6.1 Problem 6.1

6.1.1

Substituting $f(\vec{x}', t') = \delta(x')\delta(y')\delta(t')$ into equation 6.47 in Jackson:

$$\begin{aligned}\Psi(\vec{x},t) &= \int \frac{[f(\vec{x}',t')]_{\text{ret}}}{|\vec{x}-\vec{x}'|} d^3x' \\ &= \int \frac{[\delta(x')\delta(y')\delta(t')]_{\text{ret}}}{|\vec{x}-\vec{x}'|} dx'dy'dz'\end{aligned}$$

Noting that $[t']_{\text{ret}} = t - |\vec{x}-\vec{x}'|/c$,

$$\begin{aligned}\Psi(\vec{x},t) &= \iiint \frac{\delta(x')\delta(y')\delta\left(t - \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}/c\right)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} dx'dy'dz' \\ &= \int \frac{\delta\left(t-\sqrt{x^2+y^2+(z-z')^2}/c\right)}{\sqrt{x^2+y^2+(z-z')^2}} dz'\end{aligned}$$

Letting $\rho = \sqrt{x^2+y^2}$ and $\tilde{z} = z - z'$:

$$\Psi(\vec{x},t) = \int \frac{\delta\left(t-\sqrt{\rho^2+\tilde{z}^2}/c\right)}{\sqrt{\rho^2+\tilde{z}^2}} d\tilde{z}$$

We will use the following identity:

$$\delta(f(z)) = \sum_i \frac{1}{|f'(z)|}\delta(z-z_i) \tag{6.1}$$

where z_i are the zeroes of $f(z)$: $z_i = \pm\sqrt{c^2t^2-\rho^2}$. Hence, the delta function our expression for $\Psi(\vec{x},t)$ is equal to:

$$\delta\left(t-\sqrt{\rho^2+z^2}/c\right) = \sum_i \frac{c\sqrt{\rho^2+z^2}}{|z|}\delta(z-z_i)$$

$$\begin{aligned}\Psi(\vec{x},t) &= \int \frac{1}{\sqrt{\rho^2+\tilde{z}^2}}\left[\frac{c\sqrt{\rho^2+\tilde{z}^2}}{|\tilde{z}|}\delta\left(\tilde{z}-\sqrt{c^2t^2-\rho^2}\right) + \frac{c\sqrt{\rho^2+\tilde{z}^2}}{|\tilde{z}|}\delta\left(\tilde{z}+\sqrt{c^2t^2-\rho^2}\right)\right] d\tilde{z} \\ &= \frac{1}{\sqrt{\rho^2+\left(\sqrt{c^2t^2-\rho^2}\right)^2}}\left[\frac{c\sqrt{\rho^2+\left(\sqrt{c^2t^2-\rho^2}\right)^2}}{\left|\sqrt{c^2t^2-\rho^2}\right|} + \frac{c\sqrt{\rho^2+\left(-\sqrt{c^2t^2-\rho^2}\right)^2}}{\left|-\sqrt{c^2t^2-\rho^2}\right|}\right] \\ &= \frac{1}{\cancel{|ct|}}2\frac{c\,\cancel{|ct|}}{\sqrt{c^2t^2-\rho^2}}\end{aligned}$$

Note that this solution is imaginary for $ct < \rho$ as a result of the delta function we're using. However, it is important to note that we're integrating over the real number line– therefore, the imaginary solutions are forbidden. Hence, $\Psi(\vec{x}, t)$ is zero for $ct < \rho$. We will multiply it by the unit step function:

$$\boxed{\Psi(\vec{x}, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}}$$

6.1.2

Substituting $f(\vec{x}', t') = \delta(x')\delta(t')$ into equation 6.47 in Jackson:

$$\begin{aligned}\Psi(\vec{x}, t) &= \int \frac{[f(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|} d^3x' \\ &= \int \frac{[\delta(x')\delta(t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|} dx'dy'dz'\end{aligned}$$

Noting that $[t']_{\text{ret}} = t - |\vec{x} - \vec{x}'|/c$,

$$\begin{aligned}\Psi(\vec{x}, t) &= \iiint \frac{\delta(x')\delta\left(t - \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}/c\right)}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} dx'dy'dz' \\ &= \iint \frac{\delta\left(t - \sqrt{x^2 + (y - y')^2 + (z - z')^2}/c\right)}{\sqrt{x^2 + (y - y')^2 + (z - z')^2}} dy'dz'\end{aligned}$$

