7.3 Problem 7.3

We have two semi-infinite slabs of dielectric material with $\mu = \mu_0$ and equal indices of refraction $n > 1$, with an air gap ($n = 1$) of thickness d between them. Let the surfaces be in the x, y plane, with the gap being $z \in [0, d]$ and the incident wave coming from $z < 0$. In the first material we have an incident wave

$$\begin{aligned}\vec{E}(\vec{x}) &= \vec{E}_0 e^{i\vec{k}\cdot\vec{x}-i\omega t}\\ \vec{B}(\vec{x}) &= \vec{k} \times \vec{E}(\vec{x})/\omega\end{aligned}$$

but we also have a reflected wave

$$\begin{aligned}\vec{E}_R(\vec{x}) &= \vec{E}_R e^{i\vec{k}_R\cdot\vec{x}-i\omega t}\\ \vec{B}_R(\vec{x}) &= \vec{k}_R \times \vec{E}_R(\vec{x})/\omega\end{aligned}$$

In the air gap, we may have two waves with oscillatory behavior or exponential behavior in z. We may write either case as

$$\begin{aligned}\vec{E}(\vec{x}) &= \vec{E}_{g1} e^{i\vec{k}_{g1}\cdot\vec{x}-i\omega t} + \vec{E}_{g2} e^{i\vec{k}_{g2}\cdot\vec{x}} - i\omega t\\ \vec{B}(\vec{x}) &= \vec{k}_{g1} \times \vec{E}_{g1}(\vec{x})/\omega + \vec{k}_{g2} \times \vec{E}_{g2}(\vec{x})/\omega\end{aligned}$$

but remembering that the z components of the wavenumbers may be imaginary.

The second slab has only an outgoing wave

$$\begin{aligned}\vec{E}(\vec{x}) &= \vec{E}_2 e^{i\vec{k}_2\cdot\vec{x}-i\omega t}\\ \vec{B}(\vec{x}) &= \vec{k}_2 \times \vec{E}_2(\vec{x})/\omega\end{aligned}$$

The squares of the wavenumbers are determined by the indices of refraction and ω:

$$\begin{aligned}k = k_R = k_2 &= n\omega/c\\ k_{g1}^2 = k_{g2}^2 &= \omega^2/c^2\end{aligned}$$

but we must keep in mind that $\left(\vec{k}_{g1}\right)_z$ may be imaginary, in which case $k_{g1}^2 = k_{g1\ x}^2 + k_{g1\ y}^2 - |k_{g1\ z}|^2$.

As for the single interface discussion, we may chose x so that the incident wave is in the x, z plane. As the wave equations and the boundary conditions are invariant under translations in the x and y directions, we can Fourier transform in those directions and see that the equations involve only the same values for the k_x's and for the k_y's, so

$$\begin{aligned}k_x &= k_{R\ x} = k_{g1\ x} = k_{g2\ x} = k_{2\ x}\\ k_y &= k_{R\ y} = k_{g1\ y} = k_{g2\ y} = k_{2\ y} = 0\end{aligned}$$

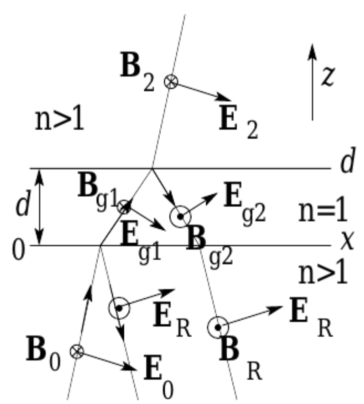
From the equality of the k_x's and the relations among the k^2'2 we have $k_z = -k_{R\ z} = k_{2\ z} = k\cos i$, and $k_{g1\ z} = -k_{g2\ z} = \sqrt{k_{g1}^2 - k_x^2} = k\sqrt{n^{-2} = \sin^2 i} = k\cos r/n$, with the angle of reflection for $\vec{k}_R$ and the angle of $\vec{k}_2$ equal to the angle of incidence i, and the angle of reflection, r given by Snell's law $n\sin i = \sin r$. Note $k_{gi\ x}$ will be imaginary if $n\sin i > 1$, and r will then be complex.

