi}{2R\rho}\cos\left(\frac{\pi\rho}{2R}\right)\sin\left(\frac{\pi\rho}{2R}\right) + \frac{\pi^2}{4R^2}\cos^2\left(\frac{\pi\rho}{2R}\right)\right]\rho d\rho}{\int_0^R \cos^2\left(\frac{\pi\rho}{2R}\right)\rho d\rho}$$
$$= \frac{\frac{1}{16}\left(4+\pi^2\right)}{\frac{1}{4\pi^2}R^2\left(-4+\pi^2\right)}$$
$$= \frac{\pi^2}{4R^2}\frac{\pi^2+4}{\pi^2-4}$$

$$\implies Rk = \frac{\pi}{2}\sqrt{\frac{\pi^2+4}{\pi^2-4}} = 2.4146$$

Note that we can ignore the integrals in φ and z. Since neither of the integrands depend on either of these variables, the values of the φ and z integrals will cancel.

The first root of $J_0(x)$ is 2.4048, so this approximate solution is about 0.4% away from the exact answer.

8.9.3

Given that $\vec{E} = E_0\left[1+\alpha(\rho/R)^2 - (1+\alpha)(\rho/R)^4\right]$,

$$\nabla\times\left(\nabla\times\vec{E}\right) = -\nabla^2\vec{E} = -\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\vec{E}}{\partial\rho}\right) + \frac{1}{\rho^2}\cancelto{0}{\frac{\partial^2\vec{E}}{\partial\varphi^2}} + \cancelto{0}{\frac{\partial^2\vec{E}}{\partial z^2}}\right)$$
$$= E_0\frac{4}{R^4}\left(\alpha R^2 - 4r^2 - 4r^2\alpha\right)\hat{z}$$

Plugging this into the definition of k^2 yields:

$$k^2 = \frac{\int_0^R \left(1+\alpha(\rho/R)^2-(1+\alpha)(\rho/R)^4\right)\frac{4}{R^4}\left(\alpha R^2-4r^2-4r^2\alpha\right)\rho d\rho}{\int_0^R \left(1+\alpha(\rho/R)^2-(1+\alpha)(\rho/R)^4\right)^2\rho d\rho}$$
$$= \frac{2+\frac{4}{3}\alpha+\frac{1}{3}\alpha^2}{\frac{R^2}{60}\left(16+7\alpha+\alpha^2\right)}$$
$$= \frac{20}{R^2}\frac{6+4\alpha+\alpha^2}{16+7\alpha+\alpha^2} \tag{2}$$

We want to find the value of α which minimizes the above expression:

$$\frac{dk^2}{d\alpha} = \frac{20}{R^2}\frac{(16+7\alpha+\alpha^2)(4+2\alpha)-(6+4\alpha+\alpha^2)(7+2\alpha)}{(16+7\alpha+\alpha^2)^2} = 0$$

$$\implies \alpha_\pm = -\frac{1}{3}(10\pm\sqrt{34})$$

α_+ is a maximum while a_- is a minimum. Plugging α_- into equation (2) yields:

$$kR = \sqrt{80\frac{17-2\sqrt{34}}{68+\sqrt{34}}} = 2.4050$$

This approximate solution is about 0.009% away from the exact answer.

8.14 Problem 8.14

8.14.1

We start with equation 8.117 in Jackson:

$$\bar{n}^2\left(\frac{dx}{dz}\right)^2 = n^2(x) - \bar{n}^2$$

Letting $n(x) = n(0)\,\text{sech}(\alpha x)$, $\bar{n} = n(0)\,\text{sech}(\alpha x_{\max})$, and $\sinh(\alpha x) = \sinh(\alpha x_{\max})\sin(\alpha z)$:

$$n^2(0)\,\text{sech}^2(\alpha x_{\max})\left(\frac{d}{dz}\sinh^{-1}\left[\sinh(\alpha x_{\max})\sin(\alpha z)\right]\right)^2 = n^2(0)\,\text{sech}^2(\alpha x) - n^2(0)\,\text{sech}^2(\alpha x_{\max})$$

$$\text{sech}^2(\alpha x_{\max})\left(\frac{d}{dz}\sinh^{-1}\left[\sinh(\alpha x_{\max})\sin(\alpha z)\right]\right)^2 = \frac{1}{1+\sinh^2(\alpha x_{\max})\sin^2(\alpha z)} - \text{sech}^2(\alpha x_{\max}) \tag{3}$$

I then used following Maple commands:

```
x:=x->arcsinh(sinh(a*xmax)*sin(a*z)/a);
simplify(sech(a*xmax)^2*diff(x(z),z)^2\
   - 1/(1+sinh(a*xmax)^2*sin(a*z)^2) + sech(a*xmax)^2);
```

The output was zero. Hence, the two sides of equation (3) are equal and $\alpha x = \sinh^{-1}\left[\sinh(\alpha x_{\max}\sin(\alpha x))\right]$ is a solution to Jackson's equation 8.117.