Letting $\tilde{y} = y - y'$ and $\tilde{z} = z - z'$:

$$\Psi(\vec{x}, t) = \iint \frac{\delta\left(t - \sqrt{x^2 + \tilde{y}^2 + \tilde{z}^2}/c\right)}{\sqrt{x^2 + \tilde{y}^2 + \tilde{z}^2}} d\tilde{y}d\tilde{z}$$

Converting to polar coordinates in the $\tilde{y}$-$\tilde{z}$ plane:

$$\begin{aligned}\Psi(\vec{x}, t) &= \iint \frac{\delta\left(t - \sqrt{x^2 + \rho^2}/c\right)}{\sqrt{x^2 + \rho^2}} \rho d\rho d\varphi \\ &= 2\pi \int \frac{\delta\left(t - \sqrt{x^2 + \rho^2}/c\right)}{\sqrt{x^2 + \rho^2}} d\rho\end{aligned}$$

Again, we will use the identity in equation (6.1) to determine that the delta function in the above equation is equal to:

$$\delta\left(t - \sqrt{x^2 + \rho^2}/c\right) = \sum_i \frac{c\sqrt{x^2 + \rho^2}}{|\rho|} \delta(\rho - \rho_i)$$

Noting that the zero of the argument of the our delta function is $\rho_i = \sqrt{c^2t^2 - x^2}$ (there is only one root since ρ is strictly nonnegative) and plugging this identity into our expression for $\Psi(\vec{x}, t)$:

$$\begin{aligned}
\Psi(\vec{x}, t) &= 2\pi \int \frac{1}{\cancel{\sqrt{x^2+\rho^2}}} \left[\frac{c\cancel{\sqrt{x^2+\rho^2}}}{\cancel{\rho}} \delta\left(\rho - \sqrt{c^2t^2 - x^2}\right) \right] \cancel{\rho} d\rho \\
&= 2\pi \int c\delta\left(\rho - \sqrt{c^2t^2 - x^2}\right) d\rho \\
&= 2\pi \int c
\end{aligned}$$

Again, we have imaginary roots for $ct < |x|$. For this reason, we again multiply $\Psi(\vec{x}, t)$ by the unit step function:

$$\boxed{\Psi(\vec{x}, t) = 2\pi c \Theta(ct - |x|)}$$

6.4 Problem 6.4

6.4.1

We are given that the sphere is uniformly magnetized with $\vec{m} = (4\pi/3)\vec{M}R^3$ (equation 5.107 in Jackson). We will pick a coordinate system such that the sphere is rotating about the z-axis. Hence, $\vec{m} = m\hat{z}$. Solving for $\vec{M}$ and plugging into equation 5.105 in Jackson yields:

$$\begin{aligned}
\vec{B} &= \frac{2\mu_0}{3}\vec{M} \\
&= \frac{2\mu_0}{3}\left(\frac{3m\hat{z}}{4\pi R^3}\right) \\
&= \frac{\mu_0 m}{2\pi R^3}\hat{z}
\end{aligned}$$

Equation 5.142 in Jackson states that $\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$. Assuming there is no external electric field, $\vec{E}' = 0$ and hence:

$$\begin{aligned}
\vec{E} &= -\vec{v} \times \vec{B} \\
&= -\left(\omega\hat{z} \times \vec{R}\right) \times \hat{z}\frac{\mu_0 m}{2\pi R^3} \\
&= -\frac{\mu_0 m\omega}{2\pi R^3}\left[\vec{R}\underbrace{(\hat{z}\cdot\hat{z})}_{1} - \hat{z}\underbrace{\left(\hat{z}\cdot\vec{R}\right)}_{R\cos\theta}\right] \\
&= -\frac{\mu_0 m\omega}{2\pi R^3}\left[\vec{R} - \hat{z}R\cos\theta\right]
\end{aligned}$$

In cylindrical coordinates:

$$E_z = E_\varphi = 0$$
$$E_r = -\frac{\mu_0 m\omega r}{2\pi R^3}$$

Using the differential form of Gauss' Law:

$$\begin{aligned}\frac{\rho}{\varepsilon_0} &= \nabla \cdot \vec{E} \\ &= \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\cancelto{0}{\frac{\partial E_\varphi}{\partial \varphi}} + \cancelto{0}{\frac{\partial E_z}{\partial z}} \\ &= -\frac{1}{r}\left(\frac{2\pi R^3 \mu_0 m\omega 2r - \mu_0 m\omega r^2 2\pi R^3}{4\pi^2 R^6}\right) \\ &= -\frac{\mu_0 m\omega}{\pi R^3} + \frac{\mu_0 m\omega r}{2\pi R^3}\end{aligned}$$

$$\boxed{\rho = -\frac{m\omega}{\pi c^2 R^3} + \frac{m\omega r}{2\pi R^3}}$$

6.4.2

As has already been given, the monopole moments ($l = 0$) vanish because the sphere is electrically neutral. In addition, because the electric field found in the previous part is odd ($E(r) = -E(-r)$), we note that the $l = 1$ terms will also vanish (in fact, all the odd l terms will vanish). Because the quadrupole moment ($l = 2$) is nonvanishing (as will be shown next), the lowest nonvanishing moments are quadrupole.