Finally, we can divide the problem into a part ($E_\perp$) for which $\vec{E}_0$ is perpendicular to the plane of incidence ($\vec{E} \parallel \pm\hat{e}_y$) and part (E) in which it lies in the plane of incidence ($E_{0\ y} = 0$), in which case $\vec{B} \parallel \pm\hat{e}_y$ in the first material.

As the problem is invariant under reflection in the $y = 0$ plane, in the ($E_\perp$) case all fields are reversed, so all of the $\vec{E}$'s are in the $\pm y$ direction, and all the $\vec{B}$s are in the x, z plane. In the ($E_{||}$) case reflection in the $y = 0$ plane changes none of the incident fields, and therefore none of the others, so all the $\vec{E}_y$'s vanish, and all the $\vec{B}$'s are perpendicular to the plane of incidence. The boundary conditions are continuity of D_z, B_z, E_x, E_y, H_x, and H_y at each of the two boundaries.

The $\vec{E}_{||}$ case:

We will now consider the case where all the $\vec{E}$ fields lie in the plane of incidence. See below for a picture with all of the E_x's positive if the corresponding amplitudes E are positive. This is different from what Jackson did.



From the continuity of E_x,

$$(E_0 + E_R)\cos i = (E_{g1} + E_{g2})\cos r$$

$$\left(E_{g1}e^{ikd(\cos r)/n} + E_{g2}e^{-ikd(\cos r)/n}\right)\cos r = E_2 e^{ikd\cos i}\cos i$$

and continuity of H_y gives

$$n(E_0 - E_R) = E_{g1} - E_{g2}$$

$$nE_2 e^{ikd\cos i} = E_{g1}e^{ikd(\cos r)/n} - E_{g2}e^{-ikd(\cos r)/n}$$

To simplify our algebra, let $A = e^{ikd(\cos r)/n}$ and $B = e^{ikd\cos i}$, so the second and fourth equations become

$$E_2 B\cos i = \left(E_{g1}A + E_{g2}A^{-1}\right)\cos r$$
$$E_2 Bn = E_{g1}A - E_{g2}A^{-1}$$

giving

$$E_{g1}A\cos i - E_{g2}A^{-1}\cos i = E_{g1}An\cos r - E_{g2}A^{-1}n\cos r$$

$$\implies E_{g2} = -A^2 E_{g1}\frac{n\cos r - \cos i}{n\cos r + \cos i} = -A^2 E_{g1}\frac{1-\rho}{1+\rho}$$

with:

$$\rho = \frac{\cos i}{n\cos r}$$

Plugging back into the first interface forms,

$$(E_R + E_0) = \left(-A^2\frac{1-\rho}{1+\rho} + 1\right)\frac{E_{g1}}{np} = \frac{(1-A^2)+\rho(1+A^2)}{n\rho(1+\rho)}E_{g1}$$
$$n\left(E_0 - E_R\right) = E_{g1} - E_{g2} = \left(A^2\frac{1-\rho}{1+\rho} + 1\right)E_{g1}$$

so

$$\frac{E_0+E_R}{E_0-E_R} = \frac{1}{\rho}\frac{1-A^2+\rho(1+A^2)}{1+A^2+\rho(1-A^2)}$$

Note as $A = e^{i\varphi}$ with $\varphi = kd(\cos r)/n$, $\frac{1+A^2}{1-A^2} = \frac{1+e^{2i\varphi}}{1-e^{2i\varphi}} = i\cot\varphi$, we get:

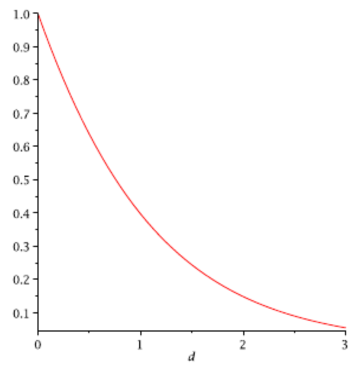
$$\frac{E_R}{E_0} = \frac{(1-\rho^2)(1-A^2)}{1-A^2+2\rho(1+A^2)+\rho^2(1-A^2)} = \frac{1-\rho^2}{1+2i\rho\cos\varphi+\rho^2}$$