Because $\bar{n} = n(0)\,\text{sech}(\alpha x_{\max}) = n(0)\cos(\theta(0))$, $\text{sech}(\alpha x_{\max}) = \cos(\theta(0))$. Hence,

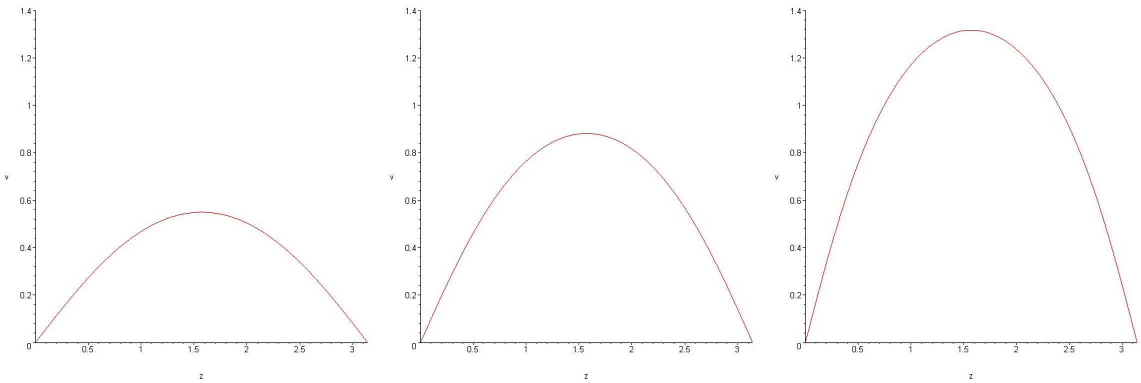
$$x_{\max} = \frac{1}{\alpha}\cosh^{-1}\left[\cos(\theta(0))\right]$$

To plot the rays, we note:

$$\begin{aligned}
\text{sech}(\alpha x) &= \cos(\theta(0)) \\
\sqrt{1+\sinh^2(\alpha x)} &= \frac{1}{\cos\theta(0)} \\
\sinh(\alpha x) &= \sqrt{\frac{1}{\cos^2\theta(0)} - 1} = \sqrt{\frac{1-\cos^2\theta(0)}{\cos^2\theta(0)}} = |\tan\theta(0)|
\end{aligned}$$

$$\implies \alpha x = \sinh^{-1}\left[\,|\tan\theta(0)|\sin(\alpha z)\right]$$

These paths are plotted in figure 1 for three different launch angles $\theta(0)$.



$\theta(0) = \pi/6$ $\theta(0) = \pi/4$ $\theta(0) = \pi/3$

Figure 1: Plots of $x(z)$ for three values of $\theta(0)$ where $\alpha = 1$.

8.14.2

To find the half-period of the ray $Z = 2z(x_{\text{max}})$, we first solve our given equation for $x(z)$ for z in terms of x:

$$z = \frac{1}{\alpha}\sin^{-1}\left(\frac{\sinh(\alpha x)}{\sinh(\alpha x_{\text{max}})}\right)$$

Solving for Z yields:

$$Z = \frac{2}{\alpha}z(x_{\text{max}}) = \frac{2}{\alpha}\sin^{-1}(1) = \frac{\pi}{\alpha}$$

8.14.3

Starting with the paragraph above equation 8.119 in Jackson:

$$\begin{aligned}
L_{\text{opt}} &= 2\int_0^{x_{\max}} n(x)ds \\
&= 2\int_0^{Z/2} n(x)\frac{n(x)}{\bar{n}}dz \\
&= 2\int_0^{Z/2} n(0)\,\text{sech}(\alpha x)\frac{\text{sech}(\alpha x)}{\text{sech}(\alpha x_{\max})}dz \\
&= 2n(0)\int_0^{Z/2} \frac{\cosh(\alpha x_{\max})}{\cosh^2(\alpha x)}dz \\
&= 2n(0)\int_0^{\pi/2\alpha} \frac{\cosh(\alpha x_{\max})}{1+\sinh^2(\alpha x)}dz \\
&= 2n(0)\cosh(\alpha x_{\max})\int_0^{\pi/2\alpha} \frac{1}{1+\sinh^2(\alpha x_{\max})\sin^2(z)}dz
\end{aligned}$$

where we've used the identity Jackson gives, $\sinh(\alpha x) = \sinh(\alpha x_{\max})\sin(z)$. Using the integral identity $\int_0^{\pi/2\alpha} \frac{dz}{1+C^2\sin^2(z)} = \frac{\pi}{2\alpha\sqrt{1+C^2}}$ yields:

$$\begin{aligned}
L_{\text{opt}} &= 2n(0)\cosh(\alpha x_{\max})\left[\frac{\pi}{2\alpha\underbrace{\sqrt{1+\sinh^2(\alpha x_{\max})}}_{\cosh(\alpha x_{\max})}}\right] \\
&= n(0)\frac{\pi}{\alpha} = n(0)Z
\end{aligned}$$

The optical path length is independent of $x_{\max}$, and hence it is also independent of $\theta(0)$. Thus, optical path length is the same for all launch angles.