We begin by find the electrostatic potential in cylindrical coordinates:

$$\Phi(\vec{x}) = -\int \vec{E} \cdot d\vec{\ell} = -\left(-\frac{\mu_0 m\omega r^2}{2\pi R^3}\right)$$

Converting to spherical coordinates:

$$\Phi(\vec{x}) = \frac{\mu_0 m\omega r^2 \sin^2\theta}{2\pi R^3}$$

Noting that $\sin^2\theta = \frac{1}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right]$:

$$\Phi(\vec{x}) = \frac{\mu_0 m\omega r^2}{2\pi R^3}\frac{1}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right]$$

We're particularly interested in the $\ell = 2$ term:

$$\Phi_{\ell=2}(r = R) = -\frac{\mu_0 m\omega}{6\pi R}P_2(\cos\theta)$$

Comparing this with the $\ell = 2, m = 0$ term of equation 4.1 in Jackson yields:

$$\begin{aligned} q_{2,0} &= \frac{\varepsilon_0 5R^3}{Y_{1,0}(\theta,\varphi)}\left(-\frac{\mu_0 m\omega}{6\pi R}P_2(\cos\theta)\right) \\ &= -\frac{5m\omega R^2}{6\pi c^2}\frac{P_2(\cos\theta)}{Y_{1,0}(\theta,\varphi)} \\ &= -\frac{5m\omega r^3}{6\pi c^2 R^3}\frac{\frac{1}{2}\left(3\cos^2\theta - 1\right)}{\frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta - 1\right)} \\ &= -\frac{5m\omega R^2}{3c^2\pi}\sqrt{\frac{\pi}{5}} \end{aligned}$$

From equation 4.6 in Jackson, we can see that $Q_{3,3} = 2\sqrt{\frac{4\pi}{5}}q_{2,0}$:

$$Q_{3,3} = 2\sqrt{\frac{4\pi}{5}}\left(-\frac{5m\omega R^2}{3c^2\pi}\sqrt{\frac{\pi}{5}}\right)$$

$$\boxed{Q_{3,3} = -\frac{4m\omega R^2}{3c^2}}$$

Because the quadrupole moment tensor is traceless, $Q_{1,1} + Q_{2,2} + Q_{3,3}$. By x-y symmetry, $Q_{1,1} = Q_{2,2}$. Hence, $Q_{1,1} = Q_{1,1} = -\frac{1}{2}Q_{3,3}$.

6.4.3

The electrostatic potential inside the sphere is as found in the previous part:

$$\begin{aligned} \Phi_{\text{in}}(\vec{x}) &= \frac{\mu_0 m\omega r^2}{2\pi R^3}\frac{1}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right] \\ \therefore \vec{E}^r_{\text{in}} &= -\frac{\mu_0 m\omega r}{\pi R^3}\frac{1}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right] \end{aligned}$$

Because everything lower than $\ell = 2$ vanishes outside the sphere, the electrostatic potential outside the sphere is:

$$\begin{aligned} \Phi_{\text{out}}(\vec{x}) &= -\frac{\mu_0 m\omega R^2}{2\pi r^3}\frac{1}{3}P_2(\cos\theta) \\ \therefore \vec{E}^r_{\text{out}} &= -\frac{\mu_0 m\omega R^2}{2\pi r^4}P_2(\cos\theta) \end{aligned}$$

$$\begin{aligned} \sigma(\theta) &= \varepsilon_0\left[E^r_{\text{out}} - E^r_{\text{in}}\right]_{r=R} \\ &= \varepsilon_0\left[-\frac{\mu_0 m\omega R^2}{2\pi r^4}P_2(\cos\theta) - \left(-\frac{\mu_0 m\omega r}{\pi R^3}\frac{1}{3}\left[1 - P_2(\cos\theta)\right]\right)\right]_{r=R} \\ &= \frac{m\omega}{\pi c^2 R^2}\left(-\frac{1}{2}P_2(\cos\theta) + \frac{1}{3}\left[1 - P_2(\cos\theta)\right]\right) \end{aligned}$$

$$\boxed{\sigma(\theta) = \frac{m\omega}{3\pi c^2 R^2}\left(1 - \frac{5}{2}P_2(\cos\theta)\right)}$$