Provided the angle of incidence is less than the angle of total reflection, ϕ and $\cot\varphi$ are real, and the reflection coefficient is

$$\left|\frac{E_R}{E_0}\right|^2 = \frac{(1-\rho^2)^2}{(1+\rho^2)^2 + r\rho^2\cot^2\varphi}$$

This will have maxima and minima according to the phase ϕ, maxima when it is a multiple of π and minima halfway between those. On the other hand, if $n\sin i > 1$, $\rho = in\sqrt{n^2\sin^2 i - 1}$ and $\cot\phi$ is imaginary and goes to i, without oscillations, Of course in either case the transmission coefficient is 1 the reflection coefficient.

Below is a plot of the transmission coefficient with $n = 1.5$ and $\angle i = 1$ rad, for $kd \in [0, 3]$.



7.16 Problem 7.16

7.16.1 Part a

Starting with equations 7.1 and assuming that $\vec{B}$ and $\vec{E}$ have solutions with harmonic time dependence:

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \Longrightarrow \qquad \nabla \times \vec{E} - i\omega \vec{B} = 0$$
$$\frac{1}{\mu_0}\nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = 0 \qquad \Longrightarrow \qquad \frac{1}{\mu_0}\nabla \times \vec{B} + i\omega \vec{D} = 0$$

Fourier transforming the above two equations, noting that $\nabla \Leftrightarrow i\vec{k}$:

$$i\vec{k} \times \vec{E} - i\omega \vec{B} = 0$$
$$i\vec{k} \times \vec{B} + i\omega\mu_0 \vec{D} = 0$$

Solving the first equation for $\vec{B}$ and plugging it into the second equation yields:

$$i\vec{k} \times \left(\frac{\vec{k} \times \vec{E}}{\omega}\right) + i\omega\mu_0 \vec{D} = 0$$

$$\boxed{\vec{k} \times \left(\vec{k} \times \vec{E}\right) + \omega^2 \mu_0 \vec{D} = 0}$$

7.16.2 Part b

We start by expanding the result from the previous problem using the BAC-CAB rule:

$$k^2\hat{n}\left(\hat{n}\times\vec{E}\right)+\omega^2\mu_0\vec{D}=0$$

$$k^2\left[\hat{n}\left(\hat{n}\cdot\vec{E}\right)-\vec{E}\underbrace{(\hat{n}\cdot\hat{n})}_{1}\right]+\omega^2\mu_0\vec{D}=0$$

$$n_i(n_jE_j)-E_i+\underbrace{\frac{\omega^2}{k^2}}_{v^2}\mu_0\underbrace{D_i}_{\varepsilon_iE_i}=0$$

$$n_in_jE_j-E_i+v^2\underbrace{\mu_0\varepsilon_i}_{1/v_i^2}E_i=0$$

$$n_in_jE_j-E_i+\frac{v^2E_i}{v_i^2}=0$$

We now find the three components of both sides of the above equation, remembering that we're implicitly summing over j in the above equation.

$$\begin{bmatrix} n_1\left(n_1E_1+n_2E_2+n_3E_3\right)-E_1\\ n_2\left(n_1E_1+n_2E_2+n_3E_3\right)-E_2\\ n_3\left(n_1E_1+n_2E_2+n_3E_3\right)-E_3\end{bmatrix}+v^2\begin{bmatrix}E_1/v_1^2\\ E_2/v_2^2\\ E_3/v_3^2\end{bmatrix}=0$$

$$\begin{bmatrix}\left(n_1n_1-1\right)E_1+n_1n_2E_2+n_1n_3E_3\\ n_2n_1E_1+\left(n_2n_2-1\right)E_2+n_2n_3E_3\\ n_3n_1E_1+n_3n_2E_2+\left(n_3n_3-1\right)E_3\end{bmatrix}+v^2\begin{bmatrix}E_1/v_1^2\\ E_2/v_2^2\\ E_3/v_3^2\end{bmatrix}=0$$

