Physics 506 **Winter 2008**

Homework Assignment #12 — Solutions

Textbook problems: Ch. 14: 14.4, 14.5, 14.8, 14.11

14.4 Using the Liénard-Wiechert fields, discuss the time-averaged power radiated per unit solid angle in nonrelativisic motion of a particle with charge e, moving

a) along the z axis with instantaneous position $z(t) = a\cos\omega_0 t$.

In the non-relativisitic limit, the radiated power is given by

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c}|\hat{n}\times\dot{\vec{\beta}}|^2 \tag{1}$$

In the case of harmonic motion along the z axis, we take

$$\vec{r} = \hat{z}a\cos\omega_0 t, \qquad \vec{\beta} = -\hat{z}\frac{a\omega_0}{c}\sin\omega_0 t, \qquad \dot{\vec{\beta}} = -\hat{z}\frac{a\omega_0^2}{c}\cos\omega_0 t$$

By symmetry, we assume the observer is in the x-z plane tilted with angle θ from the vertical. In other words, we take

$$\hat{n} = \hat{x}\sin\theta + \hat{z}\cos\theta$$

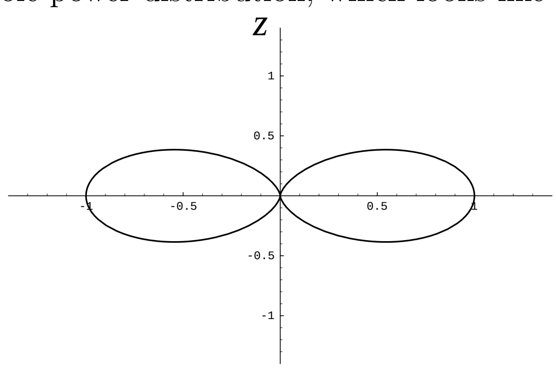
This provides enough information to simply substitute into the power expression (1)

$$\hat{n}\times\dot{\vec{\beta}} = \hat{y}\frac{a\omega_0^2}{c}\sin\theta\cos\omega_0 t \qquad \Rightarrow \qquad \frac{dP(t)}{d\Omega} = \frac{e^2a^2\omega_0^4}{4\pi c^3}\sin^2\theta\cos^2\omega_0 t$$

Taking a time average ($\cos^2\omega_0 t \to 1/2$) gives

$$\frac{dP}{d\Omega} = \frac{e^2a^2\omega_0^4}{8\pi c^3}\sin^2\theta$$

This is a familiar dipole power distribution, which looks like



Integrating over angles gives the total power

$$P = \frac{e^2a^2\omega_0^4}{3c^3}$$

b) in a circle of radius R in the x-y plane with constant angular frequency ω_0.

Sketch the angular distribution of the radiation and determine the total power radiated in each case.

Here we take instead

$$\vec{r} = R(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t) \qquad \rightarrow \qquad \vec{\beta} = \frac{R\omega_0}{c}(-\hat{x}\sin\omega_0 t + \hat{y}\cos\omega_0 t)$$

$$\dot{\vec{\beta}} = -\frac{R\omega_0^2}{c}(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t)$$

Then

$$\hat{n}\times\dot{\vec{\beta}} = -\frac{R\omega_0^2}{c}[\hat{y}\cos\theta\cos\omega_0 t + (\hat{z}\sin\theta - \hat{x}\cos\theta)\sin\omega_0 t]$$

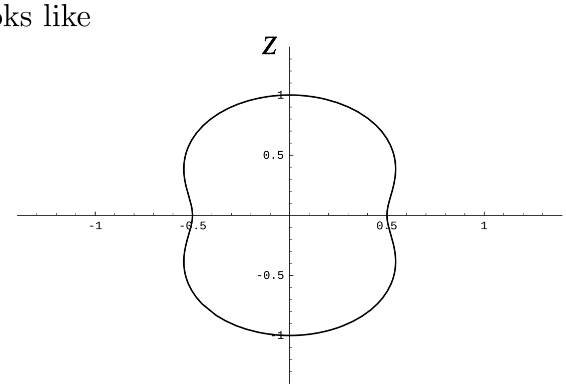
which gives

$$\frac{dP(t)}{d\Omega} = \frac{e^2R^2\omega_0^4}{4\pi c^3}(\cos^2\theta\cos^2\omega_0 t + \sin^2\omega_0 t)$$