6.4.4

$$\begin{aligned}
\mathcal{E} &= \int_{\theta=\pi/2}^{0} \vec{E}\cdot d\vec{\ell} = [-\Phi_{\text{out}}]_{\theta=\pi/2}^{0}\Big|_{r=R} \\
&= \left[\frac{\mu_0 m\omega R^2}{2\pi r^3}\frac{1}{3}\underbrace{P_2(\cos(0))}_{1} - \left(-\frac{\mu_0 m\omega R^2}{2\pi r^3}\frac{1}{3}\underbrace{P_2\left(\cos\left(\frac{\pi}{2}\right)\right)}_{-1/2}\right)\right]_{r=R} \\
&= \frac{\mu_0 m\omega}{6\pi R} + \frac{\mu_0 m\omega}{12\pi R}
\end{aligned}$$

$$\boxed{\mathcal{E} = \frac{\mu_0 m\omega}{4\pi R}}$$

6.5 Problem 6.5

6.5.1

Starting with equation 6.117 in Jackson:

$$\begin{aligned}
\vec{P}_{\text{field}} &= \frac{1}{c^2}\int_V \vec{E}\times\vec{H}d^3x \\
&= \frac{1}{c^2}\int_V (-\nabla\Phi)\times\vec{H}d^3x \\
P^i_{\text{field}} &= -\frac{1}{c^2}\sum_{i,j}\varepsilon_{ijk}\int_V \frac{\partial\Phi}{\partial x_i}H_j d^3x
\end{aligned}$$

Integrating by parts:

$$\begin{aligned}
P^i_{\text{field}} &= \frac{1}{c^2}\sum_{i,j}\left[-\varepsilon_{ijk}\int_S \Phi H_j dS_i + \varepsilon_{ijk}\int_V \Phi\frac{\partial H_j}{\partial x_i}d^3x\right] \\
\vec{P}_{\text{field}} &= -\frac{1}{c^2}\int_S \Phi d\vec{S}\times\vec{H} + \frac{1}{c^2}\int_V \Phi\underbrace{\nabla\times\vec{H}}_{\vec{J}}d^3x \\
&= -\frac{1}{c^2}\int_S \Phi d\vec{S}\times\vec{H} + \frac{1}{c^2}\int_V \Phi\vec{J}d^3x
\end{aligned}$$

The surface integral vanishes if $\Phi d\vec{S}\times\vec{H}\to 0$ as $r\to\infty$. Since $dS\propto r^2$, the surface integral vanishes if $r^2\Phi\vec{H}\to 0$ as $r\to\infty$.

6.5.2

We start by Taylor expanding Φ:

$$\Phi = \cancelto{0}{\Phi(\vec{0})} + \vec{x} \cdot \underbrace{\nabla\Phi(\vec{0})}_{-\vec{E}(\vec{0})} + \dots$$

Plugging this into our solution for $\vec{P}_{\text{field}}$ from the previous part yields:

$$\begin{aligned}
\vec{P}_{\text{field}} &= \frac{1}{c^2}\int \left(-\vec{x}\cdot\vec{E}\right)\vec{J}d^3x \\
P^i_{\text{field}} &= -\frac{1}{c^2}\sum_j \int J_i x_j E_j(0) d^3x \\
&= -\frac{1}{c^2}\sum_j E_j(0)\int x_j J_i d^3x
\end{aligned}$$

Using the equation two equations below 5.52 in Jackson:

$$\int x_j J_i d^3x = -\int x_i J_j d^3x$$

$$\implies \int x_j J_i d^3x = \frac{1}{2}\left(\int x_j J_i d^3x - \int x_i J_j d^3x\right)$$

Plugging this into our expression for P^i_{field}:

$$\begin{aligned}
P^i_{\text{field}} &= -\frac{1}{c^2}\sum_j E_j(0)\int \frac{1}{2}\left(x_j J_i - x_i J_j\right) d^3x \\
&= -\frac{1}{c^2}\sum_{j,k} \varepsilon_{ijk} E_j(0)\frac{1}{2}\int \left(-\vec{x}\times\vec{J}\right)_k d^3x \\
\vec{P}_{\text{field}} &= \frac{1}{c^2}\vec{E}(\vec{0}) \times \underbrace{\frac{1}{2}\int \left(\vec{x}\times\vec{J}\right) d^3x}_{\vec{m}}
\end{aligned}$$

$$\boxed{\vec{P}_{\text{field}} = \frac{1}{c^2}\vec{E}(\vec{0}) \times \vec{m}}$$