$$\left(\underbrace{\begin{bmatrix}n_1^2-1 & n_1n_2 & n_1n_3\\ n_2n_1 & n_2^2-1 & n_2n_3\\ n_3n_1 & n_3n_2 & n_3^2-1\end{bmatrix}}_{A}+v^2\underbrace{\begin{bmatrix}1/v_1^2 & 0 & 0\\ 0 & 1/v_2^2 & 0\\ 0 & 0 & 1/v_3^2\end{bmatrix}}_{B}\right)\begin{bmatrix}E_1\\ E_2\\ E_3\end{bmatrix}=0$$

Hence, solutions to v^2 are eigenvalues of A. We can solve for v^2 using the characteristic equation, $\det(A+v^2B)=0$. Taking the determinant of $A+v^2B$ in Maple yields the following (where we have rearranged some terms for reasons which will become immediately apparent), and using the fact that $n_1^2+n_2^2+n_3^2=1$:

$$\frac{v^2}{v_1v_2v_3}\Big(v^4+\underbrace{v_2^2v_3^2-n_2^2v_2^2v_3^2-n_3^2v_2^2v_3^2}_{n_1^2v_2^2v_3^2}+\underbrace{n_2^2v^2v_2^2-v^2v_2^2}_{-v^2v_2^2(n_1^2+n_3^2)}+\underbrace{n_3^2v^2v_3^2-v^2v_3^2}_{v^2v_3^2(n_1^2+n_2^2)}+\underbrace{v_1^2v_3^2-n_1^2v_1^2v_3^2-n_3^2v_1^2v_3^2}_{n_2^2v_1^2v_3^2}$$
$$+\underbrace{n_1^2v^2v_1^2-v^2v_1^2}_{v^2v_1^2(n_2^2+n_3^2)}+\underbrace{v_1^2v_2^2-n_1^2v_1^2v_2^2-n_2^2v_1^2v_2^2}_{n_3^2v_1^2v_2^2}\Big)+\underbrace{n_1^2+n_2^2+n_3^2-1}_{0}=0$$

$$\begin{aligned}\frac{v^2}{v_1 v_2 v_3}&\left[n_1^2(v^4 + v_2^2 v_3^2 - v^2 v_2^2 - v^2 v_3^2)\right. \\ &+n_2^2(v^4 + v^2 v_3^2 + v_1^2 v_3^2 - v^2 v_1^2) \\ &\left.+n_3^2(v^4 + v^2 v_2^2 - v^2 v_1^2 + v_1^2 v_2^2)\right] = 0\end{aligned}$$

$$\frac{v^2}{v_1 v_2 v_3}\left[n_1^2(v^2 - v_2^2)(v^2 - v_3^2) + n_2^2(v^2 - v_1^2)(v^2 - v_3^2) + n_3^2(v^2 - v_1^2)(v^2 - v_2^2)\right] = 0$$

Dividing both sides of this equation by $(v^2 - v_1^2)(v^2 - v_2^2)(v^2 - v_3^2)$ yields:

$$\begin{aligned}\frac{v^2}{v_1 v_2 v_3}\left[\frac{n_1^2}{v^2 - v_1^2} + \frac{n_2^2}{v^2 - v_2^2} + \frac{n_3^2}{v^2 - v_1^2}\right] &= 0 \\ \frac{v^2}{v_1 v_2 v_3}\left[\sum_{i=1}^{3}\frac{n_i^2}{v^2 - v_i^2}\right] &= 0\end{aligned}$$

There are three solutions for v: 0, v_+, and v_- (where the last two solutions can be found using the quadratic equation, which yields two solutions). The two nontrivial solutions satisfy the Fresnel equation, which is when the term in brackets in the above equation is zero. That is:

$$\boxed{\sum_{i=1}^{3}\frac{n_i^2}{v_\pm^2 - v_i^2} = 0}$$

7.16.3 Part c

From our solution to part a, we obtain, for a wave in mode a:

$$\left(\vec{n}\cdot\vec{E}_a\right)\vec{n} - \vec{E}_a = -\mu_0\frac{\omega^2}{k^2}\varepsilon\cdot\vec{E}_a = -v_a^2\vec{D}_a$$

where $\vec{n} = \vec{k}/k$ is a unit vector in the $\vec{k}$ direction, and $v_a^2 = \mu_0\omega^2/k^2$, which may be different for the different modes, as the phase velocity may depend on the polarization.