Taking a time average gives

$$\frac{dP}{d\Omega} = \frac{e^2R^2\omega_0^4}{8\pi c^3}(1+\cos^2\theta)$$

This distribution looks like



The total power is given by integration over angles. The result is

$$P = \frac{2e^2R^2\omega_0^4}{3c^3}$$

14.5 A *nonrelativistic* particle of charge ze, mass m, and kinetic energy E makes a *head-on* collision with a fixed central force field of finite range. The interaction is repulsive and described by a potential $V(r)$, which becomes greater than E at close distances.

a) Show that the total energy radiated is given by

$$\Delta W = \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\left|\frac{dV}{dr}\right|^2\frac{dr}{\sqrt{V(r_{\min})-V(r)}}$$

where $r_{\rm min}$ is the closest distance of approach in the collision.

In the non-relativistic limit, we may use Lamour's formula written in terms of $\dot{\vec{p}}$

$$P(t) = \frac{2(ze)^2}{3m^2c^3}\left|\frac{d\vec{p}}{dt}\right|^2 = \frac{2(ze)^2}{3m^2c^3}\left(\frac{dV(r)}{dr}\right)^2 \tag{2}$$

where we have used Newton's second law to write

$$\frac{d\vec{p}}{dt} = \vec{F} = -\hat{r}\frac{dV(r)}{dr}$$

The radiated energy is given by integrating power over time

$$\Delta W = \int_{-\infty}^{\infty} P(t)\, dt$$

However, this can be converted to an integral over the trajectory of the particle. By symmetry, we double the value of the integral from closest approach to infinity

$$\Delta W = 2\int_{r_{\rm min}}^{\infty} \frac{P}{dr/dt} dr \tag{3}$$

The velocity dr/dt can be obtained from energy conservation. For a head-on collision, we have simply

$$E = \frac{1}{2}m\dot{r}^2 + V(r) \qquad \Rightarrow \qquad \frac{dr}{dt} = \sqrt{\frac{2(E - V(r))}{m}}$$

Substituting $P(t)$ from (2) as well as dr/dt into (3) then yields

$$\Delta W = \frac{4z^2e^2}{3m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\rm min}}^{\infty}\left(\frac{dV}{dr}\right)^2 \frac{dr}{\sqrt{E - V(r)}}$$

Since the velocity (and hence kinetic energy) vanishes at closest approach, the total energy E is the same as the potential energy at closest approach, $E = V(r_{\rm min})$. Using this finally gives

$$\Delta W = \frac{4z^2e^2}{3m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\rm min}}^{\infty}\left(\frac{dV}{dr}\right)^2 \frac{dr}{\sqrt{V(r_{\rm min}) - V(r)}} \tag{4}$$

b) If the interaction is a Coulomb potential $V(r) = zZe^2/r$, show that the total energy radiated is

$$\Delta W = \frac{8}{45}\frac{zmv_0^5}{Zc^3}$$

where v_0 is the velocity of the charge at infinity.

Substituting

$$V(r) = \frac{zZe^2}{r}, \qquad \frac{dV}{dr} = -\frac{zZe^2}{r^2}$$

into (4) gives

$$\begin{aligned}
\Delta W &= \frac{4z^3Ze^5}{3m^2c^3}\sqrt{\frac{zZmr_{\min}}{2}}\int_{r_{\min}}^{\infty}\frac{1}{r^{7/2}}\frac{dr}{\sqrt{r-r_{\min}}} \\
&= \frac{4z^3Ze^5}{3m^2c^3r_{\min}^3}\sqrt{\frac{zZmr_{\min}}{2}}\int_1^{\infty}\frac{1}{r^{7/2}}\frac{dr}{\sqrt{r-1}} \\
&= \frac{4z^3Ze^5}{3m^2c^3r_{\min}^3}\sqrt{\frac{zZmr_{\min}}{2}}\times\frac{16}{15} \\
&= \frac{32z^3Ze^5}{45m^2c^3r_{\min}^3}\sqrt{2zZmr_{\min}}
\end{aligned}$$