6.5.3

We start by dividing both sides of equation 5.56 in Jackson by μ_0:

$$\vec{H}(\vec{x}) = \frac{1}{4\pi}\left[\frac{3\hat{r}\left(\hat{r}\cdot\vec{m}\right) - \vec{m}}{|\vec{r}|^3}\right]$$

Substituting this into the surface integral from the first part of this problem yields:

$$-\frac{1}{c^2}\int_S \underbrace{\Phi}_{\approx -\vec{r}\cdot\vec{E}_0} d\vec{S}\times\vec{H} = -\frac{1}{c^2}\int_S \left(-\vec{r}\cdot\vec{E}_0\right)\left(dS\hat{r}\times\frac{1}{4\pi}\left[\frac{3\hat{r}\left(\hat{r}\cdot\vec{m}\right)-\vec{m}}{|\vec{r}|^3}\right]\right)$$

$$= \frac{1}{4\pi c^2}\int_S \left(\vec{r}\cdot\vec{E}_0\right)\left[-\frac{\vec{r}\times\vec{m}}{|\vec{r}|^4}\right] dS$$

$$= -\frac{1}{4\pi c^2}\int_S \left(\vec{r}\cdot\vec{E}_0\right)\left[\frac{\vec{r}\times\vec{m}}{\cancelto{r^2}{|\vec{r}|^4}}\right]\cancel{r^2}d(\cos\theta)d\varphi$$

$$= -\frac{1}{4\pi c^2}\int_S \left(\vec{r}\cdot\vec{E}_0\right)\left[\frac{\vec{r}\times\vec{m}}{|\vec{r}|^2}\right] d(\cos\theta)d\varphi$$

$$\Leftrightarrow -\frac{1}{4\pi}\int_S \frac{1}{r^2}r_\ell\vec{E}_{0,\ell}\varepsilon_{ijk}r_j m_k d(\cos\theta)d\varphi$$

$$= -\frac{1}{4\pi c^2}\varepsilon_{ijk}E_{0,\ell}m_k\int_S \frac{r_\ell r_j}{r^2}d(\cos\theta)d\varphi$$

$$= -\frac{1}{4\pi c^2}\varepsilon_{ijk}E_{0,\ell}m_k\int_S \frac{r_\ell r_j}{r^2}d(\cos\theta)d\varphi$$

The integral is zero unless $\ell = j$. Hence:

$$= -\frac{1}{4\pi c^2}\varepsilon_{ijk}E_{0,j}m_k\int_S \frac{r_j^2}{r^2}d(\cos\theta)d\varphi$$

$$= -\frac{1}{4\pi c^2}\varepsilon_{ijk}E_{0,j}m_k\left[\frac{1}{3}\underbrace{\int_S \frac{\cancel{r^2}}{\cancel{r^2}}d(\cos\theta)d\varphi}_{4\pi}\right]$$

$$= -\frac{1}{4\pi c^2}\varepsilon_{ijk}E_{0,j}m_k\frac{1}{3}4\pi$$

$$= -\frac{1}{3c^2}\varepsilon_{ijk}E_{0,j}m_k$$

$$\Leftrightarrow -\frac{1}{3c^2}\vec{E}_0\times\vec{m}$$

Adding this to the volume integral (which is equal to the solution found in the second part to this problem) yields the final answer:

$$\vec{P}_{\text{field}} = \left(-\frac{1}{3c^2}\vec{E}_0\times\vec{m}\right)+\left(\frac{1}{c^2}\vec{E}_0\times\vec{m}\right)$$

$$\boxed{\vec{P}_{\text{field}} = \frac{2}{3c^2}\vec{E}_0\times\vec{m}}$$

The same result can be obtained by plugging equation equation 5.62 ($\int_V \vec{H} d^3x = \frac{2}{3}\vec{m}$) into equation 6.117:

$$\begin{aligned}
\vec{P}_{\text{field}} &= \frac{1}{c^2}\vec{E}_0 \times \int_V \vec{H} d^3x \\
&= \frac{1}{c^2}\vec{E}_0 \times \left(\frac{2}{3c^2}\vec{m}\right) \\
&= \frac{2}{3c^2}\vec{E}_0 \times \vec{m}
\end{aligned}$$