Dot this into $\vec{D}_b$ for another mode with the same $\vec{n}$, giving:

$$\vec{D}_b\cdot\vec{D}_a v_a^2 = \vec{D}_b\cdot\left(\vec{E}_a - \left(\vec{n}\cdot\vec{E}_a\vec{n}\right)\right) = \vec{D}_b\cdot\vec{E}_a$$

because $\vec{n}\cdot\vec{E} = 0$. Of course, the same applies with $a \leftrightarrow b$, so:

$$\vec{D}_b\cdot\vec{D}_a\left(v_a^2 - v_b^2\right) = \vec{D}_b\cdot\vec{E}_a - \vec{D}\cdot\vec{E}_b$$

But:

$$\vec{D}_b \cdot \vec{E}_a = \sum_i \varepsilon_i E_i^a E_i^b = \vec{D}_a \cdot \vec{E}_b$$

so,

$$\vec{D}_b \cdot \vec{D}_a \left(v_a^2 - v_b^2\right) = 0$$

and if the two modes have different phase velocities ($v_a \neq v_b$), then:

$$\vec{D}_b \cdot \vec{D}_a = 0$$

8.5 Problem 8.5

8.5.1 Part a

For the simply-connected triangular region the modes will be either TE or TM, with the longitudinal B_z or E_z given by a solution ψ of the Helmholtz equation $(\nabla_t^2 + \gamma^2)\psi = 0$ with boundary conditions $\partial\psi/\partial n|_S = 0$ or $\psi|_S = 0$ respectively.

Any such solution on the triangle $0 \leq x \leq a$, $y \leq x \leq a$ can be extended to a solution on the square $0 \leq x \leq a$, $0 \leq y \leq a$ by defining $\psi(x,y)|_{y>x} = \pm\psi(y,x)$, with the plus sign for the Neumann (TE) case and the minus sign for the Dirichlet (TM) case. The vanishing on $x = y$ in the TM case insures continuity, which is automatic with the plus sign in the TE case. The normal derivative is is zero and continuous due to the plus sign in the TE case, but is automatic with the minus sign for the TM case.

So the solutions for the triangle must be combinations of solutions for the square, but with symmetry or antisymmetry under $x \leftrightarrow y$ for TE and TM modes respectively. In terms of the functions in 8.135 and 8.136, this means

$$\begin{aligned} TM: \qquad E_{zmn} &= E_0 \left[\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) - \sin\left(\frac{m\pi y}{a}\sin\left(\frac{n\pi x}{a}\right)\right)\right] \\ TE: \qquad H_{zmn} &= H_0 \left[\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) - \cos\left(\frac{m\pi y}{a}\cos\left(\frac{n\pi x}{a}\right)\right)\right] \end{aligned}$$

as $\gamma_{mn} = \frac{\pi}{a}\sqrt{m^2+n^2}$. For our purposes we do not need to calculate the normalization constants E_0 and H_0.

Thus all the modes are the same as for the square of side a, except that $m = n$ is forbidden for the TM mode, and for each pair $m = n$, there is only one mode rather than two. As for the rectangle, $m = n = 0$ is forbidden as it leads to zero transverse fields. The cutoff frequencies are

$$\omega_{mn} = \frac{\pi}{a}\sqrt{\frac{m^2+n^2}{\mu_0\varepsilon_0}}$$

8.5.2 Part b

The lowest modes are $\mathrm{TM}_{1,2}$ and $\mathrm{TE}_{0,1}$. The attenuation coefficients depend on ξ_λ and η_λ, which involve the ratio of integrals over the boundary to those over the area.