We may relate $r_{\min}$ to the velocity v_0 at infinity using energy conservation

$$\frac{zZe^2}{r_{\min}} = \frac{1}{2}mv_0^2 \qquad \Rightarrow \qquad r_{\min} = \frac{2zZe^2}{mv_0^2}$$

Substituting this in the above radiated energy expression gives

$$\Delta W = \frac{8zmv_0^5}{45Zc^3}$$

14.8 A swiftly moving particle of charge ze and mass m passes a fixed point charge Ze in an approximately straight-line path at impact parameter b and nearly constant speed v. Show that the total energy radiated in the encounter is

$$\Delta W = \frac{\pi z^4Z^2e^6}{4m^2c^4\beta}\left(\gamma^2+\frac{1}{3}\right)\frac{1}{b^3}$$

This is the relativistic generalization of the result of Problem 14.7.

We start with the Liénard result for the radiated power of a relativistic accelerated charge

$$P = \frac{2}{3}\frac{(ze)^2}{c}\gamma^6[(\dot{\vec\beta})^2 - (\vec\beta\times\dot{\vec\beta})^2]$$

We may remove the cross-product by rewriting the second term using the identity $(\vec\beta\times\dot{\vec\beta})^2 = \beta^2(\dot{\vec\beta})^2 - (\vec\beta\cdot\dot{\vec\beta})^2$. The result is

$$P = \frac{2}{3}\frac{(ze)^2}{c}\gamma^4[(\dot{\vec\beta})^2 + \gamma^2(\vec\beta\cdot\dot{\vec\beta})^2] \tag{5}$$

We now compute the acceleration $\dot{\vec{\beta}}$ for a particle obeying Newton's second law. Starting with

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{v}) = mc\frac{d}{dt}\frac{\vec{\beta}}{\sqrt{1-\beta^2}} = mc\frac{(1-\beta^2)\dot{\vec{\beta}} + \vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})}{(1-\beta^2)^{3/2}}$$

we write

$$(1-\beta^2)\dot{\vec{\beta}} + \vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}}) = \frac{1}{mc\gamma^3}\vec{F} \tag{6}$$

In order to solve this expression for $\dot{\vec{\beta}}$ we may first take the dot product of both sides with the velocity $\vec{\beta}$ to obtain

$$\vec{\beta}\cdot\dot{\vec{\beta}} = \frac{1}{mc\gamma^3}\vec{\beta}\cdot\vec{F}$$

Note that, physically, this gives the parallel component of the acceleration in terms of the parallel component of the force. Substituting this back into (6) gives the desired expression

$$\dot{\vec{\beta}} = \frac{1}{mc\gamma}[\vec{F} - \vec{\beta}(\vec{\beta}\cdot\vec{F})]$$

We now insert this into the Liénard result, (5), to get

$$\begin{aligned} P &= \frac{2}{3}\frac{(ze)^2\gamma^2}{m^2c^3}[(\vec{F} - \vec{\beta}(\vec{\beta}\cdot\vec{F}))^2 + \gamma^{-2}(\vec{\beta}\cdot\vec{F})^2] \\ &= \frac{2}{3}\frac{(ze)^2\gamma^2}{m^2c^3}[F^2 - (\vec{\beta}\cdot\vec{F})^2] \end{aligned} \tag{7}$$

If desired, we can break this up into force components parallel and perpendicular to the velocity

$$F^2 = F_\perp^2 + F_\parallel^2, \qquad \vec{\beta}\cdot\vec{F} = \beta F_\parallel$$