For the $\mathrm{TE}_{0,1}$ mode, $\gamma = \pi/a$. Take $\psi(x, y) = \cos(\pi x/a) = \cos(\pi x/a) + \cos(\pi y/a)$

$$\begin{aligned}
\int_\Gamma \left|\psi^2\right| &= a \int_0^1 du \left(\left|\psi(au, 0)\right|^2 + \left|\psi(a, au)\right|^2 + \sqrt{2} \left|\psi(au, au)\right|^2 \right) \\
&= a \int_0^1 du \left((1 + \cos(\pi u))^2 + (\cos(\pi u) - 1)^2 + \sqrt{2}[2\cos(\pi u)]^2 \right) \\
&= (3 + 2\sqrt{2})a
\end{aligned}$$

$$\int_A \left|\psi^2\right| = \frac{1}{2} \int_0^a dx \int_0^a dy \left|\psi(x, y)^2\right| = \frac{1}{2} \int_0^a dx \int_0^a dy [\cos(\pi x/a) + \cos(\pi y/a)]^2 = \frac{a^2}{2}$$

$$\begin{aligned}
C &= 2 + \sqrt{2}a \\
A &= \frac{1}{2}a^2
\end{aligned}$$

$$\begin{aligned}
\zeta_{0,1}^{\mathrm{TE}} &= \frac{A}{C} \int_\Gamma \left|\psi^2\right| \Big/ \int_A \left|\psi^2\right| \\
&= \frac{a^2/2}{(2 + \sqrt{2})a} \frac{(3 + 2\sqrt{2})a}{a^2/2} = \frac{3 + 2\sqrt{2}}{2 + \sqrt{2}} = \frac{\sqrt{2} + 1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\int_\Gamma \left|\hat{n} \times \nabla_t \psi\right|^2 &= \int_0^a \left|\frac{\partial \psi}{\partial x}\right|^2 (x, 0) dx + \int_0^a \left|\frac{\partial \psi}{\partial y}\right|^2 (a, y) dy \\
&\quad + \int_0^a dx \sqrt{2} \left(\frac{1}{\sqrt{2}} \left[\frac{\partial \psi}{\partial x}_{\frac{\partial \psi}{\partial y}} \right] (x, x) \right)^2 \\
&= \frac{\pi^2}{a} \int_0^1 \sin^2(\pi u)\, du \left[1 + 1 + 2\sqrt{2} \right] = (\sqrt{2} + 1)\frac{\pi^2}{a}
\end{aligned}$$

Also, of course, $\int_A \left|\vec{\nabla}_t \psi\right|^2 = \gamma^2 \int_A \psi^2$, $\gamma = \pi/a$. So

$$\begin{aligned}
\xi &= \frac{A}{C} \int_\Gamma \left|\hat{n} \times \vec{\nabla}\psi\right|^2 \Big/ \int_A \left|\vec{\nabla}\psi\right|^2 \\
&= \frac{a^2/2}{(2 + \sqrt{2})a} (\sqrt{2} + 1) \frac{\pi^2}{a} \frac{a^2}{\pi^2} \frac{2}{a^2} = \frac{1}{\sqrt{2}}
\end{aligned}$$

$\eta_\lambda = \zeta_\lambda - \xi_\lambda = 1$. Thus,

$$\beta_{01}^{\mathrm{TE}} = \sqrt{\frac{\varepsilon}{\mu}} \frac{1}{\sigma\delta_\lambda} \frac{2+\sqrt{2}}{a} \frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\omega_\lambda^2/\omega^2}} \left(\frac{1}{\sqrt{2}} + \frac{\omega_\lambda^2}{\omega^2}\right)$$