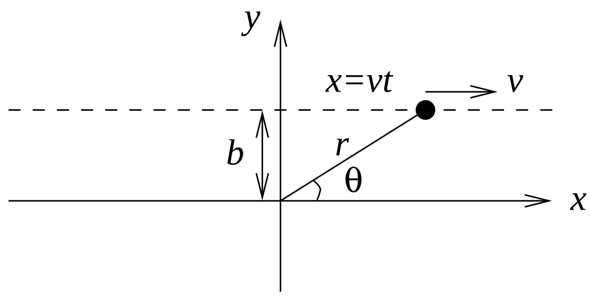
to arrive at

$$P = \frac{2}{3}\frac{(ze)^2}{m^2c^3}[\gamma^2 F_\perp^2 + F_\parallel^2] \tag{8}$$

For the Coulomb potential $V = (ze)(Ze)/r$, the force is radially directed

$$\vec{F} = -\hat{r}\frac{dF}{dr} = \hat{r}\frac{zZe^2}{r^2}$$

Assuming the particle moves in an approximately straight-line path with impact parameter b



the parallel and perpendicular components of the force are

$$F_\perp = \frac{zZe^2}{r^2}\frac{b}{r}, \qquad F_\parallel = \frac{zZe^2}{r^2}\frac{\sqrt{r^2-b^2}}{r}$$

Inserting this into (8) gives

$$P = \frac{2}{3}\frac{z^4Z^2e^6}{m^2c^3}\frac{b^2\gamma^2+(r^2-b^2)}{r^6} = \frac{2}{3}\frac{z^4Z^2e^6}{m^2c^3}\left(\frac{b^2(\gamma^2-1)}{r^6}+\frac{1}{r^4}\right)$$

The total radiated energy is given by integrating

$$\Delta W = \int_{-\infty}^{\infty} P\,dt = 2\int_b^\infty \frac{P}{dr/dt}dr = 2\int_b^\infty P\frac{r}{v}\frac{dr}{\sqrt{r^2-b^2}}$$

where we have used the relation $r^2 = x^2+b^2 = (vt)^2+b^2$ to change from time to radial integration. Substituting in the explicit formula for the power gives

$$\begin{aligned}\Delta W &= \frac{4}{3}\frac{z^4Z^2e^6}{m^2c^3v}\int_b^\infty \left(\frac{b^2(\gamma^2-1)}{r^5}+\frac{1}{r^3}\right)\frac{dr}{\sqrt{r^2-b^2}} \\ &= \frac{4}{3}\frac{z^4Z^2e^6}{m^2c^4\beta b^3}\int_1^\infty \left(\frac{(\gamma^2-1)}{u^5}+\frac{1}{u^3}\right)\frac{du}{\sqrt{u^2-1}}\end{aligned}$$

where we have changed to a dimensionless variable u by letting $r = bu$. The integral can now be performed by trig substitution $u = \sec\theta$, $\sqrt{u^2-1} = \tan\theta$ and $du = \tan\theta\sec\theta d\theta$

$$\begin{aligned}\Delta W &= \frac{4}{3}\frac{z^4Z^2e^6}{m^2c^4\beta b^3}\int_0^{\pi/2}[(\gamma^2-1)\cos^4\theta+\cos^2\theta]d\theta \\ &= \frac{4}{3}\frac{z^4Z^2e^6}{m^2c^4\beta b^3}\left((\gamma^2-1)\frac{3\pi}{16}+\frac{\pi}{4}\right) \\ &= \frac{\pi z^4Z^2e^6}{4m^2c^4\beta b^3}\left((\gamma^2-1)+\frac{4}{3}\right) = \frac{\pi z^4Z^2e^6}{4m^2c^4\beta b^3}\left(\gamma^2+\frac{1}{3}\right)\end{aligned}$$

14.11 A particle of charge ze and mass m moves in external electric and magnetic fields $\vec{E}$ and $\vec{B}$.

a) Show that the classical relativistic result for the instantaneous energy radiated per unit time can be written

$$P = \frac{2}{3}\frac{z^4e^4}{m^2c^3}\gamma^2[(\vec{E}+\vec{\beta}\times\vec{B})^2 - (\vec{\beta}\cdot\vec{E})^2]$$

where $\vec{E}$ and $\vec{B}$ are evaluated at the position of the particle and γ is the particle's instantaneous Lorentz factor.

We start with the Liénard result, written in terms of the force, which was obtained above in (7)

$$P = \frac{2}{3}\frac{(ze)^2\gamma^2}{m^2c^3}[F^2 - (\vec{\beta}\cdot\vec{F})^2]$$

The Lorentz force for a particle of charge ze is given by

$$\vec{F} = ze(\vec{E}+\vec{\beta}\times\vec{B})$$

so that

$$F^2 = (ze)^2(\vec{E}+\vec{\beta}\times\vec{B})^2, \qquad (\vec{\beta}\cdot\vec{F})^2 = (ze)^2(\vec{\beta}\cdot\vec{E})^2$$

Substituting this into the expression for the radiated power immediately yields the desired result

$$P = \frac{2}{3}\frac{(ze)^4\gamma^2}{m^2c^3}[(\vec{E}+\vec{\beta}\times\vec{B})^2 - (\vec{\beta}\cdot\vec{E})^2] \tag{9}$$

b) Show that the expression in part a can be put into the manifestly Lorentz-invariant form

$$P = \frac{2z^4r_0^2}{3m^2c}F^{\mu\nu}p_\nu p^\lambda F_{\lambda\mu}$$

where $r_0 = e^2/mc^2$ is the classical charged particle radius.

We can perform an explicit calculation with

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad p^\mu = \gamma mc(1, \beta_x, \beta_y, \beta_z)$$

to obtain

$$F^{\mu\nu}p_\nu = \gamma mc \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\beta_x \\ -\beta_y \\ -\beta_z \end{pmatrix} = \gamma mc \begin{pmatrix} -\vec{\beta}\cdot\vec{E} \\ [\vec{E}+\vec{\beta}\times\vec{B}]_x \\ [\vec{E}+\vec{\beta}\times\vec{B}]_y \\ [\vec{E}+\vec{\beta}\times\vec{B}]_z \end{pmatrix}$$

For $p^\lambda F_{\lambda\mu}$, we may use antisymmetry of the Maxwell tensor along with a lowering of the μ index to deduce that

$$p^\lambda F_{\lambda\mu} = \gamma mc \begin{pmatrix} \vec{\beta}\cdot\vec{E} \\ [\vec{E}+\vec{\beta}\times\vec{B}]_x \\ [\vec{E}+\vec{\beta}\times\vec{B}]_y \\ [\vec{E}+\vec{\beta}\times\vec{B}]_z \end{pmatrix}$$

As a result, we see that

$$F^{\mu\nu}p_\nu p^\lambda F_{\lambda\mu} = \gamma^2 m^2 c^2[(\vec{E}+\vec{\beta}\times\vec{B})^2 - (\vec{\beta}\cdot\vec{E})^2]$$

This allows us to rewrite (9) in the manifestly Lorentz-invariant form

$$P = \frac{2}{3}\frac{(ze)^4}{m^4c^5}F^{\mu\nu}p_\nu p^\lambda F_{\lambda\mu} = \frac{2}{3}\frac{z^4 r_0^2}{m^2 c}F^{\mu\nu}p_\nu p^\lambda F_{\lambda\mu} \tag{10}$$

where we have introduced $r_0 = e^2/mc^2$.

Alternatively, note that the relativistic generalization of Larmor's formula is given by

$$P = -\frac{2}{3}\frac{(ze)^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right)$$

Using the manifestly Lorentz covariant form of the Lorentz force law

$$\frac{dp^\mu}{d\tau} = \frac{ze}{c}F^{\mu\nu}U_\nu = \frac{ze}{mc}F^{\mu\nu}p_\nu$$

then directly gives (10).