For the full square of side a, the mode is the same, the area $= \int_A |\psi^2| = a^2$, both twice as big, $C = 4a$, $\int_\Gamma |\psi^2| = 6a$, and

$$\begin{aligned}\int_\Gamma |\hat{n}\times\nabla_t\psi|^2 = \int_0^a \left|\frac{\partial\psi}{\partial x}\right|^2 (x,0)dx + \int_0^a \left|\frac{\partial\psi}{\partial y}\right|^2 (a,y)dy \\ + \int_0^a \left|\frac{\partial\psi}{\partial x}\right|^2 (x,a)dx + \int_0^a \left|\frac{\partial\psi}{\partial y}\right|^2 (0,y)dy = 2\pi^2/a\end{aligned}$$

so

$$\begin{aligned}\xi_{01}^{\square\mathrm{TE}} &= \frac{A}{C}\frac{2\pi^2/a}{(\pi^2/a^2)a^2} = \frac{a^2}{4a}\frac{2}{a} = \frac{1}{2} \\ \xi_{01}^{\square\mathrm{TE}} &= \frac{6a}{a^2}\frac{a^2}{4a} = \frac{3}{2}\end{aligned}$$

$$\eta_{01}^{\square\mathrm{TE}} = 1$$

$$\beta_{01}^{\square\mathrm{TE}} = \sqrt{\frac{\varepsilon}{\mu}} \frac{1}{\sigma\delta_\lambda a} \frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\omega_\lambda^2/\omega^2}} \left(1 + 2\frac{\omega_\lambda^2}{\omega^2}\right)$$

The ratio of attenuation of the triangle to the square is not frequency independent, but at all frequencies it is greater than 1.

$\mathrm{TM}_{1,2}$ mode:

We can take $\psi = \sin(\pi x/a)\sin(2\pi y/a) - \sin(2\pi x/a)\sin(\pi y/a)$, $\gamma_{1,2} = \sqrt{5}\pi/a$,

$$\begin{aligned}\int_A \psi^2 &= \frac{1}{2}\int_\square \left(\sin^2(\pi x/a)\right)\sin^2(2\pi y/a) + \sin^2(2\pi x/a)\sin^2(xy/a) \\ &\qquad - 2\sin(\pi x/a)\sin(2\pi y/a)\sin(2\pi x/a)\sin(\pi y/a) \\ &= \frac{a^2}{2}\left(\frac{1}{4}+\frac{1}{4}+0\right) = \frac{a^2}{4}\end{aligned}$$

$$
\begin{aligned}
\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 &= \int_0^1 adu \left[\left(\frac{\partial\psi}{\partial y}\right)^2 (au,0) + \left(\frac{\partial\psi}{\partial x}\right)^2 (a,au) \right.\\
&\qquad \left. + \frac{1}{\sqrt{2}}\left(\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y}\right)^2 (au,au)\right]\\
&= \frac{\pi^2}{a}\int_0^1 du \left[(2\sin(\pi u) - \sin(2\pi u))^2 + (-\sin(2\pi u) - 2\sin(\pi u))^2\right.\\
&\qquad \left. + \frac{1}{2}(2\cos(\pi u)\sin(2\pi u) - 4\sin(\pi u)\cos(2\pi u))^2\right]\\
&= \frac{5\pi^2}{\sqrt{2}a}(1+\sqrt{2})
\end{aligned}
$$

where it is useful to write the expression to be squared in the last term as $3\sin(\pi u)\sin(3\pi u)$. Thus

$$
\beta_{12}^{\mathrm{TM}} = \frac{\varepsilon}{\mu}\frac{1}{2\sigma\delta_\lambda}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Big/ \gamma^2 \int_A |\psi|^2 = \frac{4+2\sqrt{2}}{a}\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{2\sigma\delta_\lambda}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}
$$

For the square $\int_A \psi^2 = a^2/2$ and

$$
\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 = 4\int_0^1 adu\left(\frac{\pi}{a}\right)^2 (2\sin(\pi u) - \sin(2\pi u))^2 = 10\frac{\pi^2}{a}
$$

so

$$
\xi_{12}^{\square\mathrm{TM}}\frac{a}{4}\frac{10\pi^2}{a}\frac{2}{a^2}\frac{a^2}{5\pi^2} = 1
$$

$$
\gamma_{12}^{\square\mathrm{TM}} = \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{\sigma\delta_\lambda}\frac{4a}{2a^2}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\xi_\lambda = \sqrt{\frac{\varepsilon}{\mu}}\frac{2}{a\sigma\delta_\lambda}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}
$$

Thus the triangle attenuation is $1 + 1/\sqrt{2} = 1.71$ times that of the square.